SOME PARTICLE DISTRIBUTIONS AND PLASMA–MAGNETIC FIELD BOUNDARIES

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Summary

Three properties of the plasma sheath in quadrupole and hexapole geometries are investigated: the variation of the magnetic field, particle density, and electrostatic potential in the transition layer. Some comparison is made with experimental results.

I. Introduction

We wish to find a model to describe the steady-state behaviour of a semi-infinite slab of diamagnetic plasma contained by a magnetic field. Initially we do not restrict the degree of diamagnetism or the sharpness of the boundary but modify these conditions later when describing some practical examples. We assume that the component w of the particle's mean velocity tangential to the plasma–magnetic field boundary is the dominant one and has the same value for each kind of particle present, say protons and electrons. Figure 1 shows the situation for the case of a plane boundary.

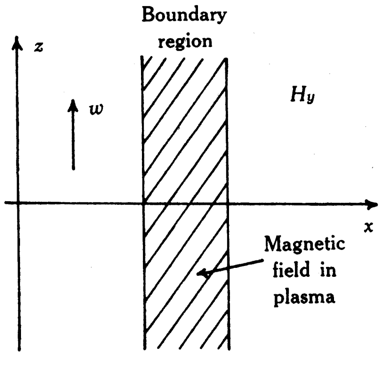

Fig. 1.—Plane plasma–magnetic field boundary of arbitrary thickness. The z component of particle velocity, w, is parallel to the boundary, while the magnetic field H_y and particle density both vary with x.

Physically this problem arises when a plasma blob is fired from a source into a magnetic guide field. Experimental results (e.g. Demichev *et al.* 1966; Tuckfield and Scott 1966) have been obtained using multipole guide fields. Once inside the guide the plasma blob spreads along the axis and may be considered longitudinally uniform. Tuckfield and Scott indicated that their source produced particles with a mean longitudinal velocity much greater than the ion thermal velocity. We consider the simplest theoretical model to describe these conditions.

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Plasma is ejected from the source parallel to the z axis with mean velocity w. We assume that there are sufficient two-particle collisions in the source, where particle density is very high and collision cross sections are large until the gas is very hot, for the velocity distribution to relax towards an approximately Gaussian distribution about the mean. Outside the source the gas is more tenuous and collisions much less probable, but the approximately randomized distribution of particle velocities is maintained, and this is the distribution seen by an observer moving with the mean velocity w. Depending on the acceleration mechanism in the source, the random components of velocity parallel and perpendicular to the z axis may have different mean square values. The distribution function in Section II is of this type, with independent parameters β' and β''. We examine the plasma far from the entrance to the guide field where, because of the dimensions of the system, we can ignore end effects and treat it as a stream of particles in collisionless equilibrium and infinite in longitudinal extent.

The present analysis also assumes that the plasma is approximately charge neutral. Some authors (Harris 1962; Nicholson 1963) have pointed out that exact charge neutrality in the sheath is obtained if particles of either sign have the same momentum distribution; then in the magnetic field they will have the same curvature and follow the same orbits, but this is far from a state of equilibrium. For the present analysis the distribution is chosen so that the particles have the same mean velocity tangential to the boundary with a thermal spread about the mean.

For dimensional reasons the sheath thickness turns out to be scaled by a factor $\kappa = (4\pi Ne^2/m_e c^2)^{\frac{1}{2}}$, depending on the density N in the main body of the plasma. κ^{-1} is numerically equal to the conventional Debye length $(8\pi ne^2/kT)^{-\frac{1}{2}}$ calculated for temperature $2m_e c^2$ and is therefore much larger than the local Debye length for electrons of temperature kT, except in regions at the outer edge of the plasma where the particle densities are less than a fraction $kT/m_e c^2$ of the maximum, and then this neutral approximation no longer holds. Figure 5 shows that the sheath may be considered to extend for a distance of about κ^{-1} on either side of the half-density point.

Trapping of particles in the boundary layer has been discussed in the literature (Sestero 1964; Morse 1965), but it is not very important in this analysis. Morse has shown that electrons originating deep in the plasma will be trapped in the sheath by a rising magnetic field if their tangential velocity is in such a direction that the orbital curvature turns the particles deeper into the magnetic field, and if the field in the sheath is then strong enough to contain the electrons. We find that all particles from the sheath can reach all parts of the plasma except where the plasma sheath departs significantly from neutrality; the exceptional particles in these regions are then said to be trapped. Firsov (1959) and Sestero (1964) have examined the more complex problem that arises when the particle's directed velocity is much less than its thermal velocity, showing that different parts of phase space become inaccessible to the particle as it traverses the sheath.

In this paper we consider first a plasma contained in a magnetic field, using the charge-neutral approximation, and then examine in Section III the problem of plasma confined in a quadrupole cusped magnetic field, comparing our results with the expression of Berkowitz *et al.* (1958) usually quoted in the literature. In Section

IV we discuss the complete expression for the electrostatic potential and the validity of the charge-neutral approximation, and then finally examine some experimental results of Tuckfield and Scott (1966) in the light of our analysis.

II. Distribution Functions

Let us consider the Boltzmann distribution of a hydrogen plasma in equilibrium that is contained by a magnetic field independent of the z coordinate, $\boldsymbol{H} = \boldsymbol{H}(x,y)$. If we neglect individual collisions the field can be represented by a self-consistent vector potential $\boldsymbol{A} = \boldsymbol{A}(x,y)$, which represents the sum of effects of external magnetic currents and of the plasma.

The Hamiltonian for a single particle of mass m and charge q (e.s.u.) is

$$\mathscr{H} = \tfrac{1}{2}m^{-1}\{\boldsymbol{p}-(q/c)\boldsymbol{A}\}^2+q\phi\,,$$

where $\phi(x,y)$ is the electrostatic potential within the plasma and

$$\boldsymbol{v} = m^{-1}\{\boldsymbol{p}-(q/c)\boldsymbol{A}\}$$

is the particle velocity. For protons we have $q = +e$ and for electrons $q = -e$.

In this system, as $\mathscr{H}$ is independent of z and of time, $\mathscr{H}$ and p_z are constants of the motion for each particle. It is well known that a stationary particle distribution in phase space can be an arbitrary function of the constants of the motion (see e.g. Landau and Lifshitz 1938). We will consider a distribution of the form

$$\text{const.}(\beta''/2\pi m)^{\frac{1}{2}}\exp(-\beta'\mathscr{H})\exp\{-\beta''(p_z-mw')^2/2m\}\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z\,. \tag{1}$$

For mathematical simplicity it is often sufficient to consider the limiting case $\beta'' \to \infty$, in which this distribution contains a δ-function in p_z

$$\text{const.}\exp(-\beta'\mathscr{H})\,\delta(p_z-mw)\,\mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\mathrm{d}p_x\,\mathrm{d}p_y\,\mathrm{d}p_z\,. \tag{2}$$

The distribution (2) is appropriate to the case where the particles are ejected from a source all with exactly the same longitudinal velocity w in the z direction. The distribution (1) has mean longitudinal velocity w, if we write $\beta''w'/(\beta'+\beta'') = w$. These distributions may also contain a few trapped particles; we discuss this in more detail in Section IV.

Integrating over the momenta shows that both distributions give the same number densities $n_\pm$, with $A_z = A$,

$$n_+ = N\exp\left\{-\beta\left(-\frac{e}{c}wA+\frac{1}{2m_\mathrm{p}}\frac{e^2}{c^2}A^2+e\phi\right)\right\}, \tag{3a}$$

$$n_- = N\exp\left\{-\beta\left(\frac{e}{c}wA+\frac{1}{2m_\mathrm{e}}\frac{e^2}{c^2}A^2-e\phi\right)\right\}, \tag{3b}$$

where for the distribution function (1) $\beta \equiv \beta'\beta''/(\beta'+\beta'')$ and for (2) $\beta = \beta'$.

As a first approximation the plasma is assumed to be neutral, and this requires a potential ϕ_0,

$$e\phi_0 = \frac{e}{c}wA+\frac{e^2}{4c^2}A^2\left(\frac{1}{m_\mathrm{e}}-\frac{1}{m_\mathrm{p}}\right). \tag{4}$$

For this value of the potential the distribution does not contain trapped particles. In a better approximation to ϕ there is a small density of them, $kT/m_e c^2$ (Section IV). In a coordinate system moving with the mean plasma velocity w the electrostatic potential is given by the A^2 term alone, as follows from the Lorentz transformation or from the fact that (4) remains valid in the moving system for which $w = 0$; the second term in (4) depends on the mass difference between protons and electrons, which causes them to follow different orbits in the magnetic field. With the above value ϕ_0 of ϕ, the number densities reduce to

$$n = n_+ = n_- = N \exp\left\{-\beta\left(\frac{1}{m_p}+\frac{1}{m_e}\right)\frac{e^2}{4c^2}A^2\right\} = N\exp(-\lambda^2 A^2), \tag{5}$$

where

$$\lambda^2 = \frac{\beta e^2}{4c^2}\left(\frac{1}{m_p}+\frac{1}{m_e}\right) \simeq \frac{\beta e^2}{4c^2 m_e}$$

scales the vector potential. The current density derived from the original distribution functions is, with $w_\pm = m_\pm^{-1}\{p_\pm \mp (e/c)A\}$,

$$\begin{aligned} j_z &= (e/c)\langle n_+ w_+ - n_- w_-\rangle \\ &= -n\left(\frac{e}{c}\right)^2 A\left(\frac{1}{m_p}+\frac{1}{m_e}\right)\left(1+\frac{\beta'}{\beta'+\beta''}\right). \end{aligned}$$

Thus from Maxwell's equation curl $\boldsymbol{B} = 4\pi \boldsymbol{j}$

$$\nabla^2 A = 4\pi N\left(\frac{e}{c}\right)^2\left(\frac{1}{m_p}+\frac{1}{m_e}\right)\left(1+\frac{\beta'}{\beta'+\beta''}\right)A\exp\left\{-\frac{\beta e^2}{4c^2}\left(\frac{1}{m_p}+\frac{1}{m_e}\right)A^2\right\}$$

or

$$\nabla^2(\lambda A) = \kappa^2(\lambda A)\exp(-\lambda^2 A^2), \tag{6}$$

where

$$\kappa^2 = \frac{8\pi N e^2}{2c^2}\left(\frac{1}{m_p}+\frac{1}{m_e}\right)\left(1+\frac{\beta'}{\beta'+\beta''}\right)$$

determines a scale of length in the equation. In the limit of the δ-function distributio. (2) ($\beta'' \to \infty$)

$$\kappa^2 \simeq 8\pi N e^2/2m_e c^2$$

or, numerically,

$$\kappa = 3{\cdot}54 \text{ cm}^{-2} \qquad \text{for} \qquad N = 10^{12}\text{ cm}^{-3}.$$

We note that in the limit, as A tends to zero, equation (6) becomes

$$\nabla^2(\lambda A) = \kappa^2(\lambda A), \tag{7}$$

with the solution in (r, θ) coordinates

$$\lambda A(\kappa r, \theta) = \sum_n \alpha_n I_n(\kappa r)\cos n\theta,$$

where $I_n(\kappa r)$ are modified Bessel functions of order n.

III. The Quadrupole Cusp

We now examine the simple two-dimensional cusp configuration (Berkowitz *et al.* 1958); it may be generated by four line currents that are positive in the first and third quadrants and negative in the second and fourth, as shown in Figure 2. By symmetry, the lines bisecting the angles between the conductors are lines of force.

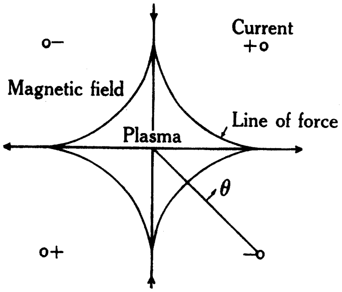

Fig. 2.—Configuration of simple cusp generated by four line currents.

For this discussion we will have $A = A(\kappa r, \theta)$, and we can expand the vector potential A in a Fourier series which has $A_z = 0$ on $\theta = \pm\frac{1}{4}\pi$ as

$$\lambda A(\kappa r, \theta) = f(\kappa r)\cos 2\theta + g(\kappa r)\cos 6\theta + \dots. \tag{8}$$

Equation (6) is then written

$$\nabla^2(\lambda A) = \kappa^2(f\cos 2\theta + g\cos 6\theta)\exp\{-(f\cos 2\theta + g\cos 6\theta)^2\} \tag{9}$$

$$= \kappa^2\exp(-\tfrac{1}{2}f^2)[f\{I_0(\tfrac{1}{2}f^2) - I_1(\tfrac{1}{2}f^2)\}\cos 2\theta + gI_1(\tfrac{1}{2}f^2)\cos 2\theta$$

$$-fI_1(\tfrac{1}{2}f^2)\cos 6\theta - gI_0(\tfrac{1}{2}f^2)\cos 6\theta]. \tag{10}$$

Equation (10) neglects harmonics above 6θ and terms in fg and g^2. The details of the expansion of the exponential in (9) are given in Appendix I. It is shown in Appendix II that the only important term on the right-hand side of equation (10) is the first one. A variational method shows that with only the first term the equation gives the best approximation of the form $\lambda A = f\cos 2\theta$.

The equation for the $\cos 2\theta$ component of (10) is then

$$\nabla^2(f\cos 2\theta) = \kappa^2 f\exp(-\tfrac{1}{2}f^2)(I_0 - I_1)\cos 2\theta, \tag{11}$$

or

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)f - \frac{4}{r^2}f = \kappa^2 f\exp(-\tfrac{1}{2}f^2)(I_0 - I_1). \tag{12}$$

We have three possible solutions of equation (12) as the magnitude of f is varied:

(i) For large f, we may equate the right-hand side of (10) to zero; the solution is then

$$f(r) = Pr^2 + Q/r^2, \tag{13}$$

where the constants P and Q are to be determined subsequently from matching solutions at the point $\kappa r = 2\cdot 8$.

(ii) In the limit as f tends to zero, we may equate the exponential to unity and obtain an equation similar to (7), namely

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)f-\frac{4}{r^2}f=\kappa^2 f. \tag{14}$$

The solution to this equation is

$$f(\kappa r)=I_2(\kappa r). \tag{15}$$

(iii) If we have a small but finite f, it is necessary to add a small correction term $\epsilon(\kappa r)$ to the solution (15). Expanding the exponential and Bessel functions of equation (12) we obtain

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)f-\frac{4}{r^2}f=\kappa^2 f(1-\tfrac{3}{4}f^2+\ldots). \tag{16}$$

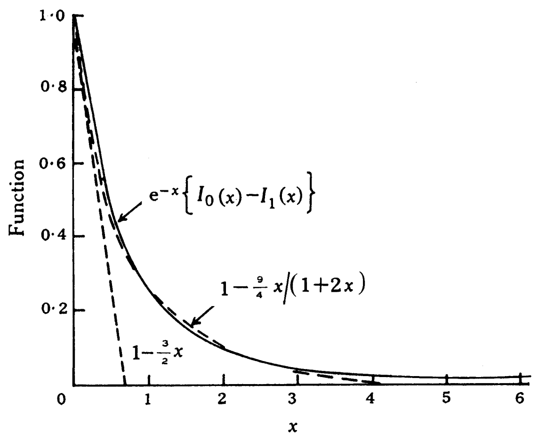

Fig. 3.—Approximations to the functions $e^{-x}\{I_0(x)-I_1(x)\}$.

However, as f increases beyond three the right-hand side of (16) is not a sufficiently good approximation and we will therefore use instead

$$\exp(-\tfrac{1}{2}f^2)\{I_0(\tfrac{1}{2}f^2)-I_1(\tfrac{1}{2}f^2)\}\simeq f\{1-af^2/(1+bf^2)\} \tag{17}$$

with $a=9/8$ and $b=1$. Figure 3 gives an indication of the error incurred by using this expression. Our equation is now

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)f-\frac{4}{r^2}f=\kappa^2 f\left(1-\frac{af^2}{1+bf^2}\right) \tag{18}$$

with the solution

$$f(\kappa r)=I_2(\kappa r)+\epsilon(\kappa r). \tag{19}$$

The small correction term $\epsilon(\kappa r)$ is obtained by solving the equation

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}}{\mathrm{d}r}\right)\epsilon-\frac{4}{r^2}\epsilon=\kappa^2\epsilon-\frac{\kappa^2 aI_2^3}{1+bI_2^2}.$$

We find that

$$\epsilon(\kappa r) = \alpha(\kappa r)^2 I_2^3/(1+I_2^2) \qquad \text{with} \qquad \alpha = -0\cdot 0188.$$

The solutions of equation (12), namely equations (13) and (19), may be matched at $\kappa r = 2\cdot 8$, as Figure 3 shows that the expression (17) is still a good approximation here. At $\kappa r = 2\cdot 8$ we have the condition that $f(\kappa r)$ and $f'(\kappa r)$ are continuous functions, and so we obtain from (13) and (19) $\kappa^2 P = 0\cdot 238$ and $Q/\kappa^2 = -2\cdot 98$. The complete solution for the vector potential is

$$\lambda A(\kappa r, \theta) = \{I_2(\kappa r)+\epsilon(\kappa r)\}\cos 2\theta, \qquad \kappa r \leqslant 2\cdot 8, \tag{20a}$$

$$\lambda A(\kappa r, \theta) = \{0\cdot 238(\kappa r)^2 - 2\cdot 98/(\kappa r)^2\}\cos 2\theta, \qquad \kappa r > 2\cdot 8, \tag{20b}$$

with

$$\epsilon(\kappa r) = -0\cdot 0188(\kappa r)^2 I_2^3(\kappa r)/\{1+(I_2(\kappa r))^2\}.$$

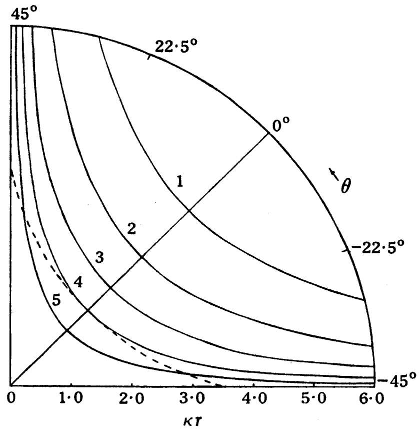

Fig. 4.—Curves of constant vector potential $f(\kappa r)\cos 2\theta$, which are also curves of constant density n, and lines of force in the $(\kappa r, \theta)$ plane are shown for the quadrupole cusp:

1, $n = 10^{-7}N$, $f = 4\cdot 0$
2, $n = 0\cdot 02\,N$, $f = 2\cdot 0$
3, $n = 0\cdot 37\,N$, $f = 1\cdot 0$
4, $n = 0\cdot 78\,N$, $f = 0\cdot 5$
5, $n = 0\cdot 94\,N$, $f = 0\cdot 25$

The dashed curve shows the classical hypocycloid

$$x^{\frac{2}{3}}+y^{\frac{2}{3}} = \text{const.}$$

The particle density n and the electrostatic potential ϕ_0 of this distribution may be calculated from the expressions (20) for A. We have

$$n = N\exp\{-(f(\kappa r)\cos 2\theta)^2\} \tag{21}$$

and, as a first approximation, equation (4) gives

$$\beta e\phi_0(\kappa r, \theta) = \{f(\kappa r)\cos 2\theta\}^2.$$

Figure 4 shows the variation of n with $(\kappa r, \theta)$; the curves correspond to different constant values of λA. The hypocycloid (dashed curve), $x^{\frac{2}{3}}+y^{\frac{2}{3}}$ = constant, of the conventional theory for a sharp plasma–magnetic field boundary is shown for comparison and is matched to these curves at the point $\lambda A(\kappa r, 0) = \frac{1}{2}$. A check on the density variation along the axis of symmetry $\theta = 0$ is provided by comparison with the solution for a plane boundary.

For a plane boundary we have $A = A(x)$ and equation (6) can be integrated to yield

$$\{\mathrm{d}(\lambda A)/\mathrm{d}x\}^2 + \kappa^2\exp(-\lambda^2 A^2) = \kappa^2 C \tag{22}$$

which, since $H = \mathrm{d}A/\mathrm{d}x$, we write as

$$H^2/8\pi + 2n/\beta = 2(N/\beta)C \tag{22a}$$

and this is the pressure balance equation for a plane boundary, that is, $H^2/8\pi +$ (particle pressure) $=$ constant. Integrating (22) we obtain

$$x = \kappa^{-1}\int_{\lambda A_0}^{\lambda A} \{C - \exp(-\lambda^2 A^2)\}^{-\frac{1}{2}}\,\mathrm{d}\lambda A\,, \tag{23}$$

with x measured from the point where $A = A_0$. Depending on the value of C there are different types of solutions which have been discussed in the literature (Harris 1962; Longmire 1963; Nicholson 1963). For the present purpose the appropriate choice is $C = 1$, which makes the magnetic field vanish in the main body of the plasma. Equation (23) has been integrated numerically with $C = 1$, starting from $A = 0\cdot 1$ at $x = 0$. The result is shown in Figure 5, together with the curves for H^2, n, and $\beta e\phi_0$ which are derived from the curve for λA.

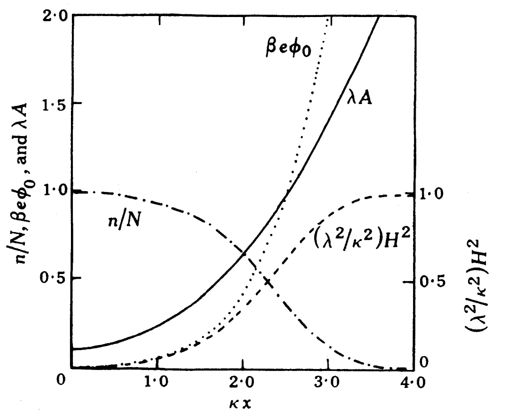

Fig. 5.—Curves of magnetic pressure $H^2/8\pi$, particle density n/N, magnetic vector potential $\lambda A(x)$, and electrostatic potential $\beta e\phi_0$ for a plane boundary when $p + H^2/8\pi = \kappa^2/8\pi\lambda^2$.

A comparison of the curves of particle density n for the two types of boundary considered above shows that the variation of n with κr along an axis of symmetry $(\theta = 0)$ in the quadrupole case is indistinguishable from the variation of n with κr in the case of a plane boundary. Thus we have a check on the accuracy of the analytical approximations made in discussing the quadrupole.

IV. The Electrostatic Potential

Our expression for $\phi(\kappa r, \theta)$, the electrostatic potential generated by the plasma, must be consistent with Poisson's equation

$$\nabla^2\phi = -4\pi\rho = -4\pi e(n_+ - n_-)\,.$$

From Section II we have expressions (3) for the number densities. We will write the electric potential as

$$\phi = \phi_0 + \phi_1\,,$$

with ϕ_0 given by equation (4), as the potential generated by the plasma in the neutral approximation. Then

$$n_+ = n\exp(-\beta e\phi_1) \qquad \text{and} \qquad n_- = n\exp(\beta e\phi_1)$$

with the local number density $n(\kappa r, \theta)$ given by equation (5). We may thus write

$$\rho(\kappa r, \theta) = -en\{\exp(\beta e\phi_1) - \exp(-\beta e\phi_1)\} = -2en\sinh\beta e\phi_1$$

and

$$\nabla^2(\phi_0+\phi_1) = 8\pi en\sinh\beta e\phi_1 \simeq K_{\mathrm{D}}^2\phi_1, \tag{24}$$

where

$$K_{\mathrm{D}}(\kappa r, \theta) = (8\pi ne^2\beta)^{\frac{1}{2}} = \lambda_{\mathrm{D}}^{-1}(\kappa r, \theta) \tag{25}$$

and λ_{D} is the Debye screening length. The formal solution of equation (24) is

$$\phi_1 = (1+\lambda_{\mathrm{D}}^2\nabla^2+\lambda_{\mathrm{D}}^2\nabla^2\lambda_{\mathrm{D}}^2\nabla^2+\ldots)\lambda_{\mathrm{D}}^2\nabla^2\phi_0. \tag{26}$$

The expansion is valid in the region where $(\lambda_{\mathrm{D}}^2\nabla^2\phi_0)/\phi_0$ is small; we can estimate this ratio using equation (4), to write ϕ_0 in terms of A, and equation (6). We have first

$$\begin{aligned}\lambda_{\mathrm{D}}^2\nabla^2\phi_0 &= \lambda_{\mathrm{D}}^2\nabla^2\left\{\frac{w}{c}A+\frac{e}{4c^2}\left(\frac{1}{m_{\mathrm{e}}}-\frac{1}{m_{\mathrm{p}}}\right)A^2\right\} \\ &= \frac{kT}{m_{\mathrm{e}}c^2}\left(\phi_0-\frac{w}{2c}A\right)+\frac{e}{4c^2}\left(\frac{1}{m_{\mathrm{e}}}-\frac{1}{m_{\mathrm{p}}}\right)2\lambda_{\mathrm{D}}^2(\nabla A)^2.\end{aligned} \tag{27}$$

(Since we have chosen $A = A_z(x,y)$ we can if we wish replace $(\nabla A)^2$ by H^2 quit generally.) From (27) the ratio $(\lambda_{\mathrm{D}}^2\nabla^2\phi_0)/\phi_0$ will contain two terms, the first o which is always small. To find the size of the second term we must choose a particula configuration, and we will take the one-dimensional plane boundary as we have see (Section III) that this is a good approximation to the boundary of a cusp near the axis of symmetry.

Where n is large λ_{D} is small and the right-hand side of (27) is negligible. The term $\phi_1 = \lambda_{\mathrm{D}}^2\nabla^2\phi_0$ becomes important in the region where n is small; here, referring to Figure 5 we see that we can write $A \simeq Hx'$, where $x' = x-1\cdot5\,\kappa^{-1}$, and

$$(\lambda_{\mathrm{D}}^2\nabla^2\phi_0)/\phi_0 \simeq 2\lambda_{\mathrm{D}}^2/(x-1\cdot5\,\kappa^{-1})^2,$$

which remains small until $n(\kappa r, \theta)$ is vanishingly small. When λ_{D} is comparable with x, ϕ stops increasing as A^2 and continues as an electrostatic field in free space, its sources being in the region containing the plasma.

The expansion (26) made above is therefore valid within the limits stated, but in this region ϕ_1 contributes negligibly to the total electrostatic potential and we can write with sufficient accuracy for most purposes

$$e\phi \simeq e\phi_0 = \frac{e}{c}wA+\frac{e^2}{4c^2}\left(\frac{1}{m_{\mathrm{e}}}-\frac{1}{m_{\mathrm{p}}}\right)A^2. \tag{28}$$

Trapped Particles

While ϕ_1 can be neglected for most cases of interest, it is important when we consider the possibility of particles being trapped in the boundary layer. The earlier remark (Section I) that electrons can be trapped in the plasma sheath once the charge-neutral approximation is dropped can now be demonstrated and we show that only a small number of electrons can be trapped. We continue to use the example of the plane boundary discussed above.

In regions where there is no magnetic field and zero electrostatic potential, the chief contribution to, and the lower limit of, the total energy of a particle is its kinetic energy $\frac{1}{2}mw^2$ along the z axis. In the magnetic field, the total energy of the system is expressed by the Hamiltonian $\mathscr{H}$ defined in Section II for a particle of mass m and charge q. Clearly, for a particle at any point in the magnetic field to be able to reach or, since the orbits are reversible, to have come from any part of the field-free region, we need

$$\mathscr{H} > \tfrac{1}{2}mw^2. \tag{29}$$

From equations (26), (27), and (28) we have the expression for $\phi = \phi_0+\phi_1$ and (29) becomes for protons

$$\frac{p_x^2+p_y^2}{2m_\mathrm{p}}+\frac{e^2}{4c^2}A^2\left(\frac{1}{m_\mathrm{e}}+\frac{1}{m_\mathrm{p}}\right) > -\frac{kT}{m_\mathrm{e}c^2}\left\{\frac{e}{2c}\,wA+\frac{e^2}{4m_\mathrm{e}c^2}A^2\left(1-\frac{m_\mathrm{e}}{m_\mathrm{p}}\right)\right\} -\frac{e^2}{2m_\mathrm{e}c^2}\left(1-\frac{m_\mathrm{e}}{m_\mathrm{p}}\right)\lambda_\mathrm{D}^2(\nabla A)^2 \tag{30}$$

and for electrons

$$\frac{p_x^2+p_y^2}{2m_\mathrm{e}}+\frac{e^2}{4c^2}A^2\left(\frac{1}{m_\mathrm{e}}+\frac{1}{m_\mathrm{p}}\right) > \frac{kT}{m_\mathrm{e}c^2}\left\{\frac{e}{2c}\,wA+\frac{e^2}{4m_\mathrm{e}c^2}A^2\left(1-\frac{m_\mathrm{e}}{m_\mathrm{p}}\right)\right\} +\frac{e^2}{2m_\mathrm{e}c^2}\left(1-\frac{m_\mathrm{e}}{m_\mathrm{p}}\right)\lambda_\mathrm{D}^2(\nabla A)^2. \tag{31}$$

Both equations are satisfied for all values of A in the charge-neutral approximation which neglects the potential term ϕ_1, that is, the right-hand sides of (30) and (31). The terms in $kT/m_\mathrm{e}c^2$ are always small when compared with $e\phi_0$, and are not significant in relation to the left-hand side of (31) unless $e\phi_0$ or $(e/c)wA$ are of order $m_\mathrm{e}c^2$.

The term in $\lambda_\mathrm{D}^2(\nabla A)^2$ can be important and may lead to trapping of electrons of small transverse kinetic energy. Since we are formally considering a steady-state system, which lasts for infinite time, electrons may have diffused into the trap by collision, for, although this process is inefficient, the trap has to be filled only once to attain a steady state. If the plasma has boundaries in the y direction, electrons from the boundaries may have run down the field lines and become trapped or electrostatic instabilities of the type suggested by Bernstein, Greene, and Kruskal (1957) may cause external electrons to become trapped in the sheath. (This second mechanism would require time-varying fields, however, and we are considering a static system.)

The thermal energy of the electrons determines the extent to which they penetrate the boundary; we have from (31) as a condition for trapping electrons

$$\frac{e}{c}\frac{kT}{c^2}\,wA-\frac{e^2}{2c^2}A^2+\frac{e^2}{c^2}\lambda_\mathrm{D}^2(\nabla A)^2 > 0 \tag{32a}$$

and for trapping protons

$$-\frac{e}{c}\frac{kT}{c^2}\,wA-\frac{e^2}{2c^2}A^2-\frac{e^2}{c^2}\lambda_{\mathrm{D}}^2(\nabla A)^2>0. \tag{32b}$$

Unless wA is very large and negative there are no protons trapped and even for the electrons the first term in (32a) is negligible. We therefore choose to illustrate explicitly for the electrons the case where $w=0$. Equation (32a) becomes, using (25),

$$2H^2>K_{\mathrm{D}}^2A^2.$$

Substituting for A from equation (5) we obtain

$$H^2/16\pi Nm_e c^2>(n/N)\log(N/n). \tag{33}$$

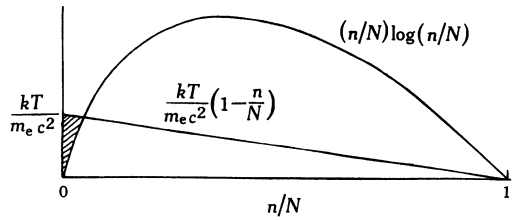

Fig. 6.—Illustrating the trapping of electrons in the sheath. The shaded portion (trapped electrons) is very small.

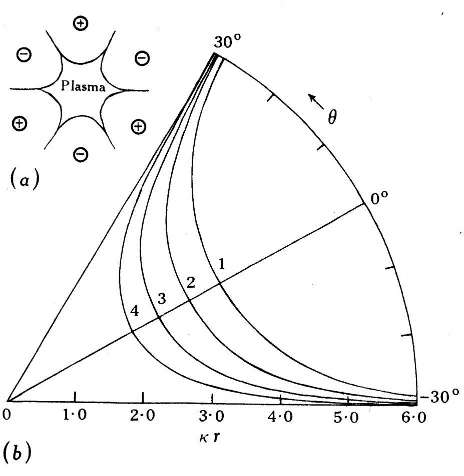

Fig. 7.—Showing for the hexapole (a) the cross section with three positive and three negative current-carrying rods, and (b) curves of constant vector potential $f(\kappa r)\cos 3\theta$ ($f=I_3(\kappa r)$) and constant density n in the $(\kappa r,\theta)$ plane:

1, $n=0\cdot02\,N, f=2\cdot0$
2, $n=0\cdot37\,N, f=1\cdot0$
3, $n=0\cdot78\,N, f=0\cdot5$
4, $n=0\cdot94\,N, f=0\cdot25$

Equation (33) is quite general. To write this condition in terms of the electron thermal energy we can use the pressure balance equation (22a) with $C=1$. We have

$$(kT/m_e c^2)(1-n/N)>(n/N)\log(N/n). \tag{34}$$

The shaded portion of Figure 6 is the region where (33) is satisfied; (34) shows that relative to the density at the centre of the field-free region, the density of trapped electrons is very small.

V. The Hexapole

Some experimental evidence for the kind of density distribution we have obtained for the cusp configuration is reported by Tuckfield and Scott (1966); they have investigated the guiding of a hydrogen plasma moving along a hexapole guide field with a cross section as indicated in Figure 7(a). They find a six-sided pattern as expected for values of the external field B_0 (measured midway between the rods) greater than about 500 G, and at lower field strengths the pattern is three-sided; the protons are attracted to the three rods which carry current $+\mathscr{I}$ away from the plasma source, and are repelled from the others which carry current $-\mathscr{I}$. The plasma density 5×10^{11} particles cm^{-3} is large enough for the plasma to behave diamagnetically in the centre and thus our general analysis can be applied.

Solving equation (7) we obtain the vector potential for a hexapole configuration. Near $r = 0$ the solutions are (approximately)

$$A = CI_3(\kappa r)\cos 3\theta\,, \qquad \kappa r \leqslant 3\,, \tag{35a}$$

$$A = \{R(\kappa r)^3+S/(\kappa r)^3\}\cos 3\theta\,, \qquad \kappa r > 3\,. \tag{35b}$$

Contours of $A(\kappa r,\theta)$ and n are shown in Figure 7(b).

From Section II the proton number density is

$$n_+ = N\exp\left\{\beta\left(\frac{e}{c}wA-\frac{1}{2m_\mathrm{p}}\frac{e^2}{c^2}A^2-e\phi\right)\right\}. \tag{36}$$

In the centre the plasma is approximately charge neutral and the electrostatic potential has the value ϕ_0 (equation (4)); but near the rods the plasma density has a negligible effect on the magnetic field and we may therefore neglect the term $e\phi$ in equation (36). The vector potential near each rod is now described most accurately by its vacuum value

$$A = 2\mathscr{I}\log(r'/a)\,, \tag{37}$$

where r' is measured from the centre of that rod.

From (36) and (37) we see that the magnitude of the current in the rods determines whether the protons are repelled from all rods or from alternate rods. For large values of $\mathscr{I}$ the term in A^2 in the exponential of (36) is the more important, and the exponential is negative for both signs of $\mathscr{I}$. Thus we obtain the familiar six-sided configuration observed by Tuckfield and Scott (1966) as protons are repelled from all the rods. At lower values of B_0 the ratio of the first two terms in the exponential, namely $eA/2m_\mathrm{p}cw$, decreases until the term in A is more important, and then the three-sided pattern emerges. The exponential is positive under those rods carrying current $+\mathscr{I}$ and negative for the others; that is, protons are attracted to the rods with $+\mathscr{I}$ and repelled from the ones with $-\mathscr{I}$.

For a hexapole geometry with radius a equal to 8 cm, we can calculate the ratio of the terms in A and A^2 at a distance of 1 cm from the centre of a rod. For a field strength B_0 of 800 G we find

$$eA/2m_\mathrm{p}cw \simeq 2\cdot 5$$

and so expect repulsion from all rods. The terms in A and A^2 are equal when $B_0 \simeq 320$ G, which is consistent with the observed change from a three-sided to a six-sided light pattern, between 265 and 530 G. When the configuration becomes six-sided and the plasma is concentrated in the centre, an analysis similar to that given in Section IV for the quadrupole cusp configuration applies, and the magnetic field in the centre is described by equations (35).

Other results of Demichev *et al.* (1966), who used strong magnetic fields of up to 6000 G in an experiment similar to the one considered here, but with a quadrupole field, also demonstrate the efficiency of this method of controlling and guiding a plasma.

VI. Acknowledgment

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Appendix I

Expansion in Terms of Modified Bessel Functions I_n

For simplification of equation (9) we use the expansion (Watson 1958)

$$\exp(\pm\lambda\cos\psi) = I_0(\lambda)\pm 2I_1(\lambda)\cos\psi+\ldots,\qquad \lambda>0.$$

Equation (9) is written in full as

$$\nabla^2\{f(\kappa r)\cos 2\theta+g(\kappa r)\cos 6\theta\}$$

$$=\kappa^2(f\cos 2\theta+g\cos 6\theta)\exp[-\{\tfrac{1}{2}(f^2+g^2)+(\tfrac{1}{2}f^2+g^2)\cos 4\theta+fg\cos 8\theta+\tfrac{1}{2}g^2\cos 12\theta\}].$$

We now expand the exponential in terms of Bessel functions I_n

$$\nabla^2(f\cos 2\theta+g\cos 6\theta)$$

$$\begin{aligned}&=\kappa^2(f\cos 2\theta+g\cos 6\theta)\exp\{-\tfrac{1}{2}(f^2+g^2)\}\\ &\quad\times[\{I_0(\tfrac{1}{2}f^2+fg)-2I_1(\tfrac{1}{2}f^2+fg)\cos 4\theta+\ldots\}\{I_0(fg)-2I_1(fg)\cos 8\theta+\ldots\}\\ &\qquad\times\{I_0(\tfrac{1}{2}g^2)-2I_1(\tfrac{1}{2}g^2)\cos 12\theta+\ldots\}]\end{aligned}$$

$$= \kappa^2 \exp\{-\tfrac{1}{2}(f^2+g^2)\}\{I_0(\tfrac{1}{2}f^2+g^2)\, I_0(fg)\, I_0(\tfrac{1}{2}g^2)\,(f\cos 2\theta + g\cos 6\theta)$$
$$-2I_1(\tfrac{1}{2}f^2+fg)\, I_0(fg)\, I_0(\tfrac{1}{2}g^2)\,(f\cos 2\theta + g\cos 6\theta)\cos 4\theta$$
$$-2I_1(fg)\, I_0(\tfrac{1}{2}f^2+fg)\, I_0(\tfrac{1}{2}g^2)\,(f\cos 2\theta + g\cos 6\theta)\cos 8\theta$$
$$-2I_1(\tfrac{1}{2}g^2)\, I_0(\tfrac{1}{2}f^2+fg)\, I_0(fg)\,(f\cos 2\theta + g\cos 6\theta)\cos 12\theta\}.$$

If we now neglect terms in fg and $\frac{1}{2}g^2$, we have equation (10)

$$\nabla^2(f\cos 2\theta + g\cos 6\theta) = \kappa^2 \exp(-\tfrac{1}{2}f^2)[f\cos 2\theta\{I_0(\tfrac{1}{2}f^2) - I_1(\tfrac{1}{2}f^2)\} + g\cos 2\theta\, I_1(\tfrac{1}{2}f^2)$$
$$- f\cos 6\theta\, I_1(\tfrac{1}{2}f^2) - g\cos 6\theta\, I_0(\tfrac{1}{2}f^2)].$$

Appendix II

Approximation to Equation (10)

We have noted in Section III the approximation to equation (10) in which all terms on the right-hand side of the equation, except the first, are ignored; it is now necessary to justify this. Let us therefore consider the $\cos 6\theta$ component

$$\nabla^2(g\cos 6\theta) = \kappa^2 \exp(-\tfrac{1}{2}f^2)\{g\, I_0(\tfrac{1}{2}f^2) - fI_1(\tfrac{1}{2}f^2)\}\cos 6\theta.$$

Then

$$\frac{d^2g}{d(\kappa r)^2} + \frac{1}{\kappa r}\frac{dg}{d(\kappa r)} - \frac{36g}{(\kappa r)^2} = \exp(-\tfrac{1}{2}f^2)(gI_0 - fI_1)$$

and we find $g \simeq fI_1/I_0$,

$$g(\kappa r) \simeq I_2(\kappa r)\, I_1(\tfrac{1}{2}I_2^2(\kappa r))/I_0(\tfrac{1}{2}I_2^2(\kappa r)) \simeq \tfrac{1}{32}(\tfrac{1}{2}\kappa r)^6, \qquad \text{for small } \kappa r.$$

With this expression for $g(\kappa r)$,

$$g(\kappa r)/f(\kappa r) \simeq \tfrac{1}{16}(\tfrac{1}{2}\kappa r)^4.$$

Hence, the $\cos 6\theta$ term in the expression (8) for $\lambda A(\kappa r, \theta)$ makes a negligible contribution to the vector potential. From equation (10) we see also that the $\cos 2\theta$ component is

$$\nabla^2(f\cos 2\theta) = \kappa^2 \exp(-\tfrac{1}{2}f^2)\{f(I_0 - I_1) + gI_1\}\cos 2\theta,$$

and the ratio of the terms on the right-hand side is

$$\mathscr{R} = gI_1/f(I_0 - I_1) \simeq \tfrac{1}{16}(\tfrac{1}{2}\kappa r)^4 I_1/(I_0 - I_1),$$

which gives

$$\mathscr{R} < \tfrac{1}{9}(\tfrac{1}{2}\kappa r)^4, \qquad \text{when } \kappa r < 2{\cdot}8.$$

Therefore we are justified in neglecting all terms except the first on the right-hand side of equation (10).