ESTIMATION OF THE PROPERTIES OF QUARKS

I. VECTOR QUARK–QUARK INTERACTION

By G. B. Smith*† and L. J. Tassie*

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Abstract

From the sum rules for hadron scattering, the nucleon form factors, and the masses of the hadrons, the properties of quarks are estimated as: 2 GeV/c^2 $\lesssim$ quark mass $\lesssim$ 30 GeV/c^2, $0 \cdot 1$ fm $\lesssim$ range of the quark–quark interaction $\lesssim 0 \cdot 25$ fm. The hadrons are described by a relativistic independent quark model using the Dirac equation with a potential energy term for the effective interaction. This model is compared with a rigid rotor model.

I. Introduction

This paper reports an attempt to estimate the properties of quarks, as indicated by the successes of the quark model. The quark model (Kokkedee 1969), in which a baryon consists of three quarks and a meson consists of a quark and antiquark, cannot give a complete description of hadrons. For instance, a nucleon will contain additional quark–antiquark pairs, some of which form mesons; because the binding energy of a meson in the nucleon is very much smaller than that of the other constituents of the nucleon, the density of mesons will extend further out of the nucleon.

The quark model can only be expected to apply to some core of a hadron, the properties concerning the outer parts of the hadron being more appropriately described by other means, such as a virtual meson cloud around a nucleon, or by more complicated models including the additional quark–antiquark pairs.

In this paper we estimate the quark mass, and the root-mean-square separation distance of the quarks in the core. By considering the scattering sum rules for the total hadronic cross sections and the nucleon form factors we estimate the root-mean-square separation distance of the quarks in the hadronic core.

We use the one-particle Dirac equation as the basis of the dynamical description of the hadrons. There are two alternatives for introducing the quark–quark interaction. We may write the interaction as a potential energy (i.e. the fourth component of a 4-vector), as would be expected if the quark–quark interaction is a vector interaction, or alternatively we may write the interaction as a scalar term, as would be expected if the quark–quark interaction is a scalar interaction. In the present paper we choose the former while in the following paper (Smith 1970; present issue, pp. 627–32) the latter interaction is used.

Using this dynamical description, and the estimate of the root-mean-square separation distance of the quarks in the core, the quark mass required to produce the observed hadron spectra is estimated.

* Research School of Physical Sciences, Australian National University, P.O. Box 4, Canberra, A.C.T. 2600.

† Present address: Canberra College of Advanced Education, P.O. Box 381, Canberra City, A.C.T. 2601.

II. Range of the Quark–Quark Interaction

Sum rules relating meson–baryon and baryon–baryon total cross sections obtained in the quark model are based on the simple postulates (Levin and Frankfurt 1965; Lipkin and Scheck 1966) that the quark forward scattering amplitudes are additive and isospin invariant. These assumptions led to the following sum rules for total cross sections (Lipkin and Scheck 1966; James and Watson 1967)

$$\sigma(\mathrm{pp})+\sigma(\bar{\mathrm{p}}\mathrm{p}) = \tfrac{3}{2}\{\sigma(\pi^{+}\mathrm{p})+\sigma(\pi^{-}\mathrm{p})\}+\tfrac{1}{2}\{\sigma(\mathrm{K}^{+}\mathrm{p})+\sigma(\mathrm{K}^{-}\mathrm{p})\}-\tfrac{1}{2}\{\sigma(\mathrm{K}^{+}\mathrm{n})+\sigma(\mathrm{K}^{-}\mathrm{n})\}\,, \tag{1}$$

$$\sigma(\bar{\mathrm{p}}\mathrm{p})+\sigma(\mathrm{pn}) = 2\sigma(\pi^{-}\mathrm{p})+\sigma(\pi^{+}\mathrm{p})\,, \tag{2}$$

$$\sigma(\bar{\mathrm{p}}\mathrm{n})+\sigma(\mathrm{pp}) = \sigma(\pi^{-}\mathrm{p})+2\sigma(\pi^{+}\mathrm{p})\,. \tag{3}$$

Since the quarks in the core of the hadron are very strongly bound, we expect

$$r_{\mathrm{c}} \sim \tfrac{1}{2}b\,,$$

where r_{c} is the radius of the core and b is the range of the quark–quark interaction. We assume that the range for the quark–quark interaction is the same as for the quark–antiquark interaction and that the size of the core is the same for mesons and baryons, for the sake of simplicity and for the lack of any experimental evidence as yet against these assumptions. The quark model should apply to hadron–hadron scattering when the distance between the two hadrons is less than $2b$. Since the investigation of this region requires a wavelength λ smaller than or comparable with the diameter of the region,

$$\lambda \lesssim 4b\,, \tag{4}$$

we expect the sum rules (1), (2), and (3) to apply only at energies high enough to satify equation (4) for the wavelength of the projectile in the centre-of-mass system. Thus it is possible to estimate b from the energy at which the sum rules (1), (2), and (3) fail. At wavelengths larger than $4b$ we expect the hadron scattering cross section to be determined by the properties of the outer regions of the hadron, as calculated using such models as the peripheral model (Jackson 1965).

Assuming that the internal velocity of the quarks inside each hadron is small compared with the relative velocity of the two hadrons, identical kinematical conditions are provided if the different cross sections are compared at the same centre-of-mass energy for the quark–quark or quark–antiquark system. However, in relating the total centre-of-mass energy of the hadrons to the total centre-of-mass energy of the constituent quarks the mass of the quark is an independent variable and this mass may be interpreted as either the bare quark mass (James and Watson 1967) or an effective quark mass (Kokkedee and Van Hove 1966). In the former case the sum rules are graphed as functions of the relative velocity of the two hadrons; this is easily accomplished since the laboratory energy E_{lab} of the projectile divided by its rest mass m is dependent only on the projectile's velocity in the laboratory frame; or in the latter case the sum rules are graphed as functions of the laboratory momentum P_{lab} of the projectile since, in the spirit of additivity, the hadronic momentum is assumed to be the sum of the momenta of the constituent quarks. For sum rules involving both meson–baryon and baryon–baryon total cross sections we must compare them at laboratory momentum in the ratio 2 : 3 (Kokkedee and Van Hove 1966).

In Figure 1 the Johnson–Treiman (1965) relations

$$\tfrac{1}{2}\{\sigma(\mathrm{K^+p})-\sigma(\mathrm{K^-p})\} = \sigma(\pi^+\mathrm{p})-\sigma(\pi^-\mathrm{p}) = \sigma(\mathrm{K^+n})-\sigma(\mathrm{K^-n}) \tag{5}$$

are shown as functions of (*a*) P_{lab} and (*b*) E_{lab}/mc^2. The data used in this paper are taken from Lindenbaum *et al.* (1961), Diddens *et al.* (1963), Citron *et al.* (1964), Galbraith *et al.* (1965), Foley *et al.* (1967), and Allaby *et al.* (1969). Figure 1 shows

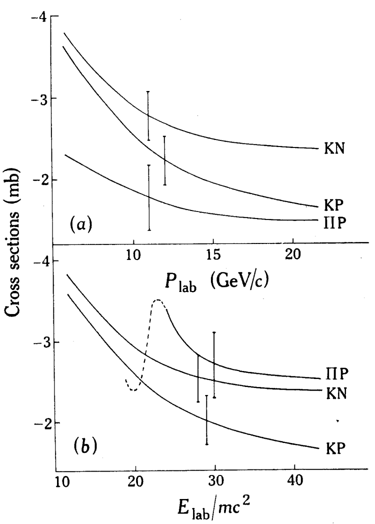

Fig. 1.—Johnson–Treiman relations (equations (5)) plotted against (*a*) P_{lab} and (*b*) E_{lab}/mc^2:

$\mathrm{KP} = \tfrac{1}{2}\{\sigma(\mathrm{K^+p})-\sigma(\mathrm{K^-p})\}$

$\Pi\mathrm{P} = \sigma(\pi^+\mathrm{p})-\sigma(\pi^-\mathrm{p})$

$\mathrm{KN} = \sigma(\mathrm{K^+n})-\sigma(\mathrm{K^-n})$

that this comparison does not resolve the question whether cross sections should be compared at the same P_{lab} or the same E_{lab}/mc^2. Other sum rules involving only meson–baryon total cross sections, namely (Lipkin and Scheck 1966) the symmetric sum rule

$$\sigma(\mathrm{K^+p})+\sigma(\mathrm{K^-p}) = \tfrac{1}{2}\{\sigma(\pi^+\mathrm{p})+\sigma(\pi^-\mathrm{p})+\sigma(\mathrm{K^+n})+\sigma(\mathrm{K^-n})\}$$

and the antisymmetric sum rule

$$\sigma(\mathrm{K^+p})-\sigma(\mathrm{K^-p})-\{\sigma(\mathrm{K^+n})-\sigma(\mathrm{K^-n})\} = \sigma(\pi^+\mathrm{p})-\sigma(\pi^-\mathrm{p})\,,$$

have been examined but they do not resolve the question either. In Figure 2 (see also James and Watson 1967) sum rule (2) is plotted for the same abscissae as in Figure 1 except that the meson–baryon and baryon–baryon total cross sections are compared at P_{lab} in the ratio of 2 : 3. Figure 2 indicates that E_{lab}/mc^2 should be used to compare sum rules, and this conclusion is supported by the results of sum rule (3) also. This comparison implies that the bare quark mass should be used as the mass variable in relating the total centre-of-mass energy of the hadrons to the total centre-of-mass energy of the constituent quarks.

In Figure 3 the sum rule (1) is plotted as a function of E_{lab}/mc^2; this relation was selected because accurate experimental data are available. By extrapolation sum rule (1) appears to hold for

$$E/mc^2 \geqslant 45$$

corresponding to a laboratory momentum $\gtrsim 6$ GeV/c for the pion. The analysis of

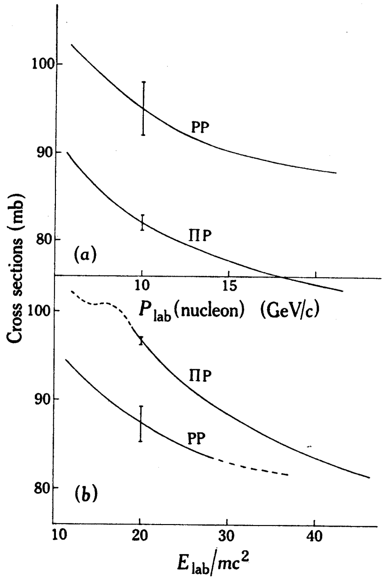

Fig. 2.—Sum rule (2):

$$\mathrm{PP} = \sigma(\bar{\mathrm{p}}\mathrm{p})+\sigma(\mathrm{pn}),$$

$$\Pi\mathrm{P} = 2\sigma(\pi^-\mathrm{p})+\sigma(\pi^+\mathrm{p}).$$

In (*a*) the meson–baryon and baryon–baryon cross sections are compared at P_{lab} in the ratio 2 : 3. In (*b*) all cross sections are compared at E_{lab}/mc^2.

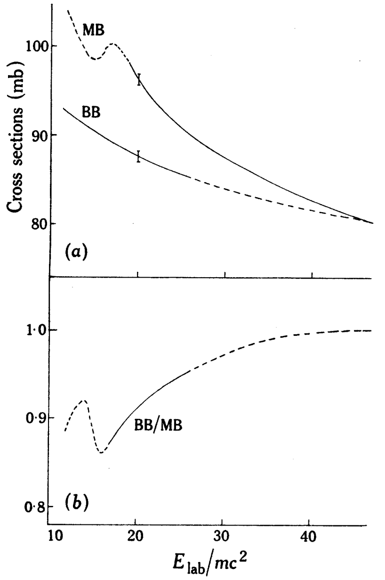

Fig. 3.—Sum rule (1):

$$\mathrm{BB} = \sigma(\mathrm{pp})+\sigma(\bar{\mathrm{p}}\mathrm{p}),$$

$$\mathrm{MB} = \tfrac{3}{2}\{\sigma(\pi^+\mathrm{p})+\sigma(\pi^-\mathrm{p})\}+\tfrac{1}{2}\{\sigma(\mathrm{K}^+\mathrm{p})+\sigma(\mathrm{K}^-\mathrm{p})\} - \tfrac{1}{2}\{\sigma(\mathrm{K}^+\mathrm{n})+\sigma(\mathrm{K}^-\mathrm{n})\},$$

shown in (*a*) with the ratio BB/MB plotted in (*b*). The dashed lines are extrapolations.

James and Watson (1967) shows that sum rules (2) and (3) are consistent with experiment for pion laboratory momenta $\gtrsim 6$ GeV/c. Then equation (4) yields a lower limit of 0·2 fm for the range of the quark–quark interaction.

Further information about the quark interaction is available from the electromagnetic form factors of the nucleon (Ishida, Konno, and Shimodaira 1966). If the nucleon consists of a core of three quarks acting as a source for a meson cloud, the nucleon form factor is

$$G(k^2) = \mathscr{G}(k^2)\, G_{\mathrm{c}}(k^2)\,,$$

where $\mathscr{G}(k^2)$ is the form factor of the meson cloud about a point source, $G_{\mathrm{c}}(k^2)$ is the form factor of the core, and k^2 is the square of the 4-momentum transfer. The form factor of the meson cloud about the core is taken from dispersion theory (Gasiorowicz

1966); for the proton the electric form factor of the meson cloud is

$$\mathscr{G}_{\mathrm{p}}(k^2) = C_\rho \frac{m_\rho^2}{m_\rho^2+k^2} + C_\omega \frac{m_\omega^2}{m_\omega^2+k^2} + C_\phi \frac{m_\phi^2}{m_\phi^2+k^2}$$

and for the neutron

$$\mathscr{G}_{\mathrm{n}}(k^2) = -C_\rho \frac{m_\rho^2}{m_\rho^2+k^2} + C_\omega \frac{m_\omega^2}{m_\omega^2+k^2} + C_\phi \frac{m_\phi^2}{m_\phi^2+k^2},$$

where C_ρ, C_ω, and C_ϕ are constants. Taking the electric form factor of the neutron as

$$G_{\mathrm{n}}^{\mathrm{E}}(k^2) \equiv 0$$

and neglecting the mass difference between the ρ and the ω mesons, the electric form factor relevant to the proton is

$$\mathscr{G}_{\mathrm{p}}^{\mathrm{E}}(k^2) = \frac{\frac{1}{2}m_\rho^2}{m_\rho^2+k^2} + \frac{\frac{1}{2}m_\omega^2}{m_\omega^2+k^2}. \tag{6}$$

The experimental nucleon form factor agrees (see e.g. Islam and Vasavada 1969) with the dipole fit for $0 < k^2 < 25\ (\mathrm{GeV}/c)^2$,

$$G_{\mathrm{p}}^{\mathrm{E}}(k^2) = \left(1+\frac{k^2}{0\cdot 71\ (\mathrm{GeV}/c)^2}\right)^{-2}. \tag{7}$$

Using

$$\langle r^2\rangle = -6\,\mathrm{d}G(k^2)/\mathrm{d}k^2 \qquad \text{at} \quad k^2 = 0\,,$$

the root-mean-square radius of the core obtained from equations (6) and (7) is

$$\langle r^2\rangle_{\mathrm{c}}^{\frac{1}{2}} = 0\cdot 52 \quad \mathrm{fm}\,. \tag{8}$$

Since other processes may contribute to the size of the nucleon, equation (8) is an upper limit to the size of the core.

For a flat-bottomed well $\langle R^2\rangle^{\frac{1}{2}}$ is approximately half the range of the potential. From the scattering sum rules we find a lower limit of $0\cdot 2$ fm for the range of the quark–quark interaction, which implies

$$\langle R^2\rangle^{\frac{1}{2}} \gtrsim 0\cdot 1 \quad \mathrm{fm}\,.$$

From the nucleon form factors

$$\langle R^2\rangle^{\frac{1}{2}} = 2\langle r^2\rangle_{\mathrm{c}}^{\frac{1}{2}} \lesssim 1\cdot 0 \quad \mathrm{fm}\,.$$

Consequently the root-mean-square separation distance of the quarks in the core is bound by

$$0\cdot 1\ \mathrm{fm} \lesssim \langle R^2\rangle^{\frac{1}{2}} \lesssim 1\cdot 0\ \mathrm{fm}\,.$$

III. Dynamical Description

To obtain further information about quarks by fitting the observed masses of the hadrons, it is necessary to calculate energy levels according to the quark model. While this should be done by solving the relativistic two-body and three-body problems, it seems better first to extract as much information about quarks as possible by simpler means. We use an independent quark model of both mesons and baryons, and consider only states which differ from the ground state by a change in quantum numbers of one quark. The quark wavefunction ψ_{nlj} is an eigenfunction of the Dirac equation

$$(c\boldsymbol{\alpha}.\mathbf{p}+\beta mc^2+V)\psi_{nlj} = E_{nlj}\psi_{nlj}\,, \tag{9}$$

where m is the reduced mass of the quark; the spherically symmetric potential V is taken to have the same form (although not necessarily the same strength) for both baryons and mesons. The discrete eigenvalues of (9) are bounded as (Rose 1961)

$$-mc^2 < E_{nlj} < +mc^2\,.$$

TABLE 1

j–j STATES DECOUPLED INTO L–S STATES

j–j State	J^{PC} of Equivalent L–S States
$s_{\frac{1}{2}}.s_{\frac{1}{2}}$	$\lvert 0^{-+}\rangle$ or $\lvert 1^{--}\rangle$
$s_{\frac{1}{2}}.p_{\frac{1}{2}}$	$\lvert 0^{++}\rangle$ or $\sqrt{\frac{2}{3}}\lvert 1^{++}\rangle-\sqrt{\frac{1}{3}}\lvert 1^{+-}\rangle$
$s_{\frac{1}{2}}.p_{3/2}$	$\lvert 2^{++}\rangle$ or $\sqrt{\frac{1}{3}}\lvert 1^{++}\rangle+\sqrt{\frac{2}{3}}\lvert 1^{+-}\rangle$

We interpret the state with the lowest possible eigenvalue $E_{nlj} = -mc^2$ as a hadron of zero mass, so that the mass of a hadron corresponding to a bound state at energy E_{nlj} will be taken as

$$M_{nlj}\,c^2 = E_{nlj}+mc^2\,. \tag{10}$$

For comparison, the Schrödinger equation was used with the same potential V, and we selected the zero of Mc^2 at $2mc^2$ below the lower bound of the energy continuum to correspond to the relativistic treatment. Disagreement between the results of the Schrödinger equation and the Dirac equation would show that the motion of a quark inside a hadron is not nonrelativistic.

In a model of elementary particles in which we describe the low lying hadrons by ground state wavefunctions and the higher hadronic multiplets as angular momentum excitations of these ground states, we are faced with the problem of the relationship between the bound states of the model and the observed hadrons. Experimentally the baryons are known to have half-integer total angular momentum and good parity. The mesons have integer total angular momentum and good parity and charge parity. L–S coupling of three spin $\frac{1}{2}$ quarks to form a baryon, and a quark and an antiquark to form a meson, have the experimentally required properties. However, if we wish to use the Dirac equation (9) to describe these hadrons then j–j coupling is appropriate.

For j–j coupling of a quark and antiquark to form a meson the parity and charge parity of a j–j state is most readily found by decoupling it into L–S states. In Table 1 the resultant j–j states of an $s_{\frac{1}{2}}$ quark (or antiquark) and another antiquark (or quark) is given with the decomposition of these states into L–S states.

Taking the splitting of the j–j states to form J^P states as proportional to $\boldsymbol{j}_1 \cdot \boldsymbol{j}_2$ the energy of the j–j state can be written as the weighted average of the J^P states it forms. For example, for the $Y = 0$, $I = 1$ mesons the mass of the $1s_{\frac{1}{2}}$ state is

$$M(1s_{\frac{1}{2}}) = \tfrac{1}{4}(3\rho + \pi) \approx 600 \text{ MeV}/c^2 .$$

For either assignment of the compound states, to fit the $J^P = 0^+$, 1^+, 1^-, and 2^+ mesons requires that

$$600 \text{ MeV}/c^2 \lesssim M(1s_{\frac{1}{2}}) \lesssim 800 \text{ MeV}/c^2 ,$$

$$400 \text{ MeV}/c^2 \lesssim M(1p_{\frac{1}{2}}) - M(1s_{\frac{1}{2}}) \lesssim 450 \text{ MeV}/c^2 ,$$

$$580 \text{ MeV}/c^2 \lesssim M(1p_{3/2}) - M(1s_{\frac{1}{2}}) \lesssim 680 \text{ MeV}/c^2 .$$

Using a model of the baryons in which we assume two quarks are coupled to zero total spin, and assuming that the low lying baryons are angular momentum excitations of the third quark, indicates that the $s_{\frac{1}{2}}$ state should describe the $\frac{1}{2}^+$ octet, the $p_{\frac{1}{2}}$ state the $\frac{1}{2}^-$ octet, and the $p_{3/2}$ state the $\frac{3}{2}^-$ octet.

This extends the bound state energy ranges to

$$600 \text{ MeV}/c^2 \lesssim M(1s_{\frac{1}{2}}) \lesssim 1320 \text{ MeV}/c^2 , \tag{11a}$$

$$400 \text{ MeV}/c^2 \lesssim M(1p_{\frac{1}{2}}) - M(1s_{\frac{1}{2}}) \lesssim 610 \text{ MeV}/c^2 , \tag{11b}$$

$$470 \text{ MeV}/c^2 \lesssim M(1p_{3/2}) - M(1s_{\frac{1}{2}}) \lesssim 680 \text{ MeV}/c^2 . \tag{11c}$$

We require that the eigenvalues of (9) satisfy (11).

IV. Numerical Procedure

For five forms of the potential V: (1) square well; (2) cut-off harmonic oscillator, namely,

$$\begin{aligned} V(r) &= -A(1 - r^2/\beta) , \qquad & r \leqslant \beta , \\ &= 0 , & r > \beta ; \end{aligned}$$

(3) Woods–Saxon potential (roughly intermediate between forms (1) and (2)), namely,

$$V(r) = -B[1 + \exp\{13 \cdot 2(r - r_0)/r_0\}]^{-1} ;$$

(4) Gaussian potential; and (5) exponential potential; eigenvalues of (9) were found numerically by a trial-and-error procedure. For a spherically symmetric potential the solution of (9) is (Rose 1961)

$$\psi_{nlj} = \begin{pmatrix} g(r) & \chi^{\mu}_{+k} \\ \mathrm{i}f(r) & \chi^{\mu}_{-k} \end{pmatrix} ,$$

where $\chi^{\mu}_{\pm k}$ are two component spinors and f and g (the radial wavefunctions) are the solutions of

$$\frac{\mathrm{d}\mu_1}{\mathrm{d}x}+\frac{K\mu_1}{x}-(\epsilon-v+1)\mu_2=0\,,\qquad \frac{\mathrm{d}\mu_2}{\mathrm{d}x}-\frac{K\mu_2}{x}+(\epsilon-v-1)\mu_1=0\,, \tag{12}$$

where $\mu_1 = rg$, $\mu_2 = rf$, $x = rmc/\hbar$, $\epsilon = E/mc^2$, and $v = V/mc^2$. Equations (12) were numerically integrated "out" from $x = 0$ to an arbitrary matching point $x = a$ and "in" from a point where the potential was assumed to be zero to $x = a$. The difference between the "out" and "in" solutions were used to alter ϵ until this difference was zero. This was accomplished using the result (see Appendix)

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\left[\frac{\mu_2}{\mu_1}\right]^{\mathrm{out}}-\left[\frac{\mu_2}{\mu_1}\right]^{\mathrm{in}}\right)_{x=a}=-\left(\left[\frac{\int_0^a(\mu_1^2+\mu_2^2)\,\mathrm{d}r}{\mu_1^2}\right]^{\mathrm{out}}+\left[\frac{\int_a^\infty(\mu_1^2+\mu_2^2)\,\mathrm{d}r}{\mu_1^2}\right]^{\mathrm{in}}\right)_{x=a},$$

which states that, between its discontinuities,

$$D=[\mu_2/\mu_1]^{\mathrm{out}}-[\mu_2/\mu_1]^{\mathrm{in}}$$

is a monotonically decreasing function of ϵ. This allows ϵ to be increased or decreased depending on whether D was positive or negative in the previous trial solution. The number of nodes in the eigenfunction determines if the eigenvalue is the lowest one.

V. Mass of the Quark

The strength of the potential in (9) was adjusted to fit the $1s_{\frac{1}{2}}$ eigenvalue to the values of $M(1s_{\frac{1}{2}})$ given in (11). In Figure 4 the energy spacings $E(1p_{3/2})-E(1s_{\frac{1}{2}})$ and $E(1p_{\frac{1}{2}})-E(1s_{\frac{1}{2}})$ are shown as functions of m for a fixed value of the root-mean-square radius of the $1s_{\frac{1}{2}}$ state, $\langle R^2\rangle^{\frac{1}{2}} = 0\cdot2$ fm. For $\langle R^2\rangle^{\frac{1}{2}} \geqslant 5\hbar/mc$ we found that $E(1p_{3/2})-E(1s_{\frac{1}{2}})$ and $E(1p_{\frac{1}{2}})-E(1s_{\frac{1}{2}})$ were almost independent of $M(1s_{\frac{1}{2}})$ for $M(1s_{\frac{1}{2}})$ as in (11). For $m \gtrsim 3$ GeV/c^2 all potentials gave essentially the same energy spacing. For $m \lesssim 3$ GeV/c^2 the variation of the energy spacing for different potentials was less than the variation for different $M(1s_{\frac{1}{2}})$. In Figure 4, curve A represents the maximum and curve B the minimum for the various potentials with $M(1s_{\frac{1}{2}})$ as in (11). As $\langle R^2\rangle^{\frac{1}{2}}$ is increased the energy spacing decreases so that for $\langle R^2\rangle^{\frac{1}{2}} = 1\cdot0$ fm the maximum of $E(1p_{3/2})-E(1s_{\frac{1}{2}})$ is ~ 100 MeV for $M(1s_{\frac{1}{2}})$ as in (11). To fit the $J^P = 0^+$, 1^+, 1^-, and 2^+ mesons and the $\frac{1}{2}^+$, $\frac{1}{2}^-$, and $\frac{3}{2}^-$ baryons as set out in (11) requires

$$\langle R^2\rangle^{\frac{1}{2}} \lesssim 0\cdot25 \quad \mathrm{fm}\,.$$

Using $\langle R^2\rangle^{\frac{1}{2}} = 0\cdot1$ fm, the reduced mass of the quark is

$$8\ \mathrm{GeV}/c^2 \lesssim m \lesssim 15\ \mathrm{GeV}/c^2$$

and, for $\langle R^2\rangle^{\frac{1}{2}} = 0\cdot25$ fm,

$$1\ \mathrm{GeV}/c^2 \lesssim m \lesssim 2\ \mathrm{GeV}/c^2$$

if the p-wave, s-wave energy spacing is to be fitted. Consequently the root-mean-square separation distance of the quarks in the core is

$$0 \cdot 1 \text{ fm} \lesssim \langle R^2 \rangle^{\frac{1}{2}} \lesssim 0 \cdot 25 \text{ fm}$$

and the quark mass $\mathscr{M}$ is

$$2 \text{ GeV}/c^2 \lesssim \mathscr{M} \lesssim 30 \text{ GeV}/c^2 .$$

For these parameter ranges the point at which the strength of the exponential potential is $1/e$ that at the origin occurs at a radius greater than $1 \cdot 5$ fm, and on these grounds the exponential well can be excluded as unrealistic.

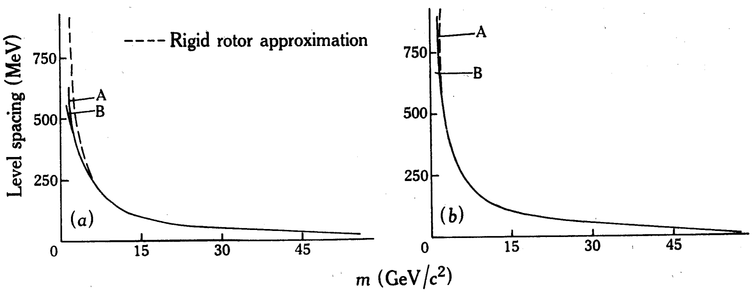

Fig. 4.—Energy spacings between the p-wave and s-wave eigenvalues for a fixed root-mean-square radius of the s-state wavefunction: (*a*) $1p_{1/2}$–$1s_{1/2}$ spacing, (*b*) $1p_{3/2}$–$1s_{1/2}$ spacing. The eigenvalues of the Schrödinger equation are well represented by the rigid rotor approximation, while the eigenvalues of the Dirac equation are distinctly different for a reduced mass $\lesssim$ 6 Compton wavelengths (see text for details).

VI. Rigid Rotor Model

The nonrelativistic results obtained from the Schrödinger equation are reproduced remarkably well by a rigid rotor approximation. The energy E of rotation of a rigid rotor is given by

$$E = L^2/2I = \hbar^2 l(l+1)/2I ,$$

where L is the total angular momentum and I the moment of inertia of the rotor. For a spherically symmetric mass distribution

$$I = \tfrac{2}{3} m \langle R^2 \rangle ,$$

where m is the total mass and $\langle R^2 \rangle$ the mean square radius of the mass distribution. We could fit the 1p–1s energy spacing with

$$E(1\text{p}) - E(1\text{s}) = 3\hbar^2/2m\langle R^2 \rangle_{1\text{s}}$$

to within $\sim 5\%$ for $m \gtrsim 1 \cdot 5\,\hbar/\langle R^2 \rangle^{\frac{1}{2}} c^2$. For $m \gtrsim 6\hbar/\langle R^2 \rangle^{\frac{1}{2}} c^2$ the nonrelativistic and relativistic p-wave, s-wave energy spacings are almost identical. Consequently the rigid rotor approximation fits the results obtained by solving the Dirac equation for $m \gtrsim 6\hbar/\langle R^2 \rangle^{\frac{1}{2}} c^2$. For $m \lesssim 6\hbar/\langle R^2 \rangle^{\frac{1}{2}} c^2$ the relativistic results are distinctly different from the nonrelativistic results, and hence, for the stated parameter range of the quark, relativistic effects are not negligible.

VII. Conclusions

For quarks whose interaction is described by a non-singular potential energy, the hadron spectrum and the scattering relations imply a quark mass $\mathscr{M}$ in the range

$$2\ \mathrm{GeV}/c^2 \lesssim \mathscr{M} \lesssim 30\ \mathrm{GeV}/c^2$$

and a root-mean-square separation distance of the quarks in the core given by

$$0\cdot 1\ \mathrm{fm} \lesssim \langle R^2\rangle^{\frac{1}{2}} \lesssim 0\cdot 25\ \mathrm{fm}\,.$$

Relativistic effects are not negligible for these values of the quark parameters. For larger values of the quark mass, the spacing of quark energy levels according to the Dirac equation agrees with that of the Schrödinger equation and with the rigid rotor model.

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Appendix

Taking equations (12) for two values of ϵ, namely ϵ^i and ϵ^j, we get

$$\frac{\mathrm{d}\mu_1{}^i}{\mathrm{d}x}+\frac{K\mu_1{}^i}{x}-(\epsilon^i-v+1)\mu_2{}^i=0\,, \tag{A1}$$

$$\frac{\mathrm{d}\mu_2{}^i}{\mathrm{d}x}-\frac{K\mu_2{}^i}{x}+(\epsilon^i-v-1)\mu_1{}^i=0 \tag{A2}$$

and

$$\frac{\mathrm{d}\mu_1{}^j}{\mathrm{d}x}+\frac{K\mu_1{}^j}{x}-(\epsilon^j-v+1)\mu_2{}^j=0\,, \tag{A3}$$

$$\frac{\mathrm{d}\mu_2{}^j}{\mathrm{d}x} - \frac{K\mu_2{}^j}{x} + (\epsilon^j - v - 1)\mu_1{}^j = 0\,. \tag{A4}$$

Multiplying (A1) by $\mu_2{}^j$ and (A3) by $\mu_2{}^i$ and substracting gives

$$\mu_2{}^j \frac{\mathrm{d}\mu_1{}^i}{\mathrm{d}x} - \mu_2{}^i \frac{\mathrm{d}\mu_1{}^j}{\mathrm{d}x} + \frac{K}{x}(\mu_2{}^j \mu_1{}^i - \mu_2{}^i \mu_1{}^j) - (\epsilon^i - \epsilon^j)\mu_2{}^i \mu_2{}^j = 0\,, \tag{A5}$$

and similarly from (A2) and (A4) we get

$$\mu_1{}^j \frac{\mathrm{d}\mu_2{}^i}{\mathrm{d}x} - \mu_1{}^i \frac{\mathrm{d}\mu_2{}^j}{\mathrm{d}x} - \frac{K}{x}(\mu_1{}^j \mu_2{}^i - \mu_1{}^i \mu_2{}^j) + (\epsilon^i - \epsilon^j)\mu_1{}^i \mu_1{}^j = 0\,. \tag{A6}$$

Substracting (A6) from (A5) gives

$$\mathrm{d}(\mu_1{}^i \mu_2{}^j - \mu_1{}^j \mu_2{}^i)/\mathrm{d}x = (\epsilon^i - \epsilon^j)(\mu_1{}^i \mu_1{}^j + \mu_2{}^i \mu_2{}^j)\,. \tag{A7}$$

Applying the boundary conditions μ_1 and μ_2 tend to zero as x tends to zero or infinity, integration of (A7) leads to

$$\Big[\mu_1{}^i \mu_2{}^j - \mu_1{}^j \mu_2{}^i\Big]_{x=a}^{\text{out}} = (\epsilon^i - \epsilon^j)\Big[\int_0^a (\mu_1{}^i \mu_1{}^j + \mu_2{}^i \mu_2{}^j)\,\mathrm{d}r\Big]^{\text{out}},$$

$$\Big[\mu_1{}^i \mu_2{}^j - \mu_1{}^j \mu_2{}^i\Big]_{x=a}^{\text{in}} = (\epsilon^i - \epsilon^j)\Big[\int_\infty^a (\mu_1{}^i \mu_1{}^j + \mu_2{}^i \mu_2{}^j)\,\mathrm{d}r\Big]^{\text{in}}.$$

Hence dividing by $\mu_1{}^i \mu_1{}^j$ and substracting we get

$$\left(\left[\frac{\mu_2{}^j}{\mu_1{}^j} - \frac{\mu_2{}^i}{\mu_1{}^i}\right]^{\text{out}} - \left[\frac{\mu_2{}^j}{\mu_1{}^j} - \frac{\mu_2{}^i}{\mu_1{}^i}\right]^{\text{in}}\right)_{x=a}$$

$$= (\epsilon^i - \epsilon^j)\left(\left[\frac{\int_0^a (\mu_1{}^i \mu_1{}^j + \mu_2{}^i \mu_2{}^j)\,\mathrm{d}r}{\mu_1{}^i \mu_1{}^j}\right]^{\text{out}} + \left[\frac{\int_a^\infty (\mu_1{}^i \mu_1{}^j + \mu_2{}^i \mu_2{}^j)\,\mathrm{d}r}{\mu_1{}^i \mu_1{}^j}\right]^{\text{in}}\right)_{x=a}.$$

Letting $\epsilon^i \to \epsilon^j$ leads to

$$\frac{\mathrm{d}}{\mathrm{d}\epsilon}\left(\left[\frac{\mu_2}{\mu_1}\right]^{\text{out}} - \left[\frac{\mu_2}{\mu_1}\right]^{\text{in}}\right)_{x=a} = -\left(\left[\frac{\int_0^a (\mu_1^2 + \mu_2^2)\,\mathrm{d}r}{\mu_1^2}\right]^{\text{out}} + \left[\frac{\int_a^\infty (\mu_1^2 + \mu_2^2)\,\mathrm{d}r}{\mu_1^2}\right]^{\text{in}}\right)_{x=a},$$

which is the required result.