QUANTIZATION OF LAGRANGIANS FOR CHARGES IN AN EXTERNAL FIELD*

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Generally Lagrangians in classical mechanics depend on the coordinates and velocities of the particles at one instant of time. Obviously such an approach assumes that all interactions are instantaneous, and hence that the field has an infinite velocity. To correct this situation we must use field theory.

However, for small v it has been shown by Darwin (1920) that to terms of order v^2/c^2 we may use the Lagrangian

$$L = -mc^2(1-v^2/c^2)^{\frac{1}{2}}-e\phi+(e/c)\boldsymbol{A}\,.\,\boldsymbol{v}\,, \tag{1}$$

for a charge in an external field, where ϕ and A are the retarded scalar and vector potentials.

Considering the case of two charges, one fixed at the origin and the other moving radially from the first at a distance R at time t, the vector potential is zero and

$$\phi = \int R^{-1}\mathscr{P}(t-R/c)\,\mathrm{d}V\,. \tag{2}$$

It may be shown by expanding $\mathscr{P}(t-R/c)$ in a Taylor series that

$$\phi = e/R+(e/2c^2)\mathrm{d}^2R/\mathrm{d}t^2\,. \tag{3}$$

Therefore, the Lagrangian to second order is

$$L = -mc^2+\tfrac{1}{2}m\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^2+\frac{m}{8c^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^4-\frac{e^2}{R}-\frac{e^2}{2c^2}\frac{\mathrm{d}^2R}{\mathrm{d}t^2}\,. \tag{4}$$

The momenta from the Ostrogradsky theory (Hayes 1969) are

$$p_1 = \frac{\partial L}{\partial(\mathrm{d}R/\mathrm{d}t)}-\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial(\mathrm{d}^2R/\mathrm{d}t^2)}\right) = m\frac{\mathrm{d}R}{\mathrm{d}t}+\frac{m}{2c^2}\left(\frac{\mathrm{d}R}{\mathrm{d}t}\right)^3,$$

$$p_2 = \frac{\partial L}{\partial(\mathrm{d}^2R/\mathrm{d}t^2)} = -\frac{e^2}{2c^2}\,.$$

It should be noted that p_1 is the usual relativistic momentum expanded to second order. Since p_1 is canonical to R and p_2 is canonical to $\mathrm{d}R/\mathrm{d}t$, we may form the

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commutation relations (Hayes 1969)

$$[p_1, R] = (\hbar/\mathrm{i}) \times \text{constant}, \tag{5}$$

$$[p_2, \mathrm{d}R/\mathrm{d}t] = (\hbar/\mathrm{i}) \times \text{constant}. \tag{6}$$

The constant in equation (5) is equal to one as would be expected. However, considering the resulting uncertainty relation we must either allow the constant in (6) to be zero or admit that there is a theoretical uncertainty in e/c or that the proposed quantization of the Ostrogradsky theory by various authors (see e.g. Podolsky and Schwed 1948) is invalid.

References

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