# "PARALLEL" VISCOUS MODIFICATION OF THE RESISTIVE "TEARING" INSTABILITY IN A CARTESIAN MODEL OF THE HARD-CORE PINCH*

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There is continuing interest in the possibility that a diffuse pinch may be used to achieve plasma containment in the laboratory, and ultimately lead to a successful thermonuclear reactor. Such a system is likely to be stabilized against hydromagnetic modes by strong magnetic field shear, but the question of stability against resistive modes remains. In this note, a Cartesian model is adopted, and the influence of "parallel" viscosity (Stringer 1970) on the resistive "tearing" instability in the hard-core pinch is examined. Previously, purely resistive calculations of growth rates in this configuration have been reported by Hosking (1967) and Lister and Hosking (1970).

The particular configuration of interest is that described by Hosking (1967), except that in the Cartesian model one adopts Cartesian coordinates $(x,y,z)$ so that the equilibrium magnetic field is

$$\boldsymbol{H}_0 = H_{0y}(x)\,\boldsymbol{e}_y + H_{0z}\,\boldsymbol{e}_z\,, \tag{1}$$

where

$$H_{0y}(x) = Ax + C/x\,, \qquad H_{0z} = \text{const.}\,,$$

and

$$\left.\begin{aligned} A &= 0\,, && r_0 < x < a\,,\\ &= -J_0/2\pi(b^2-a^2)\,, && a < x < b\,,\\ &= 0\,, && b < x < l\,; \end{aligned}\right\} \tag{2a}$$

$$\left.\begin{aligned} C &= \epsilon J_0/2\pi, && r_0 < x < a\,,\\ &= \{\epsilon + a^2/(b^2-a^2)\}J_0/2\pi, && a < x < b\,,\\ &= (\epsilon-1)J_0/2\pi\,, && b < x < l\,. \end{aligned}\right\} \tag{2b}$$

The notation adopted here is analogous to that used by Hosking (1967) and similar physical parameters are also used. Again, resistivity gradients and "gravity" are neglected, so that "rippling" and "gravitational" modes are precluded. The energy source for the tearing mode lies in the magnetic field; energy is released when the field lines shorten in length.

It is assumed that the configuration is stabilized against hydromagnetic modes. For the hard-core pinch, in the case of static equilibrium the sufficiency condition is $\epsilon \geqslant 1$, corresponding to a positive equilibrium pressure gradient. Assuming physical

* Manuscript received 18 May 1971.

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quantities of the form

$$f(x,y,z,t) = f_0(x) + f_1(x)\exp(\omega t + \mathrm{i}k_y y + \mathrm{i}k_z z),$$

the linearized perturbation equations for an incompressible viscous plasma are

$$\rho\omega v_{1x} + \mathrm{D}\pi_1 = \mu\mathrm{i}FH_{1x} - \rho\nu\,\mathrm{D}(\mathrm{i}FH_0^{-2}\mathbf{v}_1 \cdot \boldsymbol{H}_0), \tag{3a}$$

$$\rho\omega v_{1y} + \mathrm{i}k_y\pi_1 = \mu\mathrm{i}FH_{1y} + \mu\mathrm{D}(H_{0y})H_{1x} - \rho\nu(\mathrm{i}k_y - 3H_{0y}\mathrm{i}FH_0^{-2})(\mathrm{i}FH_0^{-2}\mathbf{v}_1 \cdot \boldsymbol{H}_0), \tag{3b}$$

$$\rho\omega v_{1z} + \mathrm{i}k_z\pi_1 = \mu\mathrm{i}FH_{1z} - \rho\nu(\mathrm{i}k_z - 3H_{0z}\mathrm{i}FH_0^{-2})(\mathrm{i}FH_0^{-2}\mathbf{v}_1 \cdot \boldsymbol{H}_0); \tag{3c}$$

$$\eta(\mathrm{D}^2 - k^2 - \omega/\eta)H_{1x} = -\mathrm{i}Fv_{1x}, \tag{4a}$$

$$\eta(\mathrm{D}^2 - k^2 - \omega/\eta)H_{1y} = -\mathrm{i}Fv_{1y} + \mathrm{D}(H_{0y})v_{1x}, \tag{4b}$$

$$\eta(\mathrm{D}^2 - k^2 - \omega/\eta)H_{1z} = -\mathrm{i}Fv_{1z}; \tag{4c}$$

$$\mathrm{D}v_{1x} + \mathrm{i}k_y v_{1y} + \mathrm{i}k_z v_{1z} = 0, \tag{5}$$

$$\mathrm{D}H_{1x} + \mathrm{i}k_y H_{1y} + \mathrm{i}k_z H_{1z} = 0; \tag{6}$$

where

$$\mathrm{D} = \mathrm{d}/\mathrm{d}x, \qquad F = k_y H_{0y} + k_z H_{0z},$$

$$\pi_1 = p_1 + \mu\boldsymbol{H}_0 \cdot \boldsymbol{H}_1, \qquad k = (k_y^2 + k_z^2)^{\frac{1}{2}},$$

and $k_y$ represents the cylindrical wave number $m/R_0$. In particular, it should be noted that $\mathrm{D}^2F \neq 0$.

Proceeding to the discussion of the resistive modes, we note that the equations (4) are approximately given by

$$\omega H_{1x} \approx \mathrm{i}Fv_{1x}, \qquad \omega H_{1y} \approx \mathrm{i}Fv_{1y} - \mathrm{D}(H_{0y})v_{1x}, \qquad \omega H_{1z} \approx \mathrm{i}Fv_{1z}, \tag{4'}$$

for sufficiently small $\eta$, except in the neighbourhood of $F = 0$, that is, the resistive region. Elimination between equations (3), (4′), (5), and (6) yields

$$\psi'' + f(\hat{x})\psi' + g(\hat{x})\psi = 0, \tag{7}$$

where

$$f(\hat{x}) = -2F'/F + R'/R + N'/N,$$

$$g(\hat{x}) = -\alpha^2/N - F''/F - f(\hat{x})F'/F,$$

$$\psi = 2\pi r_0 H_{1x}/J_0, \qquad \alpha = k(b-a), \qquad \hat{x} = x/(b-a)$$

are dimensionless quantities, and

$$R = \rho\omega + \mu F^2/\omega, \qquad N = 1 + M - G, \qquad G = \{1 - k^2H_{0z}/k_zF\}^{-1},$$

$$M = \frac{G(1 + JH_{0y}/k_yH_0)}{1 - k_z(H_{0z} - k_zH_{0y}/k_y)J/k^2H_0}, \qquad J = \frac{3\rho\nu(F/H_0)^3}{RG}.$$

TABLE 1

VARIATION OF MAXIMUM GROWTH RATES WITH TEMPERATURE FOR BOTH VISCOUS AND INVISCID THEORY

$a = 0 \cdot 09$ m, $b = 0 \cdot 11$ m, $J_0 = 9 \cdot 5 \times 10^4$ A, $H_{0z} = 1 \cdot 075 \times 10^5$ A m$^{-1}$, $\epsilon = 1$, $m = 1$

| $R_0$ (m) | $H_{0y}$ at $R_0$ (A m$^{-1}$) | $H_0$ ($10^5$ A m$^{-1}$) | $S$ | $(b-a)/R_0$ | $k_y$ | $k_z$ | $k$ | $T$ (K) | $n_i$ (m$^{-3}$) | $\nu$ (m$^2$ s$^{-1}$) | $\delta$ | $\Delta'$ | $\omega_{max}$ (s$^{-1}$) |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 0·1089 | $8 \cdot 358 \times 10^3$ | 1·078 | $1 \cdot 463 \times 10^3$ | 0·184 | 9·18 | −0·714 | 9·21 | $10^6$ | $10^{21}$ | $9 \cdot 21 \times 10^3$ | $1 \cdot 1 \times 10^{-3}$ | 24 | $2 \cdot 4 \times 10^5$ |
| | | | | | | | | | | 0 | $1 \cdot 0 \times 10^{-1}$ | 22 | $2 \cdot 3 \times 10^5$ |
| 0·1096 | $3 \cdot 029 \times 10^3$ | 1·075 | $3 \cdot 895 \times 10^4$ | 0·182 | 9·12 | −0·257 | 9·13 | $10^7$ | $10^{21}$ | $2 \cdot 46 \times 10^6$ | $3 \cdot 8 \times 10^{-2}$ | 93 | $9 \cdot 8 \times 10^4$ |
| | | | | | | | | | | 0 | $3 \cdot 5 \times 10^{-2}$ | 69 | $7 \cdot 7 \times 10^4$ |
| 0·1098 | $1 \cdot 513 \times 10^3$ | 1·075 | $1 \cdot 065 \times 10^6$ | 0·182 | 9·11 | −0·128 | 9·11 | $10^8$ | $10^{21}$ | $6 \cdot 72 \times 10^8$ | $1 \cdot 2 \times 10^{-2}$ | 240 | $2 \cdot 9 \times 10^4$ |
| | | | | | | | | | | 0 | $1 \cdot 2 \times 10^{-2}$ | 200 | $2 \cdot 5 \times 10^4$ |
| 0·1069 | $2 \cdot 378 \times 10^4$ | 1·101 | $6 \cdot 239 \times 10$ | 0·187 | 9·35 | −2·07 | 9·58 | $10^4$ | $10^{18}$ | $1 \cdot 22 \times 10^2$ | $3 \cdot 0 \times 10^{-1}$ | 8·4 | $2 \cdot 2 \times 10^7$ |
| | | | | | | | | | | 0 | $3 \cdot 0 \times 10^{-1}$ | 8·4 | $2 \cdot 2 \times 10^7$ |
| 0·1089 | $8 \cdot 358 \times 10^3$ | 1·078 | $1 \cdot 414 \times 10^3$ | 0·184 | 9·18 | −0·714 | 9·21 | $10^5$ | $10^{18}$ | $2 \cdot 82 \times 10^4$ | $1 \cdot 1 \times 10^{-1}$ | 22 | $7 \cdot 3 \times 10^6$ |
| | | | | | | | | | | 0 | $1 \cdot 1 \times 10^{-1}$ | 22 | $7 \cdot 3 \times 10^6$ |
| 0·1096 | $3 \cdot 029 \times 10^3$ | 1·075 | $3 \cdot 612 \times 10^4$ | 0·182 | 9·12 | −0·257 | 9·13 | $10^6$ | $10^{18}$ | $7 \cdot 21 \times 10^6$ | $3 \cdot 8 \times 10^{-2}$ | 84 | $3 \cdot 0 \times 10^6$ |
| | | | | | | | | | | 0 | $3 \cdot 7 \times 10^{-2}$ | 69 | $2 \cdot 6 \times 10^6$ |
| 0·1098 | $1 \cdot 513 \times 10^3$ | 1·075 | $9 \cdot 976 \times 10^5$ | 0·182 | 9·11 | −0·128 | 9·11 | $10^7$ | $10^{18}$ | $1 \cdot 99 \times 10^9$ | $1 \cdot 3 \times 10^{-2}$ | 240 | $9 \cdot 6 \times 10^5$ |
| | | | | | | | | | | 0 | $1 \cdot 2 \times 10^{-2}$ | 200 | $8 \cdot 2 \times 10^5$ |

In the above equations, the primes denote differentiation with respect to $\hat{x}$ and the term

$$-2F'/F+R'/R = -2F'/F(1+\mu F^2/\rho\omega^2)$$

is retained, since $\rho\omega$ may be comparable with $\mu F^2/\omega$ for the chosen parameters. As $\nu \to 0$, equation (7) reduces to the form given by Furth, Killeen, and Rosenbluth (1963), provided that $\rho\omega \ll \mu F^2/\omega$:

$$\psi''-(\alpha^2+F''/F)\psi = 0. \qquad (7')$$

The growth rates of the tearing modes were calculated by computing the increment in the logarithmic derivative of $\psi$,

$$\Delta' = (\psi'_+ - \psi'_-)/\psi,$$

across the resistive region from equation (7), and matching it with the form derived by Furth, Killeen, and Rosenbluth (1963) for the inner (resistive) region, namely

$$\Delta' = 3\delta\hat{\omega},$$

where

$$\delta^4 = \hat{\omega}/4\alpha^2 S^2(\hat{F}'\,|_{x=R_0})^2, \qquad \hat{\omega} = \omega\tau_R,$$

with

$$F = F/kH_0, \qquad \tau_R = (b-a)^2/\eta.$$

The detailed iterative matching procedure was similar to that used by Hosking (1967).

The maximum calculated growth rates occur when the resistive layer is adjacent to the outer edge of the current layer, and the values are given in Table 1. It is apparent that when parallel viscosity becomes significant ($T \gtrsim 10^6$ K), the tendency is for the growth rate to increase relative to the value given from inviscid ($\nu = 0$) theory. The greater stability at higher temperature appears to be due only to the correspondingly lower resistivity.

It is noteworthy that the Cartesian model adopted is not fully justified unless

$$|\mathrm{D}(H_{0y})| \gg |H_{0y}/x|,$$

corresponding to sufficient magnetic field shear. Although this criterion is satisfied in the plasma current layer $a < x < b$, it is clearly important to extend the present work to include geometrical effects, particularly in the outer plasma regions.

*Acknowledgments*

I am most grateful to Dr. R. J. Hosking for his advice, guidance, and encouragement. This work was performed with the assistance of an Australian Commonwealth Postgraduate Award. The computation was carried out under grants from the Australian Institute of Nuclear Science and Engineering and the Australian Research Grants Committee.

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