SOLUTIONS OF THE RELATIVISTIC TWO-BODY PROBLEM

II.* QUANTUM MECHANICS

By J. L. Cook†

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Abstract

This paper discusses the formulation of a quantum mechanical equivalent of the relative time classical theory proposed in Part I. The relativistic wavefunction is derived and a covariant addition theorem is put forward which allows a covariant scattering theory to be established. The free particle eigenfunctions that are given are found not to be plane waves. A covariant partial wave analysis is also given. A means is described of converting wavefunctions that yield probability densities in 4-space to ones that yield the 3-space equivalents. Bound states are considered and covariant analogues of the Coulomb potential, harmonic oscillator potential, inverse cube law of force, square well potential, and two-body fermion interactions are discussed.

I. Introduction

In Part I (Cook 1972, present issue pp. 117–39) the relativistic two-body problem was discussed and a system of calibrating proper times was proposed which permits the simple evaluation of many standard problems in a fully covariant way. The present paper deals with the Schrödinger quantization of the proper time theory and examines the properties of various relativistic models whose classical covariant solutions were obtained.

Section II is concerned with the derivation of the relativistic two-body wave equation and the properties of angular momentum operators. A covariant addition theorem is then derived which permits the configuration space and momentum space eigenfunctions to be coupled to give a covariant wavefunction. This theorem is applied to the construction of the two-body free particle wavefunction, which is found not to be a plane wave. This relativistic wave formalism is used to define the covariant cross section and scattering matrix and an expansion into covariant partial waves is derived.

Usually relativistic wavefunctions cannot be interpreted as defining probability densities in ordinary space. It is shown that this is because one is working in a 4-space of hyperbolic symmetry where features of wave propagation are unfamiliar. If the wavefunction is converted to those eigenfunctions appropriate to spherical symmetry in 3-space with an additional time coordinate, familiar and meaningful wavefunctions are obtained. A general symmetry conversion procedure is given in Section V and this is shown to yield plane waves in 3-space for the case of free particles. The conversion is applied to general scattering from potentials and formulae for the scattering matrix are derived.

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† AAEC Research Establishment, Private Mail Bag, Sutherland, N.S.W. 2232.

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The problem of bound states is considered in Section VI. Four standard models are treated: the covariant Coulomb field, the harmonic oscillator, motion under the influence of the inverse cube law of force, and the square well potential. Only the boson–boson model of these interactions is solved, but a model for the boson–fermion and fermion–fermion systems is proposed.

II. Schrödinger Quantization

The notation of Part I is used throughout the following work. Let us now examine the form of the proper time calibration theory when Schrödinger quantization is applied. The quantized relative 4-momentum of the two-body system is (Schiff 1949)

$$Q_{\mu} = \mathrm{i}\hbar\,\partial/\partial R_{\mu} \tag{1}$$

and when this is substituted into the component of the Hamiltonian which describes the relative motion,

$$2\mu(\mathscr{H}-\mathscr{V})-Q^2 = 0\,, \tag{2}$$

such that the equation becomes an operator equation acting on a covariant wave-function $\Psi(R)$, we find the covariant two-body wave equation

$$\{2\mu(\mathscr{H}-\mathscr{V})/\hbar^2+\Box^2\}\Psi(\mathbf{R}) = 0\,, \tag{3}$$

where

$$\Box^2 = G_{\mu\nu}\frac{\partial}{\partial R_{\mu}}\frac{\partial}{\partial R_{\nu}},$$

$G_{\mu\nu}$ is the metric tensor, μ is the reduced mass, and $\mathscr{V}$ is the covariant interaction.

Using the coordinates (30) of Part I and assuming hypercentral forces such that $\mathscr{V}$ is a function only of the hyper-radius S, we can separate (3) into the component eigenfunction equations

$$\left(S^2\frac{\mathrm{d}^2}{\mathrm{d}S^2}+3S\frac{\mathrm{d}}{\mathrm{d}S}+Q^2S^2-\Lambda^2\right)\Psi_S = \mathscr{V}\Psi_S\,, \tag{4a}$$

$$(1-y^2)\frac{\mathrm{d}^2\Psi_{\gamma}}{\mathrm{d}y^2}+\left(L^2-\frac{\Lambda^2+1}{1-y^2}\right)\Psi_{\gamma} = 0\,,\quad y = \tanh\gamma\,, \tag{4b}$$

$$(1-z^2)\frac{\mathrm{d}^2\Psi_{\theta}}{\mathrm{d}z^2}-2z\frac{\mathrm{d}\Psi_{\theta}}{\mathrm{d}z}+\left(L^2-\frac{m^2}{1-z^2}\right)\Psi_{\theta} = 0\,,\quad z = \cos\theta\,, \tag{4c}$$

$$\mathrm{d}^2\Psi_{\phi}/\mathrm{d}\phi^2+m^2\Psi_{\phi} = 0\,, \tag{4d}$$

where

$$\Psi(\mathbf{R}) = \Psi_S(S)\,\Psi_{\gamma}(\gamma)\,\Psi_{\theta}(\theta)\,\Psi_{\phi}(\phi)$$

and Heaviside units have been introduced ($\hbar = c = 1$). Putting

$$\Lambda^2 = \lambda(\lambda+2)\,,\qquad L^2 = l(l+1)\,, \tag{5}$$

we find the free particle eigenfunctions with $\mathscr{V} = 0$ as

$$\Psi_S = A_S J_{\lambda+1}(QS)/S + B_S N_{\lambda+1}(QS)/S\,, \tag{6a}$$

$$\Psi_\gamma = \{A_\gamma P_l^{\lambda+1}(\tanh\gamma) + B_\gamma Q_l^{\lambda+1}(\tanh\gamma)\}\operatorname{sech}\gamma\,, \tag{6b}$$

$$\Psi_\theta = A_\theta P_l^m(\cos\theta)\,, \tag{6c}$$

$$\Psi_\phi = A_\phi \exp(\mathrm{i}m\phi)\,, \tag{6d}$$

where J_ν and N_ν are Bessel functions of the first and second kind, P_μ and Q_μ are Legendre functions of the first and second kind, the A's and B's are constants, and the solutions (6c) and (6d) are chosen to be the same as in non-relativistic theory. In the following work $\hbar$ and c are shown explicitly wherever their significance in equations is considered to be important.

The wave equation (3) and solutions such as (6) are valid across the surface of equal proper time and are independent of proper time in systems where the centre-of-mass motion can be factorized from the total wavefunction. It is most important to realize that the relative time coordinate implicit in the definitions of S and γ applies only to ordinary times lying on the surface of equal proper time and that the wavefunction defines wave propagation relative to that particular coordinate. Therefore the wavefunction and its eigenvalues have a quite different significance from those used by Tomonaga (1946) and Feynman (1949), in which the particle times, wavefunctions, and eigenvalues apply to all possible surfaces and not just to one special surface.

The operators

$$\boldsymbol{L} = \mathrm{i}\hbar\left(\frac{1}{\sin\theta}\mathbf{e}_\theta\frac{\partial}{\partial\phi} - \mathbf{e}_\phi\frac{\partial}{\partial\theta}\right)$$

and

$$\boldsymbol{L}^2 = \hbar^2\left\{\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial}{\partial\theta}\right) + \operatorname{cosec}^2\theta\frac{\partial^2}{\partial\phi^2}\right\} = \mathbf{Q}_\theta^2 + \mathbf{Q}_\phi^2\operatorname{cosec}^2\theta$$

involve no derivatives with respect to S and γ, and

$$\boldsymbol{L}^2\,\Psi(\mathbf{R}) = l(l+1)\hbar^2\,\Psi(\mathbf{R})\,, \tag{7}$$

as in non-relativistic theory. The polar operator

$$\boldsymbol{A} = -\mathrm{i}\hbar\left\{\mathbf{e}_R\frac{\partial}{\partial\gamma} + \tanh\gamma\left(\mathbf{e}_\theta\frac{\partial}{\partial\theta} + \frac{1}{\sin\theta}\mathbf{e}_\phi\frac{\partial}{\partial\phi}\right)\right\}$$

satisfies

$$\boldsymbol{A}^2 = \hbar^2\left\{\left(\frac{\partial^2}{\partial\gamma^2} + 2\tanh\gamma\frac{\partial}{\partial\gamma}\right) + \tanh^2\gamma\,\boldsymbol{L}^2\right\},$$

so that the operator

$$\Lambda^2 = \boldsymbol{L}^2 - \boldsymbol{A}^2 = \boldsymbol{L}^2\operatorname{sech}^2\gamma + \hbar^2\left(\frac{\partial^2}{\partial\gamma^2} + 2\tanh\gamma\frac{\partial}{\partial\gamma}\right) = \boldsymbol{L}^2\operatorname{sech}^2\gamma - \mathbf{Q}_\gamma^2$$

has eigenvalues $\lambda(\lambda+2)$ when acting on $\Psi(\mathbf{R})$. Neither $\boldsymbol{L}$ nor $\boldsymbol{A}$ contain derivatives with respect to S. Hence

$$\boldsymbol{A}^2\,\Psi(\mathbf{R}) = \{l(l+1)-\lambda(\lambda+2)\}\hbar^2\,\Psi(\mathbf{R}) = a^2\,\Psi(\mathbf{R})\,. \tag{8}$$

If the Bohr correspondence principle (Schiff 1949) is to hold we should choose A^2 to have positive eigenvalues, and so $l > \lambda$.

Now consider the mathematical situation of the theory concerning the expansion of plane waves

$$\Psi = B\exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R}) = B\exp(\mathrm{i}QSZ) \tag{9}$$

into pseudospherically symmetric eigenfunctions. The following formulae are given by Erdelyi *et al.* (1953). The first is

$$\exp(\mathrm{i}QSZ) = 2^{\nu}\Gamma(\nu)\sum_{n=0}^{\infty}(\mathrm{i})^{n}(\nu+n)(QS)^{-\nu}\,J_{\nu+n}(QS)\,C_n^{\nu}(Z)\,, \tag{10}$$

where ν is arbitrary and C_n^{ν} is the Gegenbauer function. Comparing (10) with the solutions (6), it is seen that $n = \lambda$, $\nu = 1$ are the appropriate choices for a plane wave solution. Furthermore

$$Z = (\cosh\gamma\cosh\delta)z - \sinh\gamma\sinh\delta\,, \tag{11}$$

where $\sinh\delta = \epsilon/Q$, $\cosh\delta = q/Q$, and $z = \boldsymbol{q}\,.\,\boldsymbol{R}/qR$, and therefore one can use the Gegenbauer addition theorem to define solutions in terms of the $Q_l^{\lambda+1}(\tanh\gamma)$. However, these solutions have eigenvalues a^2 from (8) which are negative and must therefore be rejected as not satisfying the correspondence principle. What then is the alternative to the plane wave expansion?

III. Covariant Addition Theorem and Free Particle Solutions

The volume element in the hyperspace is

$$\mathrm{d}V = S^3\cosh^2\!\gamma\sin\theta\,\mathrm{d}S\,\mathrm{d}\gamma\,\mathrm{d}\theta\,\mathrm{d}\phi \tag{12}$$

and, using the $P_l^{\lambda+1}$ solution, we have

$$\int_{-1}^{1}\frac{\mathrm{d}t}{1-t^2}\,P_l^{\lambda+1}(t)\,P_l^{\lambda'+1}(t) = \frac{(\lambda+l+1)!\,\delta_{\lambda\lambda'}}{(\lambda+1)(l-\lambda+1)!} \tag{13}$$

from Erdelyi *et al.* (1953) and therefore these eigenfunctions form an incomplete orthonormal set with $l \geqslant \lambda+1$, and hence $a^2 > 0$.

From the expansion properties in the three-dimensional case, we expect an expansion of the form

$$\Psi(\mathbf{Q},\mathbf{R}) = \sum_{\lambda=-1}^{\infty}\sum_{l=\lambda+1}^{\infty}\sum_{m=-l}^{l} a_{\lambda lm}\frac{J_{\lambda+1}(QS)}{QS} \times P_l^{\lambda+1}(\tanh\gamma)\,\mathrm{sech}\,\gamma\,P_l^{m}(\cos\theta)\exp(\mathrm{i}m\phi) \tag{14}$$

to represent the free particle eigenfunction that is Lorentz invariant. The $a_{\lambda lm}$ are functions of the components of the relative momentum Q. In order to carry out the summations in (14) it is necessary to establish a covariant addition theorem, and this is indicated in a semi-rigorous way.

The spherical harmonic eigenfunctions

$$\mathscr{Y}_{lm}(\theta,\phi) = (-1)^m\{(2l+1)(l-m)!/4\pi(l+m)!\}P_l^m(\cos\theta)\exp(im\phi) \quad (15)$$

(Edmunds 1957) simplify calculations in non-relativistic theory. To this end, we define their covariant equivalents

$$\mathscr{Y}_{nlm}(\gamma,\theta,\phi) = \{n(l-n)!/(l+n)!\}\mathscr{Y}_{lm}(\theta,\phi)P_l^n(\tanh\gamma)\operatorname{sech}\gamma\,, \quad (16)$$

which satisfy

$$\iiint \overset{*}{\mathscr{Y}}_{nlm}(\gamma,\theta,\phi)\,\mathscr{Y}_{n'l'm'}(\gamma,\theta,\phi)\,\mathrm{d}\Omega = \delta_{nn'}\,\delta_{ll'}\,\delta_{mm'}\,, \quad (17)$$

where we have used $n = \lambda+1$. The integral is taken over the whole physical relative 4-space and is easily proved using the orthogonality relation (Edmunds 1957)

$$\int_0^\pi \sin\theta\,\mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\phi\,\overset{*}{\mathscr{Y}}_{lm}(\theta,\phi)\,\mathscr{Y}_{l'm'}(\theta,\phi) = \delta_{ll'}\,\delta_{mm'} \quad (18)$$

together with the integral (13) and the volume element (12), which gives

$$\mathrm{d}\Omega = \cosh^2\gamma\sin\theta\,\mathrm{d}\gamma\,\mathrm{d}\theta\,\mathrm{d}\phi\,. \quad (19)$$

The components of $\mathbf{R}$ are chosen to be $(S,\gamma_1,\theta_1,\phi_1)$ and those of $\mathbf{Q}$ to be $(Q,\gamma_2,\theta_2,\phi_2)$ for the purpose of the following argument. The object is to determine a Lorentz-invariant eigenfunction $g_n(Z)$ where

$$Z = \mathbf{Q}\,.\,\mathbf{R}/QS\,,$$

which is a superposition of the angular components of the wavefunction (6):

$$g_n(Z) = \sum_{l=n}^{\infty}\sum_{m=-l}^{l} b_{nlm}(\gamma_2,\theta_2,\phi_2)\,\mathscr{Y}_{nlm}(\gamma_1,\theta_1,\phi_1)\,. \quad (20)$$

However, we will postulate that because Z is invariant under the transformations $\gamma_1 \leftrightarrow \gamma_2$, $\theta_1 \leftrightarrow \theta_2$, $\phi_1 \leftrightarrow \phi_2$, the eigenfunctions on the right-hand side of (20) must be similarly invariant, provided we assume $g_n(Z)$ to be a real function. We therefore put

$$g_n(Z) = \sum_{l=n}^{\infty}\sum_{m=-l}^{l} a_{nlm}\,\overset{*}{\mathscr{Y}}_{nlm}(\gamma_2,\theta_2,\phi_2)\,\mathscr{Y}_{nlm}(\gamma_1,\theta_1,\phi_1)\,. \quad (21)$$

The usual addition theorem (Edmunds 1957)

$$P_l(\cos\omega) = \frac{4\pi}{2l+1}\sum_{m=-l}^{l}\overset{*}{\mathscr{Y}}_{lm}(\theta_2,\phi_2)\,\mathscr{Y}_{lm}(\theta_1,\phi_1) \quad (22)$$

can be used to carry out the summation over m in (21) which yields

$$g_n(Z) = \frac{1}{4\pi}\sum_{l=n}^{\infty}(2l+1)a_{nl}\left(\frac{n(l-n)!}{(l+n)!}\right)P_l^n(\tanh\gamma_1)\operatorname{sech}\gamma_1$$

$$\times P_l^n(\tanh\gamma_2)\operatorname{sech}\gamma_2\, P_l(z)\,, \tag{23}$$

where $z = \cos\omega$ and $a_{nl} = a_{nlm}$, as required by invariance under rotations in 3-space.

It was found in all applications in Part I that the geometrical physical region is defined by $|Z| \leqslant 1$. Regions where $|Z| > 1$ are actually accessible from the usual physical ranges of $(\gamma_1, \theta_1, \phi_1)$ and $(\gamma_2, \theta_2, \phi_2)$ unless the restriction on Z is taken as a separate kinematic condition. The sum on the right-hand side of (23) is therefore explicitly limited to the region $|Z| \leqslant 1$. Inserting a Heaviside function we have

$$g_n(Z)\,\theta(1-Z^2) = \frac{1}{4\pi}\sum_{l=n}^{\infty}(2l+1)a_{nl}\left(\frac{n(l-n)!}{(l+n)!}\right)$$

$$\times P_l^n(t_1)\,(1-t_1^2)^{\frac{1}{2}}P_l^n(t_2)\,(1-t_2^2)^{\frac{1}{2}}P_l(z)\,\theta(1-Z^2)\,, \tag{24}$$

where

$$\theta(X) = 1 \qquad \text{for} \qquad X > 0\,,$$

$$= 0 \qquad\qquad X < 0\,,$$

and $t_1 = \tanh\gamma_1$ and $t_2 = \tanh\gamma_2$. Multiplying both sides by $P_{l'}(z)$ and integrating over 3-space, we obtain

$$\int_{a(Z,z_1)}^{a(Z,z_2)} P_{l'}(z')\,g_n(Z')\,\theta(1-Z'^2)\,\mathrm{d}z'$$

$$= \frac{1}{2\pi}\sum_{l=n}^{\infty}\left(\frac{n(l-n)!}{(l+n)!}\right)a_{nl}\,P_l^n(t_1)\,(1-t_1^2)^{\frac{1}{2}}P_l^n(t_2)\,(1-t_2^2)^{\frac{1}{2}}(l+\tfrac{1}{2})$$

$$\times\int_{a(Z,z_1)}^{a(Z,z_2)} P_l(z')\,P_{l'}(z')\,\mathrm{d}z'\,, \tag{25}$$

where $a(Z, z)$ are the limits imposed by $z_1 \leqslant z \leqslant z_2$, since $|Z| \leqslant 1$. Now

$$z = t_1 t_2 + (1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}} \tag{26}$$

and behaves somewhat like a cosine of an azimuthal angle with respect to z-space. However, the well-known addition theorem (22) can be written

$$P_l(z) = P_l(t_1)\,P_l(t_2) + 2\sum_{n=1}^{l}\frac{\Gamma(l-n+1)}{\Gamma(l+n+1)}P_l^n(t_1)\,P_l^n(t_2)\cos(n\arccos Z)\,, \tag{27}$$

provided equation (26) is satisfied. Therefore, if equation (27) is substituted into the left-hand side of (25) we obtain

$$\sum_{n'=0}^{l} C_{n'l'} P_{l'}^{n'}(t_1) P_{l'}^{n'}(t_2) \int_{a(z,z_1)}^{a(z,z_2)} \mathrm{d}z'\, g_n(Z') \cos(n' \arccos Z')\, \theta(1-Z'^2)$$

$$= \sum_{n'=0}^{l'} C_{n'l'} P_{l'}^{n'}(t_1) P_{l'}^{n'}(t_2) \int_{-1}^{1} \mathrm{d}Z'\, (1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}} \cos(n' \arccos Z')\, g_n(Z'), \qquad (28)$$

where

$$C_{nl} = h_n \Gamma(l-n+1)/\Gamma(l+n+1), \qquad h_0 = 1, \quad h_n = 2, n \neq 0.$$

Now the eigenfunctions P_l^n form an orthonormal set, so the expression (28) cannot equal the right-hand side of (25) unless we choose $g_n(Z)$ as orthogonal to $\cos(n' \arccos Z)$. It follows that $g_n(Z)$ must be a member of this latter set with an appropriate weight function. Hence

$$g_n(Z) = (1-Z^2)^{-\frac{1}{2}} \cos(n \arccos Z). \qquad (29)$$

The Chebyshev polynomial

$$T_n(Z) = \cos(n \arccos Z)$$

satisfies (Gradshteyn and Ryzhik 1965)

$$\int_{-1}^{1} (1-Z^2)^{-\frac{1}{2}} T_n(Z)\, T_m(Z)\, \mathrm{d}Z = (\pi/h_n)\delta_{mn}. \qquad (30)$$

It is apparent that for the special case where $t_1 = t_2$, we have $a(1,z_2) = 1$, $a(-1,z_1) = -1$, and therefore

$$a_n = 2\pi^2/n. \qquad (31)$$

Although the series (28) diverges in the limit $t_1 \to t_2$, $Z \to \pm 1$, the constant a_n is correctly projected from the equation, as a factor $(1-z'^2)$ that arises from the process of evaluating the integral cancels the infinity in the limit. Obviously, the $n = 0$ case must be dealt with separately. One finds the eigenfunction expansion as a result of (31)

$$\frac{\cos(n \arccos Z)}{(1-Z^2)^{\frac{1}{2}}} = \frac{2\pi^2}{n} \sum_{l=0}^{\infty} \sum_{m=-l}^{l} \overset{*}{\mathscr{Y}}_{nlm}(\gamma_1, \theta_1, \phi_1)\, \mathscr{Y}_{nlm}(\gamma_2, \theta_2, \phi_2)$$

$$= \tfrac{1}{2}\pi \sum_{l=0}^{\infty} (2l+1) \frac{(l-n)!}{(l+n)!} P_l^n(t_1)\, P_l^n(t_2)\, (1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}}\, P_l(z). \qquad (32)$$

The second expansion is used for $n = 0$.

To find the equivalent to the plane wave expansion we consider the form of (10) normally used in two dimensions:

$$\exp(\mathrm{i}QSZ) = \sum_{n=0}^{\infty} (\mathrm{i})^n h_n\, J_n(QS) \cos(n \arccos Z). \qquad (33)$$

The wavefunction that is a superposition of free particle solutions is

$$\Psi(\mathbf{Q},\mathbf{R}) = \sum_{n=0}^{\infty} a_n \frac{J_n(QS)}{QS} \frac{\cos(n \arccos Z)}{(1-Z^2)^{\frac{1}{2}}}. \tag{34}$$

Comparing (33) with (34) we see that a choice of $a_n = (\mathrm{i})^n h_n$ leads to

$$\Psi(\mathbf{Q},\mathbf{R}) = \sum_{n=0}^{\infty} (\mathrm{i})^n h_n \frac{J_n(QS)}{QS} \frac{\cos(n \arccos Z)}{(1-Z^2)^{\frac{1}{2}}} = \frac{\exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R})}{QS(1-Z^2)^{\frac{1}{2}}}$$

$$= \{Q^2S^2-(\mathbf{Q}\,.\,\mathbf{R})^2\}^{-\frac{1}{2}} \exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R}), \qquad |Z| \leqslant 1. \tag{35}$$

This wavefunction has a plane wave period but is distorted by an amplitude that depends upon both **S** and **Q**. To test if (35) is a solution to the wave equation (3), we note that

$$\square^2\{f \exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R})\} = -Q^2 f \exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R}) + \mathrm{i}(\mathbf{Q}\,.\,\square f)\exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R}) + \exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R})(\square^2 f).$$

The factor $\{Q^2S^2-(\mathbf{Q}\,.\,\mathbf{R})^2\}^{-\frac{1}{2}}$ satisfies

$$\mathbf{Q}\,.\,\square\{S(1-Z^2)^{\frac{1}{2}}\}^{-1} = 0, \qquad \square^2\{S(1-Z^2)^{\frac{1}{2}}\}^{-1} = 0. \tag{36}$$

It is the $n = 0$ eigenfunction of the homogeneous equation

$$\square^2 f_{nlm} = 0.$$

The c.m. motion factors from the complete wavefunction. Combining all of these results leads to a physically meaningful two-boson wavefunction without interaction of

$$\Psi(\mathbf{r}_1,\mathbf{P}_1,\mathbf{r}_2,\mathbf{P}_2) = \frac{\exp\{\mathrm{i}(\mathbf{r}_1\,.\,\mathbf{P}_1+\mathbf{r}_2\,.\,\mathbf{P}_2)\}}{QS(1-Z^2)^{\frac{1}{2}}} = \frac{\exp\{\mathrm{i}(\mathbf{Q}_1\,.\,\mathbf{R}_1+\mathbf{Q}_2\,.\,\mathbf{R}_2)\}}{Q_2\,S(1-Z^2)^{\frac{1}{2}}}, \tag{37}$$

where $\mathbf{r}_1\,.\,\mathbf{P}_1+\mathbf{r}_2\,.\,\mathbf{P}_2 = \mathbf{Q}_1\,.\,\mathbf{R}_1+\mathbf{Q}_2\,.\,\mathbf{R}_2$, as explained in Part I, and

$$\mathbf{R}_1 = (m_1/M)\mathbf{r}_1+(m_1/M)\mathbf{r}_2, \qquad \mathbf{R}_2 = \mathbf{r}_1-\mathbf{r}_2,$$

$$\mathbf{Q}_1 = \mathbf{P}_1+\mathbf{P}_2, \qquad \mathbf{Q}_2 = \mu(\mathbf{P}_1/m_1-\mathbf{P}_2/m_2),$$

$$\mu = m_1m_2/(m_1+m_2), \quad \mathbf{S} = \mathbf{R}_2, \quad \mathbf{Q} = \mathbf{Q}_2, \quad Z = \mathbf{Q}_2\,.\,\mathbf{R}_2/Q_2S.$$

IV. Cross Sections

Having established an analogy between the relativistic kinematics in terms of relative coordinates and non-relativistic theory in general, one can almost write down the covariant quantities without proof. To show that this analogy holds for the scattering of bosons, covariant cross sections for scattering are derived. These are not the conventional cross sections associated with two-dimensional areas in three-space, but are three-dimensional cross sections of the volume in relative four-space. The expansion (35) is used for this purpose. The Bessel functions behave for large QS as

$$J_n(QS) \simeq (2/\pi QS)^{\frac{1}{2}} \cos(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi), \tag{38a}$$

$$N_n(QS) \simeq (2/\pi QS)^{\frac{1}{2}} \sin(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi), \tag{38b}$$

or in terms of Hankel functions

$$\begin{aligned} H_n^{(1)}(QS) &= J_n(QS) + \mathrm{i}N_n(QS) \\ &\simeq (2/\pi QS)^{\frac{1}{2}} \exp\{\mathrm{i}(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\}, \end{aligned} \tag{39a}$$

$$H_n^{(2)}(QS) = H_n^{(1)*}(QS). \tag{39b}$$

The free particle wave behaves as

$$\begin{aligned} \frac{N \exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{S})}{QS(1-Z^2)^{\frac{1}{2}}} &= N \sum_{n=0}^{\infty} (\mathrm{i})^n h_n \frac{J_n(QS)}{QS} \frac{\cos n\omega}{\sin \omega} \\ &\simeq N \sum_{n=0}^{\infty} (\mathrm{i})^n h_n (QS)^{-3/2} [\exp\{-\mathrm{i}(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\} + \exp\{\mathrm{i}(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\}] \\ &\qquad \times (\cos n\omega)/\sin \omega, \end{aligned} \tag{40}$$

where N is a normalization constant and $Z = \cos \omega$. The first term in square brackets on the right-hand side of (40) describes an incoming wave and the second an outgoing wave, propagating through the four-dimensional space–time. The presence of a scattering and reacting source modifies the outgoing component. The wavefunction for such a process becomes

$$\begin{aligned} \Psi \simeq N \sum_{n=0}^{\infty} (\mathrm{i})^n H_n (QS)^{-3/2} [\exp\{-\mathrm{i}(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\} + \eta_n \exp\{\mathrm{i}(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\}] \\ \times (\cos n\omega)/\sin \omega, \end{aligned} \tag{41}$$

where η_n is a complex amplitude. Equation (41) holds in the asymptotic region where QS is large and where it is assumed that no interaction takes place. The scattered component of the wave is therefore

$$\Psi_{\mathrm{sc}} \simeq N \sum_{n=0}^{\infty} (\mathrm{i})^n h_n (QS)^{-3/2} (1-\eta_n) \exp\{\mathrm{i}(QS - \tfrac{1}{2}n\pi - \tfrac{1}{4}\pi)\} (\cos n\omega)/\sin \omega. \tag{42}$$

Suppose we confine the region of interaction to a hypersphere of radius S_0, whose surface defines a Lorentz-invariant boundary in 4-space. With reference to the c.m. proper time τ, as defined in Part I, the number of particles F_{s}, scattered per second into the solid angle $\mathrm{d}\Omega$, is the number scattered through $S_0^3\,\mathrm{d}\Omega$. Hence

$$F_{\mathrm{s}}\,\mathrm{d}\Omega = -\mathbf{J}(\mathbf{S}_0/S_0)\,.\,S_0^3\,\mathrm{d}\Omega, \tag{43}$$

where

$$\mathbf{J} = -(\mathrm{i}\hbar/2\mu)(\Psi_{\mathrm{sc}} \Box \Psi_{\mathrm{sc}} - \Psi_{\mathrm{sc}} \Box \Psi_{\mathrm{sc}})$$

is the current out of the 4-sphere. Substituting the scattered component (42) into the equation (43) for the scattering rate, one finds

$$F_s(\Omega)\,\mathrm{d}\Omega = (Q/\mu)\,|\,\Psi_{\mathrm{sc}}\,|^2 S_0^3\,\mathrm{d}\Omega\,. \tag{44}$$

Put $V = Q/\mu$, as the magnitude of the relative 4-velocity, and define the covariant cross section as

$$\Sigma_{\mathrm{sc}} = F_s/V\,. \tag{45}$$

Using (42), (44), and (45), one obtains a cross section

$$\Sigma_{\mathrm{sc}} = \frac{N^2}{Q^3}\left|\sum_{n=0}^{\infty} h_n(1-\eta_n)\frac{\cos n\omega}{\sin\omega}\right|^2. \tag{46}$$

It is clear that equation (46) for the covariant cross section behaves as if there were a kinematic singularity at $\omega = 0$, on the boundary of the physical region. This singularity is cancelled by the zero in the Jacobian of the volume element; showing this explicitly,

$$\begin{aligned}\Sigma_{\mathrm{sc}}\,\mathrm{d}\Omega &= \frac{N^3}{Q^3}\left|\sum_{n=0}^{\infty} h_n(1-\eta_n)\frac{\cos(n\arccos Z)}{(1-Z^2)^{\frac{1}{2}}}\right|^2(1-Z^2)^{\frac{1}{2}}\,\mathrm{d}Z\,\mathrm{d}z\,\mathrm{d}\Phi \\ &= \frac{N^3}{Q^3}\left|\sum_{n=0}^{\infty} h_n(1-\eta_n)\cos n\omega\right|^2\mathrm{d}\omega\,\mathrm{d}z\,\mathrm{d}\Phi\,.\end{aligned} \tag{47}$$

Once again we note how the system behaves as if there were an additional azimuthal angle ω. The form of the cross section (47) applies in any frame of reference. A partial wave analysis of this type, when carried out in the laboratory system, has the same form in the c.m. system, or in any other frame of reference. The total scattering cross section is

$$\Sigma_{\mathrm{t,sc}} = \iiint \mathrm{d}\Omega\,\Sigma_{\mathrm{sc}} = (4\pi^2N^2/Q^2)\sum_{n=0}^{\infty} h_n\,|\,1-\eta_n\,|^2\,. \tag{48}$$

A completely analogous derivation of the reaction cross section yields

$$\Sigma_{\mathrm{r}} = (4\pi^2N^2/Q^3)\sum_{n=0}^{\infty} h_n(1-|\,\eta_n\,|^2)\,. \tag{49}$$

The normalization constant N could be chosen to be proportional to Q, giving Σ the same dimensions as ordinary cross sections.

The first term in $n = 0$ in (49) contains all of the s-wave, since $n \leqslant l$. It contains contributions from other partial waves as well. The scattered intensity cannot exceed the initial intensity, and so $|\,\eta_n\,| \leqslant 1$. The definition of the covariant scattering matrix is also wholly analogous to non-relativistic theory. Outside the region of interaction, the wavefunction satisfying (4a) is

$$\Psi = \sum_{n=0}^{\infty} C_n(\mathscr{I}_n - S_n\mathscr{O}_n)\,, \tag{50a}$$

where the C_n are constants and the incoming and outgoing components $\mathscr{I}_n$ and $\mathscr{O}_n$ respectively are given by

$$\mathscr{I}_n = (\mathrm{i})^n \frac{\cos n\omega}{\sin\omega} \frac{I_n}{(QS)^{3/2}}, \qquad \mathscr{O}_n = (\mathrm{i})^n \frac{\cos n\omega}{\sin\omega} \frac{O_n}{(QS)^{3/2}}, \tag{50b}$$

with

$$I_n = (\tfrac{1}{2}\pi QS)^{\frac{1}{2}} H_n^{(2)}(QS), \qquad O_n = (\tfrac{1}{2}\pi QS)^{\frac{1}{2}} H_n^{(1)}(QS), \tag{50c}$$

so that

$$\Psi = \sum_n C_n(\mathscr{I}_n + \mathscr{O}_n) - \sum_n (1-S_n)\mathscr{O}_n$$

$$\simeq N_0\left(\frac{\exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{R})}{S\sin\omega}\right) - \left(\frac{\exp\{\mathrm{i}(QS-\frac{1}{4}\pi)\}}{S^{3/2}} F(Q,Z)\right), \tag{51}$$

which is equal to (free wave)—(scattered component), with

$$F(Q,Z) = (2\pi)^{\frac{1}{2}} Q^{-3/2} N_0(1-S_n)(\cos n\omega)/\sin\omega\,, \tag{52a}$$

N_0 being a normalization constant, and

$$\Sigma_{\mathrm{sc}} = |F(Q,Z)|^2. \tag{52b}$$

$F(Q,Z)$ is the covariant scattering amplitude.

If the wavefunctions Ψ, $\mathscr{I}$, and $\mathscr{O}$ are now taken to be column vectors of channel wavefunctions, Q as channel momenta, and n as covariant angular momenta defined in each channel, S_n becomes a matrix in channel space. From the additivity of the $\Lambda_{\mu\nu}$, and the fact that it commutes with $\mathscr{H}$, one can conclude that n, and hence λ, is conserved throughout the reaction, just as l would be non-relativistically.

V. Symmetry Conversion

The covariant wavefunctions for bound states lead to convergent integrals for probability densities and reasonably simple expressions for covariant cross sections. However, the quantities

$$P = |\Psi|^2, \qquad \mathbf{J} = (\hbar/2\mu\mathrm{i})(\Psi\,\square\Psi^* - \Psi^*\,\square\Psi), \tag{53}$$

derived from the wave equation (3) are densities relative to both ordinary space and the relative times which define the surface of equal proper time. Therefore, the question arises as to how to convert these quantities to the conventional 3-space equivalents. The wave propagation in the relative time direction must be removed in such a way as to leave a relative energy eigenvalue in the 3-space Schrödinger equation equivalent to (3). This can be done by defining the Fourier transforms (Sneddon 1951) at a point in 3-space, with respect to a dummy variable ξ, as

$$\bar{\Psi}(\boldsymbol{q},\xi,\mathbf{R}) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{\infty} \exp(\mathrm{i}\xi T)\,\Psi(\mathbf{Q},\mathbf{R})\,\mathrm{d}T\,\theta(S)^2, \tag{54a}$$

$$\Psi(\mathbf{Q},\mathbf{R}) = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} \exp(-\mathrm{i}\xi T)\,\bar{\Psi}(q,\xi,\mathbf{R})\,\mathrm{d}\xi, \quad \psi(q,\mathbf{R}) = \bar{\Psi}(q,(q^2-Q^2)^{\frac{1}{2}},\mathbf{R}), \tag{54b}$$

that is, when $\xi = \epsilon$ the conversion to 3-space is achieved.

Taking the transform of (3), we get

$$(-\nabla^2-\epsilon^2)\psi + 2\mu(2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty} \exp(\mathrm{i}\epsilon T)\,\mathscr{V}\Psi\,\mathrm{d}T = 2\mu\mathscr{E}\psi, \tag{55}$$

where

$$\mathscr{H}\Psi = \mathscr{E}\Psi.$$

Using pseudospherical coordinates, one finds

$$\psi = (2\pi)^{-\frac{1}{2}}\int_{-R}^{R} \mathrm{d}(R\tanh\gamma)\;\Psi\exp(\mathrm{i}\epsilon R\tanh\gamma)$$

$$= R(2\pi)^{-\frac{1}{2}}\int_{-1}^{1} \mathrm{d}t\;\Psi\exp(\mathrm{i}\epsilon Rt) \tag{56}$$

and an interaction term

$$\rho = 2\mu R(2\pi)^{-\frac{1}{2}}\int_{-1}^{1} \mathrm{d}t\;\mathscr{V}(\mathbf{R},\mathbf{Q})\,\Psi(\mathbf{R},\mathbf{Q}). \tag{57}$$

With a small variation in the relative time coordinate at a fixed point $\mathbf{R}$, the interaction behaves as

$$\rho = \frac{2\mu R}{(2\pi)^{\frac{1}{2}}}\sum_{P=0}^{\infty}\frac{1}{P!}\left(\frac{\partial^P\mathscr{V}}{\partial t^P}\right)_{t=t'}\int_{-1}^{1}(t'-t)^P\,\Psi\exp(\mathrm{i}\epsilon Rt)\,\mathrm{d}t. \tag{58}$$

If the interaction decreases with increasing R, and vanishes as $R \to \infty$, then provided

$$\frac{\partial}{\partial t}\left(\ln\mathscr{V}\right)_{R\to\infty} \to 0, \qquad |t'| \ll 1, \tag{59}$$

one will have for large R

$$\rho \approx 2\mu R(2\pi)^{-\frac{1}{2}}\mathscr{V}(R,t')\int_{-1}^{1}\Psi\exp(\mathrm{i}\epsilon Rt)\,\mathrm{d}t$$

$$\approx 2\mu\,\mathscr{V}(R,0)\,\psi. \tag{60}$$

The wave equation (55) then becomes the non-relativistic Schrödinger equation (Schiff 1949)

$$\nabla^2\psi + 2\mu(E-V)\psi = 0 \tag{61}$$

for small values of q, where

$$\mathscr{E} = Q^2/2\mu \to E, \qquad H\psi = E\psi \sim (\mathscr{H} + \epsilon^2/2\mu)\psi. \tag{62}$$

Therefore, for large R, or slowly varying potentials, the wavefunction ψ becomes that applicable at low velocities, where q is small and ϵ is considered to be zero. This would be the case for any weak interaction, implying that particle velocities remain small relative to the velocity of light. The invalidity of truncating the series (58) near the null cone where $t' \sim 1$ indicates that measurements of relative time in this region affect the behaviour of the system violently and enhance the higher order terms of the relativistic interaction.

In the absence of any interaction, one would expect the covariant wavefunction (35), which represents two free particles, to transform directly to a plane wave at any velocity of the centre of mass, or any physical relative velocity. This is in fact the case:

$$\begin{aligned}\psi &= \frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-R}^{R} \mathrm{d}T \exp(\mathrm{i}\epsilon T)\frac{\exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{S})}{\{Q^2S^2-(\mathbf{Q}\,.\,\mathbf{S})^2\}^{\frac{1}{2}}}\,\theta(S^2)\,\theta(1-Z^2)\\ &= \frac{\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{R})}{(2\pi)^{\frac{1}{2}}}\int_{-R}^{R}\mathrm{d}T\,\{Q^2(R^2-c^2T^2)-(\boldsymbol{q}\,.\,\boldsymbol{R}-\epsilon T)^2\}^{-\frac{1}{2}}\\ &= \frac{\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{R})}{(2\pi)^{\frac{1}{2}}}\left[-\frac{1}{d^{\frac{1}{2}}}\left\{\arcsin\left(\frac{2dT+b}{(-\Delta)^{\frac{1}{2}}}\right)\right\}\right]_{-R}^{R},\end{aligned} \tag{63}$$

where

$$\Delta = 4ad-b^2, \qquad d = Q^2+\epsilon^2 = q^2,$$

$$b = 2\boldsymbol{q}\,.\,\boldsymbol{R}, \qquad a = Q^2R^2-(\boldsymbol{q}\,.\,\boldsymbol{R})^2.$$

The integrand simplifies to give

$$\psi = \frac{\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{R})}{(2\pi)^{\frac{1}{2}}q}\left[-\left\{\arcsin\left(\frac{t_2 z-t_1}{(1-z^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}}}\right)\right\}\right]_{t_1=-1}^{t_1=1},$$

where $z = (\boldsymbol{q}\,.\,\boldsymbol{R})/qR$, $t_1 = \tanh\gamma$, and $t_2 = \tanh\delta$. Using

$$z = t_1 t_2+(1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}}Z,$$

one obtains the value π for the integral, giving

$$\psi = (\tfrac{1}{2}\pi)^{\frac{1}{2}}\,q^{-1}\exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{R}). \tag{64}$$

All of formal non-relativistic scattering theory is based upon the free particle plane wave function (64). Therefore, allowing for relativistic kinematic factors, the non-relativistic expressions for cross sections, the S-matrix, partial wave expansions, and any formalism independent of the explicit form of the interaction, including reaction matrix theory (Wigner and Eisenbud 1947; Lane and Thomas 1958; Preston 1962), potential theory (Regge 1959), and Regge pole theory, are valid to arbitrarily high energies. These theories become covariant within the relative time formalism, provided no measurement is made to test Lorentz invariance. A test of Lorentz invariance is necessarily an experiment with pseudospherical symmetry in 4-space, and the additional λ-degeneracy becomes observable.

It is very instructive to show how the wave fronts propagating in 4-space combine to give the plane wave (64), and in doing so remove the n-degeneracy. Using the eigenfunction expansion (35) and the addition theorem (32), we find

$$\psi = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \mathrm{d}T \exp(\mathrm{i}\epsilon T)\, \Psi(\mathbf{R})\, \theta(S^2)$$

$$= \frac{R}{(2\pi)^{\frac{1}{2}}} \int_{-1}^{1} \mathrm{d}t_1 \exp(\mathrm{i}qRt_1 t_2) \times \tfrac{1}{2}\pi \sum_{n=0}^{\infty} (\mathrm{i})^n h_n \frac{J_n(qR(1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}})}{qR(1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}}}$$

$$\times \sum_{l=n}^{\infty} b_{ln} P_l^n(t_1)(1-t_1^2)^{\frac{1}{2}} P_l^n(t_2)\,(1-t_2^2)^{\frac{1}{2}} P_l(z)\,. \tag{65}$$

Now there exists a standard Fourier transform (Erdelyi *et al.* 1953)

$$(2\pi/Y)^{\frac{1}{2}}(\mathrm{i})^p(\sin\phi)^{\nu-\frac{1}{2}}\, C_p^\nu(\cos\phi)\, J_{\nu+p}(Y)$$

$$= \int_0^\pi \exp(\mathrm{i}Y\cos\theta\cos\phi)\, J_{\nu-\frac{1}{2}}(Y\sin\theta\sin\phi)\, C_p^\nu(\cos\theta)\,(\sin\theta)^{\nu+\frac{1}{2}}\,\mathrm{d}\theta\,, \tag{66}$$

where C_p^ν is the Gegenbauer function, which is related to the spherical harmonics by (Erdelyi *et al.* 1953)

$$\Gamma(2\nu)\,\Gamma(p+1)\, C_p^\nu(z) = 2^{\nu-\frac{1}{2}}\,\Gamma(p+2\nu)\,\Gamma(\nu+\tfrac{1}{2})\,(z^2-1)^{\frac{1}{4}-\frac{1}{2}\nu}\, P^{\frac{1}{2}-\nu}_{p+\nu-\frac{1}{2}}(z)\,. \tag{67}$$

Substituting $\nu = n+\frac{1}{2}$, $\cos\theta = t_1$, $Y = qR$, $p+\nu = l+\frac{1}{2}$, and $\cos\phi = t_2$, we obtain

$$(2\pi/qR)^{\frac{1}{2}}(\mathrm{i})^{l-n} P_l^n(t_2)\, J_{l+\frac{1}{2}}(qR)$$

$$= \int_{-1}^{1} \mathrm{d}t_1 \exp(\mathrm{i}qRt_1 t_2)\, J_n\big(qR(1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}} P_l^n(t_1)\big)\,,$$

which, when applied to (65), yields

$$\psi = \tfrac{1}{2}\pi \sum_{n=0}^{\infty} (\mathrm{i})^n h_n \sum_{l=n}^{\infty} b_{ln} \frac{(\mathrm{i})^{l-n}}{q} \frac{J_{l+\frac{1}{2}}(qR)}{(qR)^{\frac{1}{2}}} \{P_l^n(t_2)\}^2 P_l(z)$$

$$= q^{-1} \sum_{l=0}^{\infty} (\mathrm{i})^l \frac{J_{l+\frac{1}{2}}(qR)}{(qR)^{\frac{1}{2}}} P_l(z) \sum_{n=0}^{l} \tfrac{1}{2}\pi h_n b_{ln} \{P_l^n(t_2)\}^2\,. \tag{68}$$

Now $b_{ln} = (2l+1)(l-n)!/(l+n)!$, and the second sum in (68) is such that it equals $\frac{1}{2}\pi(2l+1)$ from the addition theorem, so that

$$\psi = (\pi/2q) \sum_{l=0}^{\infty} (\mathrm{i})^l\,(2l+1)\,(qR)^{-\frac{1}{2}} J_{l+\frac{1}{2}}(qR)\, P_l(z)$$

$$= (\tfrac{1}{2}\pi)^{\frac{1}{2}}\, q^{-1} \exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{R})\,,$$

in agreement with the direct result (64).

When the interaction $\mathscr{V}(S)$ is present, the converted wavefunction is

$$\psi = \frac{R}{(2\pi)^{\frac{1}{2}}} \int_{-1}^{1} \mathrm{d}t_1 \frac{\exp(\mathrm{i}qRt_1 t_2)}{qR} \sum_{n=0}^{\infty} a_n G_n\big(qR(1-t_1^2)^{\frac{1}{2}}(1-t_2^2)^{\frac{1}{2}}\big)$$

$$\times \sum_{l=n}^{\infty} b_{nl} P_l^n(t_1) P_l^n(t_2) P_l(z)$$

$$= \frac{1}{(2\pi)^{\frac{1}{2}} q} \sum_{l=0}^{\infty} (2l+1) g_l(q, R, t_2) P_l(z) , \tag{69}$$

where

$$g_l = \sum_{n=0}^{l} a_n \{(l-n)!/(l+n)!\} \bar{G}_n(q, R, t_2) P_l^n(t_2)$$

is the 3-space radial wavefunction, with

$$\bar{G}_n = \int_{-R}^{R} \mathrm{d}T \exp(\mathrm{i}\epsilon T) G_n(QS) P_l^n(cT/R) , \tag{70}$$

and $G_n(QS)/QS$ is the solution to the 4-space radial wave equation. Bertram (personal communication) has proved the important result

$$2^{-\frac{1}{2}} \pi^{\frac{1}{2}} a^{\frac{1}{2}-\nu} \int_{-1}^{1} (1-Y^2)^{\frac{1}{2}\nu-\frac{1}{4}} C_n^\nu(Y) N_{\nu-\frac{1}{2}}\big(a(1-Y^2)^{\frac{1}{2}}\big) \exp(\mathrm{i}xY) \,\mathrm{d}Y$$

$$= (\mathrm{i})^n (x^2+a^2)^{\frac{1}{2}n} C_n^\nu\big(x(x^2+a^2)^{-\frac{1}{2}}\big) \big\{(x^2+a^2)^{-\frac{1}{2}(\nu+n)} N_{\nu+n}\big((a^2+x^2)^{\frac{1}{2}}\big)\big\} , \tag{71}$$

where $N_\nu(x)$ is the associated Bessel function. Using this result, we find

$$\frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{\infty} \mathrm{d}T \exp(\mathrm{i}\epsilon T) \sum_{n=0}^{\infty} (\mathrm{i})^n h_n \frac{B_n(QS)}{QS} \frac{\cos(n \arccos Z)}{(1-Z^2)^{\frac{1}{2}}}$$

$$= \frac{2}{\pi}\frac{1}{q} 2\pi \sum_{l=0}^{\infty} (\mathrm{i})^l (l+\tfrac{1}{2}) (qR)^{-\frac{1}{2}} B_{l+\frac{1}{2}}(qR) P_l(z) , \tag{72}$$

where B_ν is any of the Bessel functions J_ν, N_ν, $H_\nu^{(1)}$, or $H_\nu^{(2)}$.

The ordinary representation of the delta function (Goertzel and Tralli 1960) is not useful in this scattering theory. Instead we use the distorted form (40) to obtain

$$\int_0^\infty Q^3 \,\mathrm{d}Q \int_{-1}^{1} \mathrm{d}Z_1 \,(1-Z_1^2)^{\frac{1}{2}} \int_{-1}^{1} \mathrm{d}z_1 \int_0^{2\pi} \mathrm{d}\Phi_1 \frac{\exp\{\mathrm{i}\mathbf{Q}\,.\,(\mathbf{S}-\mathbf{S}')\}}{Q^2 SS' \sin\omega \sin\omega'} \frac{1}{1-Z_1^2}$$

$$= 4\pi^2 \sum_{n=0}^{\infty} h_n \frac{\cos\{n(\omega-\omega')\}}{\sin\omega \sin\omega'} \times \int_0^\infty \frac{J_n(QS) J_n(QS') Q \,\mathrm{d}Q}{SS'}$$

$$= \frac{4\pi\, \delta(S-S')\, \delta(Z-Z')}{S^3(1-Z^2)^{\frac{1}{2}}} = \frac{4\pi\, \delta(S-S')\, \delta(\omega-\omega')}{S^3 \sin^2\omega} , \tag{73}$$

with

$$\mathbf{Q}\,.\,\mathbf{S}/QS = \cos(\omega-\omega_1) , \qquad \mathbf{Q}\,.\,\mathbf{S}'/QS' = \cos(\omega'-\omega_1) ,$$

where the rotation theorem

$$\int_0^{\pi} \mathrm{d}\omega_1 \cos\{n(\omega_1-\omega)\}\cos\{n'(\omega_1-\omega')\} = (\pi/h_n)\delta_{nn'}\cos\{n(\omega-\omega')\} \tag{74}$$

has been used. In these equations

$$\mathbf{Q}\,.\,\mathbf{S}_1/QS_1 = Z_1 = \cos\omega_1\,, \qquad Z = \cos\omega\,, \qquad Z' = \cos\omega'\,,$$

and $\mathbf{S}_1$ is a reference 4-vector.

Following standard derivations (Goertzel and Tralli 1960) the equation for the scattering state is

$$\Psi(\mathbf{S},\mathbf{Q}) = \frac{\exp(\mathrm{i}\mathbf{Q}\,.\,\mathbf{S})}{QS(1-Z^2)^{\frac{1}{2}}} + \iiiint S'^3\,\mathrm{d}S'\,\mathrm{d}\omega'\,\mathrm{d}z'\,\mathrm{d}\Phi' \times G(\mathbf{S},\mathbf{S}')\,\Psi(\mathbf{S}',\mathbf{Q}) \tag{75}$$

and G is the Green's function

$$\begin{aligned} G(\mathbf{S},\mathbf{S}') &= \frac{1}{4\pi}\int_0^{\infty} Q'^3\,\mathrm{d}Q' \int_{-1}^{1}\frac{\mathrm{d}Z_1}{(1-Z_1^2)^{\frac{1}{2}}}\int_{-1}^{1}\mathrm{d}z_1\int_0^{2\pi}\mathrm{d}\Phi_1 \\ &\qquad \times \frac{\exp\{\mathrm{i}\mathbf{Q}\,.\,(\mathbf{S}-\mathbf{S}')\}}{Q'^2SS'\sin\omega\sin\omega'}\left(\frac{1}{Q^2-Q'^2}\right) \\ &= \tfrac{1}{4}\sum_{n=0}^{\infty} g_n(S,S')\,h_n\frac{\cos\{n(\omega-\omega')\}}{\sin\omega\sin\omega'} \\ &= (1/4\pi)H_0^{(1)}(Q\,|\,\mathbf{S}-\mathbf{S}'\,|)/SS'\sin\omega\sin\omega'\,, \end{aligned} \tag{76}$$

where

$$\begin{aligned} g_n(S,S') &= \int_0^{\infty}\frac{J_n(Q'S)\,J_n(Q'S')\,Q'\,\mathrm{d}Q'}{SS'(Q'^2-Q^2)} \\ &= \mathrm{Re}\{H_n^{(1)}(QS')\,J_n(QS)/SS'\}\,, \qquad S<S' \\ &= \mathrm{Re}\{H_n^{(1)}(QS)\,J_n(QS')/SS'\}\,, \qquad S>S'\,. \end{aligned}$$

Taking the Fourier transform of (75) and using (64), we find the 3-space wavefunction

$$\psi(\boldsymbol{q},\boldsymbol{R}) = \exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{R}) - \frac{1}{4\pi}\iiiint \mathrm{d}S'\,\frac{\exp(\mathrm{i}q\,|\,\boldsymbol{R}-\boldsymbol{R}'\,|)}{|\,\boldsymbol{R}-\boldsymbol{R}'\,|}\mathscr{V}(\mathbf{S}')\,\Psi(\mathbf{S}',\mathbf{Q})\,. \tag{77}$$

From (77) one finds the conventional scattering amplitude as in Schiff (1949) of

$$f(\theta,\phi) = \frac{1}{4\pi}\iiiint \mathrm{d}\mathbf{S}'\,\exp(\mathrm{i}\boldsymbol{q}'\,.\,\boldsymbol{R}')\,\mathscr{V}(\mathbf{S}')\,\Psi(\mathbf{S}',\mathbf{Q})\,, \tag{78}$$

where $\boldsymbol{q}' = \boldsymbol{q}-\boldsymbol{q}_0$, $\boldsymbol{q}_0$ is the vector representing the initial beam momentum, and $\boldsymbol{q}$ is the final beam momentum in the direction (θ,ϕ). Models for scattering and perturbation expansions can be evaluated from equation (78).

VI. Bound States

There are some specific points concerning the eigenvalues of the covariant angular momentum tensor which require elucidation. The most important is the question of its role in the energy eigenvalues for discrete levels in bound states. Five models are considered here to make this role apparent.

(a) *Coulomb Two-boson Atom*

The Hamiltonian for this problem as in equation (3) can be written in the notation of Part I as

$$\mathscr{H}\Psi = \mathscr{E}\Psi = \left(\frac{(\mathbf{Q}_1-\mathbf{A}_1)^2}{2M}+\frac{(\mathbf{Q}_2-\mathbf{A}_2)^2}{2\mu}\right)\Psi. \tag{79}$$

We now assume, as in Part I, that the electromagnetic potentials are given by

$$\mathbf{A}_1 = (e^2/c)\mathbf{G}_1/S, \tag{80a}$$

where $\mathbf{G}_1$ is an operator which is assumed to obey the eigenvalue equation

$$\mathbf{G}_1\Psi = \mathbf{U}_{\mathrm{cm}}G_1\Psi, \qquad [\mathbf{G}_1,\mathscr{H}] \equiv \mathbf{G}_1\mathscr{H}-\mathscr{H}\mathbf{G}_1 = 0, \tag{80b}$$

the square brackets denoting the commutation relationship, with G_1 a scalar and $\mathbf{U}_{\mathrm{cm}}$ the 4-velocity of the centre of mass; and similarly

$$\mathbf{A}_2 = (e^2/c)\mathbf{G}_2/S, \qquad \mathbf{G}_2\Psi = \mathbf{U}_{\mathrm{cm}}G_2\Psi. \tag{80c}$$

When these potentials are substituted into (79) we find

$$\begin{aligned}\mathscr{H} &= \frac{\mathbf{Q}_1^2}{2M}+\frac{1}{2M}\Big(\mathbf{A}_1\,.\,\mathbf{Q}_1+\mathbf{Q}_1\,.\,\mathbf{A}_1\Big)-\frac{\mathbf{A}_1^2}{2M}+\frac{\mathbf{Q}_2^2}{2\mu}-\frac{1}{2\mu}\Big(\mathbf{A}_2\,.\,\mathbf{Q}_2+\mathbf{Q}_2\,.\,\mathbf{A}_2\Big)+\frac{\mathbf{A}_2^2}{2\mu}\\ &= \frac{\mathbf{Q}_1^2}{2M}+\frac{\mathbf{Q}_2^2}{2\mu}+f_1\frac{e^2}{c}\frac{1}{S}+f_2\frac{e^4}{c^2}\frac{1}{S^2}\end{aligned} \tag{81}$$

to order e^6 at least. Omitting the suffix cm from the 4-velocity, the following commutation relations must be valid for the above equality to hold:

$$[\mathbf{Q}_1,\mathbf{U}] = 0, \qquad [\mathbf{Q}_2,\mathbf{U}] = 0, \qquad [\mathbf{Q}_1,S^{-1}\mathbf{G}_1] = 0, \qquad [\mathbf{Q}_2,S^{-1}\mathbf{G}_2] = 0,$$

$$[\mathbf{Q}_2,\mathbf{G}_2] = -iS^{-1}\mathbf{e}_S G_2, \quad \mathbf{e}_S = \mathbf{S}/S\,; \qquad [\mathbf{G}_2,S^{-1}] = 0\,;$$

and therefore

$$[\mathbf{Q}_1,\mathbf{A}_1] = 0, \qquad [\mathbf{Q}_2,\mathbf{A}_2] = 0.$$

These conditions are really only necessary to simplify equation (79).

We define two operators (F_1,F_2) with constant eigenvalues (f_1,f_2),

$$F_1\Psi = f_1\Psi, \qquad F_2\Psi = f_2\Psi,$$

such that

$$(Q_1/M)G_1 - \mathbf{V}\,.\,\mathbf{U}\,G_2 = F_1,$$

with $\mathbf{V} = \mathbf{Q}_2/\mu$, and

$$(\mu/M)G_1^2 + G_2^2 = F_2.$$

With these operators, upon separating variables, equation (81) yields the radial wave equation

$$\frac{\mathrm{d}^2\Psi_S}{\mathrm{d}S^2} + \frac{3}{S}\frac{\mathrm{d}\Psi_S}{\mathrm{d}S} - \frac{\{\lambda(\lambda+2) - f_2\,\alpha^2\}}{S^2}\,\Psi_S = \left(-|E| + \frac{2f_1\,\alpha\mu}{S}\right)\Psi_S, \tag{82}$$

where $E = 2\mu(\mathscr{H} - Q_1^2/2M) = \mu(W^2 - M^2)/M$.

Putting $\beta^2 = 4E$ and $\rho = \beta S$ we obtain the equation

$$\frac{\mathrm{d}^2\Psi_S}{\mathrm{d}\rho^2} + \frac{3}{\rho}\frac{\mathrm{d}\Psi_S}{\mathrm{d}\rho} + \left(-\frac{\lambda'(\lambda'+2)}{\rho^2} - \frac{1}{4} + \frac{K}{\rho}\right)\Psi_S = 0, \tag{83}$$

in which

$$K = 2f_1\,\alpha\beta\mu = f_1(e^2/c)\mu(-E)^{-\frac{1}{2}}$$

and

$$\lambda'(\lambda'+2) = n'^2 - 1 = \lambda(\lambda+2) - f_2\,\alpha^2 = n^2 - 1 - f_2\,\alpha^2.$$

Substituting $\Psi_S = W\rho^{-3/2}$ into (83) we obtain

$$W'' + W(-\tfrac{1}{4} + K\rho^{-1} + (\tfrac{1}{4} - n'^2)\rho^{-2}) = 0. \tag{84}$$

This is Whittaker's equation (Erdelyi *et al.* 1953). The solution which tends to zero as ρ tends to infinity is

$$\begin{aligned} W_{K,n'} &= \exp(-\tfrac{1}{2}\rho)\,\rho^K\,{}_2\mathrm{F}_0(\tfrac{1}{2} - K + n', \tfrac{1}{2} - K - n'; -\rho^{-1}) \\ &= \exp(-\tfrac{1}{2}\rho)\,\rho^{n'+\frac{1}{2}}\,\Psi(\tfrac{1}{2} - K + n', 2n'+1; \rho), \end{aligned} \tag{85}$$

where $\Psi(a, c; x)$ is the confluent hypergeometric function. It behaves asymptotically for large ρ such that

$$\begin{aligned} \Psi_S &\sim \exp(-\tfrac{1}{2}\rho)\,\rho^{-n'-1}\left(\sum_{m=0}^{\infty}(-1)^m\frac{\Gamma(n'+\frac{1}{2}+m-K)}{\Gamma(n'+\frac{1}{2}-K)}\frac{1}{m!}\frac{\Gamma(\frac{1}{2}-K+m-n')}{\Gamma(\frac{1}{2}-K-n')}\rho^{K-n'-\frac{1}{2}-m}\right) \\ &\sim \text{const.}\exp(-\tfrac{1}{2}\rho)\,\rho^{K-3/2}. \end{aligned} \tag{86}$$

Near the origin, no nonsingular solution exists unless $(\frac{1}{2} - K + n')$ is a negative integer. We have there

$$\Psi_S \sim \{\Gamma(2n')/\Gamma(\tfrac{1}{2} - K + n')\}\rho^{-n'-\frac{1}{2}}.$$

Two discernible cases arise:

(i) $N = -(\frac{1}{2} - K + n')$ is an integer. This solution becomes the same as in the non-relativistic case, namely the Laguerre polynomials times factors. This is because

$$L_N^{2n'}(\rho) = \{(-1)^N/N!\}\Psi(-N, 2n'+1; \rho)$$

and these functions are nonsingular at the origin. However, we would then have

$$K = \tfrac{1}{2}+N+n'.$$

If the integer solutions for n are chosen, K is approximately half-integer, in which case the energy levels are

$$E = -f_1\,\alpha^2/K^2$$

and do not tend to the correct non-relativistic limit of the Bohr levels. There are two possible answers to this dilemma. Firstly, we could choose the solutions where n is half-integer. This would imply that the scattering theory developed in Sections III and IV was not applicable to such a pair of particles. An equivalent scattering theory in which n is half-integer is possible to derive.

(ii) The second solution is to note from the relation

$$K_{n'}(\tfrac{1}{2}x) = \pi^{\frac{1}{2}}\exp(\tfrac{1}{2}x)\,x^{n'}\,\Psi(n'+\tfrac{1}{2}, 2n'+1\,;\,x) \tag{87}$$

that as $\alpha \to 0$ the solution with integer N cannot give the free particle eigenfunctions $K_n(\frac{1}{2}\rho)$ at negative energies. We can therefore choose N to be half-integer to preserve this relationship. In this case the singularity at the origin does not allow one to normalize the solution over the physical volume element. Such two-boson atoms cannot therefore admit point source potentials which yield the Bohr levels and are nonsingular at the origin. If we assume the two bosons to be extended sources, it is possible to introduce a surface cutoff at very small values of S. Using Green's theorem, we find

$$[W_1\,W_2' - W_2\,W_1'] = (K_1-K_2)\int_{\rho_0}^{\infty} W_1\,W_2\,\rho^{-1}\,\mathrm{d}\rho\,. \tag{88}$$

Making use of the radial wave equation (82) we see that a cutoff at ρ_0 allows the integral to be written

$$\operatorname*{Lim}_{\rho\to\rho_0}[W_1\,W_2' - W_2\,W_1'] = (K_1-K_2)\int_{S_0}^{\infty} W_1\,W_2\,S^{-1}\,\mathrm{d}S\,. \tag{89}$$

This integral will vanish if

$$\rho_0\,W_i'(\rho_0)/W_i(\rho_0) = B_n\,, \qquad \partial B_n/\partial K = 0\,.$$

The higher terms in the expansion near the origin are found from the relation (85) and the equations (Erdelyi *et al.* 1953)

$$\Psi(a,c;x) = \frac{\Gamma(1-c)}{\Gamma(a-c+1)}\,\Phi(a,c;x) + x^{1-c}\frac{\Gamma(c-1)}{\Gamma(a)}\,\Phi(a-c+1, 2-c; x)\,, \tag{90}$$

where

$$\Phi(a,c;x) = 1+\frac{a}{c}x+\frac{a(a+1)}{c(c+1)}\frac{x^2}{2!}+\dots$$

is the confluent hypergeometric function. For small ρ_0 we obtain

$$K\rho_0 = \frac{(1-\frac{1}{2}c-B_n)(2-c)}{B_n-2+\frac{1}{2}c} = \frac{(2n'+1)(B_n+n'-\frac{1}{2})}{B_n+n'-\frac{3}{2}}. \tag{91}$$

Right at the origin, we would have $\rho_0 = 0$ and $B_n = -n'+\frac{1}{2}$. However, S_0 and B_n both have to be independent of K and this can only occur if

$$\rho_0 = 2(-E)^{\frac{1}{2}} S_0 \propto K^{-1}, \tag{92a}$$

which is satisfied if the energy levels appropriate to the problem are given by

$$(-E)^{\frac{1}{2}} = (\alpha/K)\mu, \qquad f_1 = \mu. \tag{92b}$$

The surface cutoff

$$S_0 = \frac{1}{2\alpha}\frac{(2n'-1)(B_n+n'-\frac{1}{2})}{B_n+n'-\frac{3}{2}} \tag{93}$$

can be chosen as small as desired, provided the actual limit is never taken to the origin. The normalizations are found from the relation

$$\int_{S_0}^{\infty} W_K W_{K'} S^{-1}\,\mathrm{d}S = (2n'^2-3n'+2)(\rho_0)^{1-2n'}\delta_{KK'} \tag{94}$$

and hence

$$\Psi_{S,K,n} = S^{-3/2}(2n^2-3n+2)^{-\frac{1}{2}}(\rho_0)^{n-\frac{1}{2}} W_{K,n'}(\rho). \tag{95}$$

The energy levels are found to be

$$E = -\frac{\alpha^2\mu^2}{K^2} = -\mu^2\left(\frac{\alpha^2}{(N+n)^2}+\frac{\alpha^4 f_2^2}{(N+n)^3 n}+O(\alpha^6)\right). \tag{96}$$

When one mass becomes very large, we find for $m_2 \gg m_1$

$$E_1 = m_1\left(1-\frac{\alpha^2}{2(N+n)^2}-\frac{\alpha^4 f_2^2}{2(N+n)^3 n}+O(\alpha^6)\right). \tag{97}$$

Two points of importance arise:

(1) The term proportional to $\frac{3}{8}\alpha^4$ is missing.

(2) The quantum number n has replaced the $(l+\frac{1}{2})$ in the normal case of one light boson in a central field.

(b) Linear Harmonic Oscillator

The force

$$\mathbf{F} = -\sum_{\nu} K_\nu X_\nu \tag{98}$$

can be represented by the potential

$$\mathscr{V} = \sum_{\nu} \tfrac{1}{2} K_\nu X_\nu^2 \tag{99}$$

with a wave equation

$$\left(-\Box^2/2\mu+\tfrac{1}{2}\sum_{\nu} K_\nu X_\nu^2\right)\Psi = \mathscr{E}\Psi, \tag{100}$$

which separates to give four equations

$$(\partial^2/\partial X_\nu^2+\tfrac{1}{2}K_\nu X_\nu^2)\Psi_\nu = \mathscr{E}_\nu \Psi_\nu \tag{101}$$

with

$$\mathscr{E} = \sum_{\nu} \mathscr{E}_\nu, \qquad \Psi = \prod_{\nu} \Psi_\nu .$$

The K_ν must transform as tensors. Putting

$$\xi_\nu = \alpha_\nu X_\nu, \qquad \alpha_\nu^4 = \tfrac{1}{2}\mu K_\nu, \qquad \sigma_\nu = \frac{2\mathscr{E}_\nu}{\hbar}\left(\frac{\mu}{K_\nu}\right)^{\frac{1}{2}} = \frac{2\mathscr{E}_\nu}{\hbar\omega_\nu},$$

we obtain

$$\mathrm{d}^2\Psi_\nu/\mathrm{d}\xi_\nu^2+(\sigma_\nu-\xi_\nu^2)\Psi_\nu = 0. \tag{102}$$

The standard solutions are the Hermite polynomials such that

$$\Psi_\nu(X_\nu) = N_{n\nu}H_n(\alpha_\nu X_\nu)\exp(-\tfrac{1}{2}\alpha_\nu^2 X_\nu^2), \tag{103}$$

with

$$\sigma_\nu = 2n+1, \qquad \mathscr{E}_{n\nu} = (n+\tfrac{1}{2})\hbar\omega_\nu, \qquad n = 0,1,2,\ldots,$$

$$N_{n\nu} = (\alpha_\nu/\pi\, 2^n\, n!)^{\frac{1}{2}}.$$

There are therefore zero-point energies corresponding to all modes of vibration along the four axes. The time-like vibrations have not been observed and it is not known how they would manifest themselves. These functions form an orthonormal set over the physical volume element.

(c) *Inverse Cube Law of Force*

When Goldstein (1953) formulated the Bethe–Salpeter (1951) wavefunctions for two spinors interacting via the ladder exchange of neutral bosons, he obtained a radial wave equation from the quantum field theory of the form

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}R^2}+\frac{3}{R}\frac{\mathrm{d}}{\mathrm{d}R}+1+\frac{4\eta}{R^2}\right)\Psi_S = 0 \tag{104}$$

in the case of equal masses. This is a special example of the radial wave equation (4a) for a hypercentral inverse cube law of 4-force. Putting $\mathscr{V} = -K/S^2$ in equation (4a) we find

$$\left(\frac{\mathrm{d}^2}{\mathrm{d}S^2}+\frac{3}{S}\frac{\mathrm{d}}{\mathrm{d}S}+Q^2-\frac{\Lambda^2+K}{S^2}\right)\Psi_S = 0, \tag{105}$$

which is identical with Goldstein's (1953) equation provided $R = QS$ and $4\eta = \Lambda^2+K = n^2+K-1$. In the non-relativistic limit, this force gives the solutions appropriate to the r^{-2} potential. It is therefore not surprising that he found no bound state, as no non-relativistic bound states exist (Morse and Feshbach 1953). Goldstein's solutions

$$\Psi_S = R^{-1} H_{n'}^{(2)}(R), \qquad n' = (1-4\eta)^{\frac{1}{2}}, \tag{106}$$

are obviously those for the inverse cube law of 4-force in the relative time theory. No bound states exist here as well for $0 \leqslant S \leqslant \infty$.

(d) Square Well Potential

With an interaction in equation (4a) of the form

$$\left.\begin{aligned} \mathscr{V} &= -\mathscr{V}_0, & S &< a, \\ &= 0, & S &> a, \end{aligned}\right\} \tag{107}$$

where $\mathscr{V}_0$ and a are constants, we have the covariant analogue of the square well potential. The solutions are

$$\left.\begin{aligned} \Psi_S &= A J_n(\alpha S)/\alpha S, & S &< a, \\ &= B H_n^{(1)}(\beta S)/\beta S, & S &> a, \end{aligned}\right\} \tag{108}$$

where

$$\alpha = \{2\mu(\mathscr{V}_0 - \mathscr{E})\}^{\frac{1}{2}}, \qquad \beta = (2\mu\mathscr{E})^{\frac{1}{2}}.$$

In the bound state region, $\mathscr{E}$ is less than zero and hence

$$\Psi_S = \mathrm{i}\beta H_n^{(1)}(\mathrm{i}\,|\,\beta\,|\,S)/|\,\beta\,|\,S = (\mathrm{i}\beta/|\,\beta\,|\,S) K_n(|\,\beta\,|\,S) \tag{109}$$

is the solution. These functions tend to zero as S tends to infinity. By choosing boundary conditions on each solution such that

$$a\left(\frac{\partial \Psi_S/\partial S}{\Psi_S}\right)_{S=a} = -(n+1), \tag{110}$$

one also ensures that the internal eigenfunctions form an orthonormal set in the interval $0 \leqslant S \leqslant a$. The energies of the bound states are obtained by noting that the boundary condition (110) is equivalent to the conditions

$$\rho J_n'(\rho) + n J_n(\rho) = \rho J_{n-1}(\rho) = 0, \tag{111}$$

when $\rho = \rho_\nu$ are zeros of $J_{n-1}(\rho)$. Hence the energy levels are

$$\mathscr{E}_\nu = \rho_\nu/a. \tag{112}$$

(e) Two-fermion Atom

There are many possible models of the two-fermion atom and the one derived here is chosen mainly for its relative simplicity. The main problem is in finding how to linearize equation (2). We choose the form

$$(\mathscr{E} - \mathscr{V} + \rho_1 \mathscr{M} + \rho_3 \boldsymbol{\gamma}\cdot\mathbf{Q})\Psi_8 = 0, \tag{113}$$

where Ψ_8 is an eight-component spinor, four components appropriate to one ordinary spinor and four to the other, with

$$\mathscr{E} = \frac{\beta}{(1-\beta^2)^{\frac{1}{2}}}\left(\frac{\mu(W^2-M^2)}{M}\right)^{\frac{1}{2}}, \qquad \mathscr{M} = \frac{1}{(1-\beta^2)^{\frac{1}{2}}}\left(\frac{\mu(W^2-M^2)}{M}\right)^{\frac{1}{2}}, \tag{114a}$$

$$\mathbf{Q} = \mathbf{Q}_2, \qquad \beta = (1+|V|^2/c^2)^{-\frac{1}{2}} = 1+(W^2-M^2)/2M\mu, \tag{114b}$$

V being the relative velocity. The Hamiltonian for the relative motion is

$$\begin{aligned}\mathscr{H} &= \rho_3\gamma_\mu Q^\mu - \rho_1\mathscr{M} + \mathscr{V} \\ &= \rho_3\gamma_S Q_S - \rho_3\,\mathrm{i}(\gamma_S/S)\rho_3 K + \mathscr{V} - \rho_1\mathscr{M},\end{aligned} \tag{115}$$

in which

$$K = \rho_2(\tfrac{1}{2}\sigma_{\mu\nu}\Lambda^{\mu\nu}+\tfrac{3}{2}), \qquad \gamma_S = \gamma_\nu X^\nu/S, \tag{116}$$

where we have (Corinaldesi and Strocchi 1963)

$$\begin{aligned}\gamma_\mu\gamma_\nu + \gamma_\nu\gamma_\mu &= 2\delta_{\mu\nu}, \\ \sigma_{\mu\nu} &= (\gamma_\mu\gamma_\nu - \gamma_\nu\gamma_\mu)/2\mathrm{i},\end{aligned} \tag{117}$$

and the γ_μ's are the usual Dirac matrices. From (117) it can be shown that

$$\gamma_\mu Q^\mu = (\gamma_\nu X^\nu/S^2)\mathbf{S}\cdot\mathbf{Q} + (\mathrm{i}/2S^2)\gamma_\nu X^\nu\sigma_{\mu\nu}\Lambda^{\mu\nu}, \tag{118}$$

and using

$$Q_S = (\mathbf{S}\cdot\mathbf{Q} - \tfrac{3}{2}\mathrm{i})/S = \mathrm{i}(\partial/\partial S + 3/2S) \tag{119}$$

it can be readily proved that the following relations hold

$$\gamma_S^2 = \rho_1^2 = \rho_2^2 = \rho_3^2 = 1, \qquad \tfrac{1}{2}\sigma_{\mu\nu}\sigma^{\mu\nu} = 12, \tag{120a}$$

$$\rho_1\rho_2\gamma_S + \rho_2\gamma_S\rho_1 = 0, \qquad K\gamma_S - \gamma_S K = 0, \qquad K\rho_2 - \rho_2 K = 0, \tag{120b}$$

$$[\mathscr{H}, K] = 0, \qquad [\mathscr{H}, K^2] = 0, \tag{120c}$$

where we choose $\rho_3 = 1$ and $\rho_1 = \rho_2 = \gamma_5$.

The analogue of the total angular momentum is

$$J_{\mu\nu} = \Lambda_{\mu\nu} + \tfrac{1}{2}\sigma_{\mu\nu}, \tag{121}$$

where $\frac{1}{2}\sigma_{\mu\nu}$ is the spin tensor and

$$J^2 = \tfrac{1}{2}J_{\mu\nu}J^{\mu\nu} = \tfrac{1}{2}\Lambda_{\mu\nu}\Lambda^{\mu\nu} + \tfrac{1}{2}\sigma_{\mu\nu}\Lambda^{\mu\nu} + \tfrac{1}{8}\sigma_{\mu\nu}\sigma^{\mu\nu} = K^2 - \tfrac{3}{4}. \tag{122}$$

We choose $\gamma_S = \gamma_1$. Putting

$$\Psi_S = \begin{pmatrix}\psi_4 \\ \psi_4\end{pmatrix}, \qquad \psi_4 = S^{-1}\begin{pmatrix}F_1 \\ F_2 \\ G_1 \\ G_2\end{pmatrix}, \tag{123}$$

we obtain the following set of differential equations

$$-\mathrm{i}S^{3/2}Q_S(G_2/S^{3/2})-(K/S)F_2+(\mathscr{E}+\mathscr{M}-\mathscr{V})F_1 = 0, \tag{124a}$$

$$-\mathrm{i}S^{3/2}Q_S(G_1/S^{3/2})-(K/S)F_1+(\mathscr{E}-\mathscr{M}-\mathscr{V})F_2 = 0, \tag{124b}$$

$$\mathrm{i}S^{3/2}Q_S(F_2/S^{3/2})+(K/S)G_2+(\mathscr{E}+\mathscr{M}-\mathscr{V})G_1 = 0, \tag{124c}$$

$$\mathrm{i}S^{3/2}Q_S(F_1/S^{3/2})+(K/S)G_1+(\mathscr{E}-\mathscr{M}-\mathscr{V})G_2 = 0, \tag{124d}$$

and with the aid of (119), while assuming that as in non-relativistic theory K is integer, we obtain the equations

$$G_2 = F_2, \qquad G_1 = F_1, \tag{125a}$$

$$(\mathscr{E}+\mathscr{M}-\mathscr{V})F_1 - \mathrm{d}F_2/\mathrm{d}S - (K/S)F_2 = 0, \tag{125b}$$

$$(\mathscr{E}-\mathscr{M}-\mathscr{V})F_2 + \mathrm{d}F_1/\mathrm{d}S - (K/S)F_1 = 0. \tag{125c}$$

These are identical with Dirac's (1958) radial equations, and he has shown that they have solutions for bound states provided

$$\mathscr{V} = -e^2/S, \qquad \mathscr{E} = \mathscr{M}c^2\{1+\alpha^2/(P+n')^2\}^{-\frac{1}{2}}, \tag{126}$$

where $P = (K^2-\alpha^2)^{\frac{1}{2}}$ and n' is an integer. When one mass becomes very large, it can be seen from equations (114) that

$$\mathscr{E} \to E_1, \qquad \mathscr{M} \to m_1, \qquad \text{as} \quad m_2 \to \infty, \tag{127}$$

and we are left with Dirac's formula for the fine structure of the hydrogen atom. The relative time theory therefore works very well in this case.

There is one important feature of this model. Using equations (122) and (116) we find

$$K^2-\tfrac{3}{4} = \lambda(\lambda+2) + \rho_2 K. \tag{128}$$

In the non-relativistic approximation, we replace ρ_2 by unity and obtain

$$\lambda = K-\tfrac{3}{2}, \quad -K-\tfrac{1}{2} \tag{129}$$

and λ must be half-integer if we are to obtain the correct fine structure. This has the effect of making the radial wave equations the same as in the non-relativistic theory, but the angular eigenfunctions become, for example,

$$\mathscr{Y}_{nlm} = c(1-t^2)^{\frac{1}{4}} P_l^{K-\frac{1}{2}}(t)\, \mathscr{Y}_{lm}(\theta,\phi). \tag{130}$$

The boson theory given in previous sections does not apply to spinors and the mathematical theorems such as the covariant addition theorem and the equations for symmetry conversion for these eigenfunctions need to be investigated.

VII. References

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Appendix

It is of some interest to show how the partial waves of covariant angular momentum are related to the partial waves of ordinary angular momentum. In order that the wavefunction Ψ in equation (50a) should be converted by the Fourier transform to the equivalent wavefunction ψ in a particular Lorentz frame, we must have

$$\psi = \sum_l C_l(\mathscr{I}_l + S_l \mathscr{O}_l),$$

where

$$\mathscr{I}_l = (qR)^{-\frac{1}{2}} h^{(2)}_{l+\frac{1}{2}}(qR)P_l(z), \qquad \mathscr{O} = (qR)^{-\frac{1}{2}} h^{(1)}_{l+\frac{1}{2}}(qR)P_l(z),$$

$$S_l = \sum_{n=0}^{l} h_n \frac{(l-n)!}{(l+n)!}\{P_l^n(t_2)\}^2 S_n$$

and therefore, from equations (32) and (46),

$$\begin{aligned}\Sigma_{\mathrm{sc}} &= \frac{1}{Q^3}\left|\sum_{n=-\infty}^{\infty}(1-S_n)\frac{\cos n\omega}{\sin\omega}\right|^2\\ &= \frac{\pi^2}{4Q^3}\left|\sum_{l=0}^{\infty}(2l+1)\sum_{n=0}^{l}(1-S_n)h_n\frac{(l-n)!}{(l+n)!}\{P_l^n(t_1)\}^2(1-t_1^2)P_l(z)\right|^2\\ &= (\pi^2/4Q)(1-\epsilon^2/q^2)\sigma_{\mathrm{sc}}(q,z),\end{aligned}$$

where

$$\sigma_{\mathrm{sc}} = \left|\sum_{l=0}^{\infty}(2l+1)(1-S_l)P_l(z)\right|^2$$

is the conventional differential cross section.