LARMOR RADIUS AND COLLISIONAL EFFECTS ON THE DYNAMIC STABILITY OF A COMPOSITE MEDIUM

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Abstract

The combined effects of a finite ion Larmor radius and collisions with neutral atoms on the dynamic stability of a composite medium are investigated. The stability analysis has been carried out for a semi-infinite composite medium of variable density in the presence and absence of a uniform streaming motion. Wave propagations transverse to the direction of the uniform horizontal magnetic field have been considered. It is found that the effects of the collisions as well as the finite ion Larmor radius are stabilizing on both streaming and non-streaming composite media.

I. Introduction

The effects of collisions with neutral atoms on the stability of the well-known Rayleigh–Taylor and Kelvin–Helmholtz configurations have been investigated by Hans (1968). He idealized a plasma, which may not be fully ionized but also permeated with neutral atoms, as a composite medium having an infinitely conducting hydromagnetic component and a neutral component, the two interacting through mutual collisions. It has been shown by both Hans (1968) and Bhatia (1970*a*) that these collisions have a stabilizing influence on the instability of the interface between two uniform superposed static media. For the case of a Kelvin–Helmholtz configuration of two superposed media, Rao and Kalra (1967) and Hans (1968) found that the collisional effects are in fact destabilizing for a sufficiently large collision frequency.

In recent years several authors (e.g. Roberts and Taylor 1962; Rosenbluth, Krall, and Rostoker 1962; Jukes 1964) have pointed out the importance of the effects of finiteness of the ion Larmor radius, which exhibits itself in the form of "magnetic viscosity" in the fluid equations, on plasma instabilities. In his investigation Hans (1968) also considered these finite Larmor radius (FLR) effects, besides the effects of collisions, on the stability of two superposed uniform media.

It has also been pointed out by several authors (see e.g. Chandrasekhar 1961) that the case of variable density is equally interesting. Recently Bhatia (1970*b*) considered collisional effects on the dynamic stability of a composite medium of variable density and showed that the collisions have a stabilizing influence. Earlier Ariel and Bhatia (1969) had investigated FLR effects on the stability of a fully ionized plasma of varying density and found that they also have a stabilizing influence on the plasma instability.

It is therefore of interest now to study the result of simultaneous inclusion of FLR and collisional effects on the stability of a composite medium of variable

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density, and this is the aim of the present work. The medium is assumed to be incompressible and having a one-dimensional stratification in density. The effects on the stability of a streaming composite medium are also considered. Since the basic equations governing the motion of a composite medium in the presence of collisional and FLR effects have been given by Hans (1968), many details are omitted here.

II. Perturbation Equations and Dispersion Relation

The linearized perturbation equations describing the motion of the infinitely conducting hydromagnetic and the neutral components of a composite medium, in the presence of a uniform horizontal magnetic field $\boldsymbol{H}=(H_0,0,0)$ acting in a direction perpendicular to the uniform streaming velocity $\boldsymbol{U}=(0,U,0)$ and a downward gravitational field $\boldsymbol{g}=(0,0,-g)$, lead finally to the following differential equation governing the velocity component w:

$$\{\mathrm{D}(\rho\,\mathrm{D}w)-k_y^2\,\rho w\}\{n+\mathrm{i}k_y\,U+\nu_{\mathrm{c}}\,\alpha(n+\mathrm{i}k_y\,U)/(n+\mathrm{i}k_y\,U+\nu_{\mathrm{c}})\}$$
$$+2\mathrm{i}\nu k_y[\mathrm{D}\{(\mathrm{D}\rho)(\mathrm{D}w)\}-k_y^2(\mathrm{D}\rho)w]+gk_y^2(\mathrm{D}\rho)w/(n+\mathrm{i}k_y\,U)=0\,. \tag{1}$$

This equation is the same as the one obtained by Hans (1968) except that the term $2\mathrm{i}\nu k_y(\mathrm{D}^2\rho)(\mathrm{D}w)$ is missing from his equation. However, this omission does not affect his calculations as he has considered the case where the density of the hydromagnetic component ρ is constant. In equation (1) $\boldsymbol{u}(u,v,w)$ denotes the velocity and ν_{c} the collision frequency between the two components (both assumed to be macroscopic). The effects on the neutral component resulting from the presence of a magnetic field and the fields of gravity and pressure are neglected. The FLR effects are exhibited in equation (1) through $\rho\nu=NT/4\omega_H$, where ω_H is the ion gyration frequency and N and T are the ion number density and temperature respectively. Also in equation (1) $\mathrm{D}\equiv\mathrm{d}/\mathrm{d}z$ and $\alpha=\rho_{\mathrm{d}}/\rho$, ρ_{d} being the density of the neutral component.

After elimination of several quantities from the linearized perturbation equations governing the system, equation (1) is obtained by employing a normal mode analysis and seeking solutions whose dependence on y and the time t is of the form

$$\exp(\mathrm{i}k_y\,y+nt)\,, \tag{2}$$

where n is the frequency and k_y the wave number of the perturbation along the y axis. We are thus analysing here the transverse mode of wave propagation.

Equation (1) holds for all density distributions. We now consider a medium in which the density ρ (and also ρ_{d}) is stratified vertically, that is,

$$\left.\begin{aligned}\rho(z)&=\rho_1\exp(\beta z)\,, && 0\leqslant z\leqslant d\,,\\ &=0\,, && \text{elsewhere}\,,\end{aligned}\right\} \tag{3}$$

where ρ_1 and β are constants. The medium is assumed to be confined between two parallel rigid surfaces at $z=0$ and d and to be infinitely extending along both horizontal directions. For a density distribution of the form (3) equation (1) becomes

$$\mathrm{D}^2 w+\beta\,\mathrm{D}w$$

$$+k_y^2\left(\frac{g\beta/(n+\mathrm{i}k_y U)-\{n+\mathrm{i}k_y U+\alpha\nu_\mathrm{c}(n+\mathrm{i}k_y U)/(n+\mathrm{i}k_y U+\nu_\mathrm{c})+2\mathrm{i}\nu\beta k_y\}}{n+\mathrm{i}k_y U+\alpha\nu_\mathrm{c}(n+\mathrm{i}k_y U)/(n+\mathrm{i}k_y U+\nu_\mathrm{c})+2\mathrm{i}\nu\beta k_y}\right)w=0\,. \quad (4)$$

On the bounding surfaces ($z=0$ and d) w must vanish. Appropriate to the condition at $z=0$, the solution to equation (4) can be written as

$$w(z)=A\{\exp(m_1 z)-\exp(m_2 z)\}\,, \quad (5)$$

where m_1 and m_2 are the roots of the equation

$$m^2+\beta m+\left(\frac{g\beta/(n+\mathrm{i}k_y U)}{n+\mathrm{i}k_y U+\alpha\nu_\mathrm{c}(n+\mathrm{i}k_y U)/(n+\mathrm{i}k_y U+\nu_\mathrm{c})+2\mathrm{i}\nu\beta k_y}-1\right)k_y^2=0\,. \quad (6)$$

The condition that w must vanish at $z=d$ requires, for a nontrivial solution,

$$\exp\{(m_1-m_2)d\}=1 \qquad \text{or} \qquad (m_1-m_2)d=2\mathrm{i}s\pi\,, \quad (7)$$

where s is an integer.

Making use of the relations between m_1 and m_2 as given by (5) and (7), the dispersion relation for different values of the parameter s then becomes

$$(n+\mathrm{i}k_y U)^3+(n+\mathrm{i}k_y U)^2\{2\mathrm{i}\nu\beta k_y+\nu_\mathrm{c}(1+\alpha)\}$$

$$+(n+\mathrm{i}k_y U)\{2\mathrm{i}\nu\nu_\mathrm{c}\beta k_y-g\beta k_y^2/(l^2+k_y^2)\}-\nu_\mathrm{c}\,g\beta k_y^2/(l^2+k_y^2)=0\,, \quad (8)$$

where we have written

$$l^2=\pi^2 s^2/d^2+\tfrac{1}{4}\beta^2\,. \quad (9)$$

III. Discussion

Chandrasekhar (1961) has shown that, in the absence of the effects of FLR and collisions, the configuration is stable or unstable according as β is less than or greater than zero respectively. For the non-streaming ($U=0$) configuration the same conclusion was also obtained earlier when the effects of FLR (Ariel and Bhatia 1969) and collisions (Bhatia 1970*b*) were included separately. We now therefore consider the dispersion relation (8) according to whether the configuration is stable or unstable.

(*a*) *Stable Stratification* ($U=0$ *and* $U\neq 0$)

Putting $q=n+\mathrm{i}k_y U$ and $\beta=-\beta_1$ ($\beta_1>0$), we can rewrite equation (8) as

$$q^3+\nu_\mathrm{c}(1+\alpha)q^2+\{g\beta_1 k_y^2/(l^2+k_y^2)\}q+\nu_\mathrm{c}\,g\beta_1 k_y^2/(l^2+k_y^2)$$

$$=\mathrm{i}(2\nu\beta_1 k_y q^2+2\nu\nu_\mathrm{c}\beta_1 k_y q)\,, \quad (10)$$

or in the simple form

$$q^3+aq^2+cq+e=\mathrm{i}(bq^2+dq)\,, \quad (11)$$

where the constants a, b, c, d, and e are all positive and real and denote the coefficients of the various powers of q in equation (10).

If we now square equation (11) we obtain a sixth-order polynomial in q which can be written in the form

$$q^6 + A_5 q^5 + A_4 q^4 + A_3 q^3 + A_2 q^2 + A_1 q + A_0 = 0\,, \tag{12}$$

where, as is clear from (11), the coefficients A_i ($i = 0, ..., 5$) are all real and positive. Applying Hurwitz's criterion to equation (12) we find that the roots of q are either all negative and real or complex conjugates with negative real parts. This indicates that, for both streaming and non-streaming, the configuration is stable in the presence of the collisional and FLR effects, as it is in the absence of these effects.

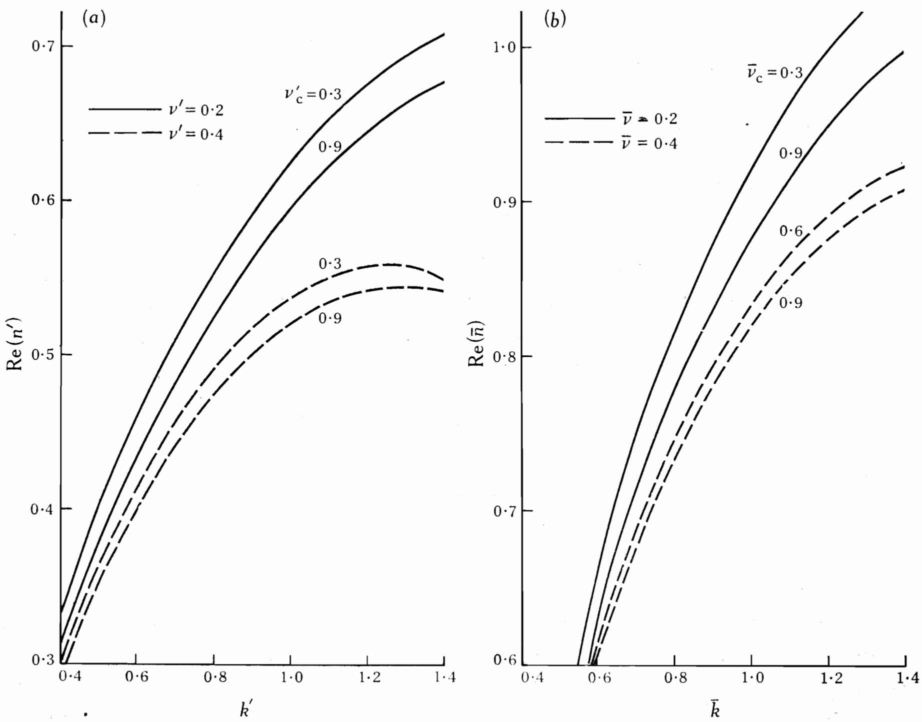

Fig. 1.—Plots of the growth rates (positive real parts of n' and $\bar{n}$) against the wave numbers k' and $\bar{k}$ respectively for (a) a non-streaming medium with $\alpha = 0{\cdot}5$ and (b) a streaming medium with $\alpha = 0{\cdot}5$ and $B = 2{\cdot}0$, for the indicated values of the ν parameters.

(b) *Unstable Stratification*

It is convenient to consider the dispersion relation (8) for unstable conditions ($\beta > 0$) separately for the cases $U = 0$ and $U \neq 0$.

(i) *Non-streaming Medium* ($U = 0$)

With $U = 0$ and the transformations

$$n' = n/(g\beta)^{\frac{1}{2}}\,, \qquad \nu' = \nu l(\beta/g)^{\frac{1}{2}}\,, \qquad \nu'_c = \nu_c/(g\beta)^{\frac{1}{2}}\,, \qquad k' = k_y/l\,, \tag{13}$$

we get the dimensionless form of equation (8)

$$n'^3+n'^2\{2\mathrm{i}\nu'k'+\nu_c'(1+\alpha)\}+n'\{2\mathrm{i}\nu'\nu_c'k'-k'^2/(1+k'^2)\}-\nu_c'k'^2/(1+k'^2)=0\,. \qquad (14)$$

Numerical calculations were performed to obtain the variation of n' with k' from equation (14) for several values of the parameters ν' and ν_c' for $\alpha = 0\cdot5$. The results of these calculations are presented in Figure 1(*a*), where the growth rate (positive real part of n') is plotted against the wave number k', for different values of ν' and ν_c'. It can be clearly seen from the figure that as ν_c' increases n' decreases, thereby indicating that the influence of collisions is stabilizing even in the presence of an FLR effect. Furthermore, the curves show that n' decreases as ν' increases for the same ν_c', and this indicates that the FLR effect is also stabilizing. We thus conclude that both effects enhance the dynamic stability of a composite medium of variable density.

(ii) *Streaming Medium* ($U \neq 0$)

In this case we make the substitutions

$$\bar{n} = n/lU\,, \qquad \bar{\nu} = \nu\beta/U\,, \qquad \bar{\nu}_c = \nu_c/lU\,, \qquad \bar{k} = k_y/l \qquad (15)$$

in equation (8) and obtain the dimensionless form of the dispersion relation

$$\bar{n}^3+\bar{n}^2\{(3+2\bar{\nu})\mathrm{i}\bar{k}+\bar{\nu}_c(1+\alpha)\}+\bar{n}\{-3\bar{k}^2-4\bar{\nu}\bar{k}^2-B\bar{k}^2/(1+\bar{k}^2)+2\mathrm{i}\bar{\nu}_c\bar{k}(1+\alpha+\bar{\nu})\}$$

$$+\{-\mathrm{i}\bar{k}^3-2\mathrm{i}\bar{\nu}\bar{k}^3-\mathrm{i}B\bar{k}^3/(1+\bar{k}^2)-\bar{\nu}_c(1+\alpha)\bar{k}^2-2\bar{\nu}\bar{\nu}_c\bar{k}^2-\bar{\nu}_cB\bar{k}^2/(1+\bar{k}^2)\}=0\,, \qquad (16)$$

where

$$B = g\beta/l^2U^2 \qquad (17)$$

is a measure of the buoyancy forces in terms of the streaming velocity.

For several values of the parameters $\bar{\nu}$, $\bar{\nu}_c$, α, and B numerical calculations were performed to locate the roots of $\bar{n}$ from equation (16) for different values of the wave number $\bar{k}$. The results are presented in Figure 1(*b*), where the growth rate (positive real part of $\bar{n}$) is plotted against $\bar{k}$ for several values of $\bar{\nu}$ and $\bar{\nu}_c$ for $\alpha = 0\cdot5$ and $B = 2\cdot0$. It can be seen that $\bar{n}$ decreases with increase in both $\bar{\nu}$ and $\bar{\nu}_c$, again indicating that both the collisional and FLR effects have a stabilizing influence on the unstable configuration.

We may thus conclude that the simultaneous inclusion of the effects of collisions and a finite Larmor radius has a stabilizing influence on the dynamics of a streaming composite medium in which there is a one-dimensional stratification in the density, at least for the range of parameters considered.

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