APPROXIMATE SOLUTIONS OF THE RELATIVISTIC GRAVITATIONAL FIELD EQUATIONS TO DESCRIBE CLUSTERS OF GALAXIES

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Abstract

Approximate solutions to the Einstein field equations are found which describe a spherically symmetric inhomogeneity in a general Robertson–Walker model, i.e. one with an arbitrary equation of state. The approximation hypothesis is that the pressure deviates only slightly from uniformity, and it is found that the density may have quite large local fluctuations, e.g. by a factor of 10^6 over a region 10^{-2} Mpc in diameter. Reference is made to observed data to determine which categories of stellar objects may be described by the results.

I. Introduction

The motivation for the present work arises from a paper by de Vaucouleurs (1970) in which attention is drawn to inhomogeneities in the distribution of matter which persist to the present limits of observation. He presents a picture of hierarchies of matter comprising galaxies, clusters, superclusters, etc. where the average density of each member differs by factors ranging from 10^2 to 10^4 from the members above and below it in the hierarchy. This represents a significant departure from the so-called "cosmological principle", and thus it seems desirable to attempt to modify the current theory of uniform (pressure and density) models of the universe, obtained by appealing to the cosmological principle, to take account of this.

Such a modified theory would aim to predict observable features of the universe which existing models lack due to their (physical) simplicity. To be worth while, a work of this nature would need to satisfy two criteria:

(1) The subject should be treated relativistically, since it is by the interaction of light and gravity (speaking loosely) peculiar to relativity that new observable features are to be predicted.

(2) The model must describe realistic variations in the cosmological density.

The preceding requirements are necessarily vague since they set out only broad and general aims for an extensive treatise to which this paper is hardly more than an introduction. Its specific aims are more limited.

A full relativistic treatment of de Vaucouleurs's hierarchies is not attempted here as this would involve considerations of anisotropy which are beyond the present scope. What is considered is the treatment of any one of de Vaucouleurs's objects in isolation as a spherically symmetric inhomogeneity within a homogeneous universe. This "background universe" is to be described by a general Robertson–Walker metric, i.e. no specific equation of state is assumed, and this provides for a more general situation than in any previous work. The aim is then to describe this situation

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by a metric which is mathematically fairly simple and easily related to the physical state variables, and which may therefore be the basis for the sort of prediction alluded to above. Finally, consideration is given as to which stellar objects might realistically be expected to satisfy the model.

The method here is to investigate spherically symmetric perturbations of the Robertson–Walker metric which permit reasonably large density fluctuations, i.e. of the order of 10^6 over a region 10^{-2} Mpc in diameter. Perturbations, involving small density fluctuations, of certain subclasses of the Robertson–Walker metric (e.g. $p = 0$ or $\frac{1}{3}\rho$) have already been extensively treated (McCrea 1939; Lifshitz 1946; Kalitzin 1961; Hawking 1966; Kristian and Sachs 1966; Sachs and Wolfe 1967).

II. Approximate Solutions for Spherically Symmetric Case

Rather than proceed in the traditional manner, which is to write the metric tensor

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + h_{\mu\nu}\,,$$

where $\bar{g}$ is the Robertson–Walker metric and the $h_{\mu\nu}$ are in some sense small, and then to seek physical interpretation of the $h_{\mu\nu}$, we shall consider the general isotropic form of a spherically symmetric line element, to represent matter whose distribution varies with the radial coordinate r alone, i.e.

$$\mathrm{d}s^2 = \exp\{\nu(r,t)\}\,\mathrm{d}t^2 - c^{-2}\exp\{\lambda(r,t)\}(\mathrm{d}r^2 + r^2\,\mathrm{d}\omega^2)\,, \tag{1}$$

where

$$\mathrm{d}\omega^2 = \mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2\,.$$

We suppose that the energy tensor describing the matter is that of a perfect fluid,

$$T^{\mu\nu} = (\rho + p/c^2)u^\mu u^\nu - g^{\mu\nu}p/c^2\,.$$

We now choose comoving coordinates and assume that matter–energy is conserved, so that $\nabla_\nu T^{\mu\nu} = 0$. In detail, the equations become

$$\mu = 0: \qquad \rho_t + \tfrac{3}{2}\lambda_t(\rho + p/c^2) = 0\,, \tag{2}$$

$$\mu = 1: \qquad p_r + \tfrac{1}{2}\nu_r(\rho c^2 + p) = 0\,, \tag{3}$$

$$\mu = 2,3: \qquad \nabla_\nu T^{\mu\nu} \equiv 0\,.$$

Equation (3) indicates that if ν is independent of r then p is also, that is, p is uniform, and the field equations imply that ρ is likewise a function of t alone.

The field equations

$$R^\mu_\nu = -\kappa(T^\mu_\nu - \tfrac{1}{2}\delta^\mu_\nu T^\lambda_\lambda)$$

reduce to the four independent equations

$$\tfrac{3}{4}\exp(-\nu)(2\lambda_{tt} + \lambda_t^2 - \nu_t\,\lambda_t) - \tfrac{1}{2}c^2\exp(-\lambda)(\nu_{rr} + \tfrac{1}{2}\nu_r^2 + 2\nu_r/r + \tfrac{1}{2}\nu_r\,\lambda_r)$$
$$= -\tfrac{1}{2}\kappa(\rho + 3p/c^2)\,, \tag{4}$$

$$-\tfrac{1}{2}\exp(-\nu)(\lambda_{tt}+\tfrac{3}{2}\lambda_t^2-\tfrac{1}{2}\lambda_t\,\nu_t)+c^2\exp(-\lambda)(\lambda_{rr}+\tfrac{1}{2}\nu_{rr}+\tfrac{1}{4}\nu_r^2-\tfrac{1}{4}\lambda_r\,\nu_r+\lambda_r/r)$$
$$= -\tfrac{1}{2}\kappa(\rho-p/c^2)\,, \tag{5}$$

$$-\exp(-\nu)(\lambda_{tt}+\tfrac{3}{2}\lambda_t^2-\tfrac{1}{2}\lambda_t\,\nu_t)+c^2\exp(-\lambda)(\lambda_{rr}+3\lambda_r/r+\tfrac{1}{2}\lambda_r^2+\nu_r/r+\tfrac{1}{2}\nu_r\,\lambda_r)$$
$$= -\kappa(\rho-p/c^2)\,, \tag{6}$$

$$\lambda_{tr}-\tfrac{1}{2}\nu_r\,\lambda_t = 0\,. \tag{7}$$

From equations (5) and (6) we have

$$(\lambda_{rr}+\nu_{rr})-(\lambda_r+\nu_r)/r-\tfrac{1}{2}(\lambda_r+\nu_r)^2 = -\nu_r^2\,. \tag{8}$$

If we assume that the pressure p varies only slightly with r, that is, $|p_r/(\rho c^2+p)|^2 \ll 1$, then by virtue of equation (3) we have $\nu_r^2 \ll 1$. We can thus neglect ν_r^2 in equation (8) which may then be integrated to give

$$\lambda+\nu = -2\ln\{1+\alpha(t)\,r^2\}+2\ln s(t)\,, \tag{9}$$

where α and s are arbitrary functions of t.

We now integrate equation (7) to get $\lambda_t = \sigma(t)\exp(\tfrac{1}{2}\nu)$, where σ is an arbitrary function of t. Combining this result with equation (9), we have

$$\lambda_t\exp(\tfrac{1}{2}\lambda) = \sigma(t)\,s(t)/\{1+\alpha(t)\,r^2\}\,,$$

or

$$\exp(\tfrac{1}{2}\lambda) = \tfrac{1}{2}\int\frac{\sigma(t)\,s(t)\,\mathrm{d}t}{1+\alpha(t)\,r^2}+h(r)$$

and

$$\exp(\tfrac{1}{2}\nu) = \frac{s(t)}{1+\alpha(t)\,r^2}\left(\int\frac{\sigma(t)\,s(t)\,\mathrm{d}t}{1+\alpha(t)\,r^2}+h(r)\right)^{-1},$$

where h is an arbitrary function of r.

We obtain a sufficiently useful subclass of possible solutions if we set $4\alpha = k$ (a constant) and rescale r so that $k = -1$, 0, and $+1$. If we now define

$$2S(t) = \int\sigma s\,\mathrm{d}t$$

and redefine the time coordinate by $s(t)\,\mathrm{d}t = S(t)\,\mathrm{d}t^*$, we have

$$\exp(\tfrac{1}{2}\nu) = S(t^*)/\{S(t^*)+h(r)(1+\tfrac{1}{4}kr^2)\}\,,$$

where $S(t^*) = S\big(t(t^*)\big)$. This leads us to replace $h(r)$ by another arbitrary function $\eta(r) = h(r)(1+\tfrac{1}{4}kr^2)$ and, omitting the asterisks from the t's, we have

$$\exp(\tfrac{1}{2}\lambda) = \frac{S(t)+\eta(r)}{1+\tfrac{1}{4}kr^2}\,,\qquad \exp(\tfrac{1}{2}\nu) = \frac{S(t)}{S(t)+\eta(r)}\,. \tag{10a, b}$$

Since $\nu_r \sim o(1)$, we have the condition $\eta/S \sim o(1)$ (see Appendix). It is also possible to give a simple geometrical interpretation of these results (Cook 1971).

As we now wish to relate the arbitrary functions η and S to the physical variables p and ρ, we use equations (4) and (5) and ultimately obtain

$$\begin{aligned}\kappa\rho &= 3S^{-2}(S_t^2+kc^2)\\ &\qquad -2c^2S^{-3}\{(1+\tfrac{1}{4}kr^2)^2\eta_{rr}+(1-\tfrac{1}{4}kr^2)(1+\tfrac{1}{4}kr^2)\eta_r/r+3k\eta\},\\ -\kappa p/c^2 &= 2S^{-1}S_{tt}+S^{-2}S_t^2+kc^2S^{-2}+\eta S^{-1}(2S^{-1}S_{tt}-2S^{-2}S_t^2-2kc^2S^{-2}).\end{aligned}$$

We may thus write $p = p_0+p_1$ and $\rho = \rho_0+\rho_1$, where p_0 and ρ_0 are respectively the pressure and density of any uniform model (represented by the Robertson–Walker metric) and p_1 and ρ_1 are given by

$$p_1 = (\rho_0 c^2+p_0)\eta S^{-1} \tag{11}$$

and

$$\kappa\rho_1 = -2c^2S^{-3}\{(1+\tfrac{1}{4}kr^2)^2\eta_{rr}+(1-\tfrac{1}{16}k^2r^4)\eta_r/r+3k\eta\}. \tag{12}$$

It is noted that ρ_1 has the form $D(r)S^{-3}(t)$, as required by the conservation equations (2) and (3), and also that, in conformity with all uniform models, the fluid satisfies to first order the isothermal expansion condition

$$\partial m/\partial t = -(p/c^2)\partial v/\partial t,$$

where m is the fluid mass enclosed by a volume element v.

We now make the standard transformation of r for the Robertson–Walker metric when $k \neq 0$, namely

$$r = 2k^{-\frac{1}{2}}\tan(\tfrac{1}{2}k^{\frac{1}{2}}\psi),$$

and obtain

$$\kappa\rho_1(\psi,t) = -2c^2S^{-3}(t)\{\eta_{\psi\psi}+k^{\frac{1}{2}}(\cos k^{\frac{1}{2}}\psi-1)\eta_\psi/\sin k^{\frac{1}{2}}\psi+3k\eta\}.$$

For small ψ this becomes

$$\kappa\rho_1 = -2c^2S^{-3}(\eta_{\psi\psi}+\eta_\psi/\psi), \tag{12a}$$

which has the same form as equation (12) for the case $k = 0$.

III. Comparison with Observed Parameters

We shall now suppose that there is a symmetric inhomogeneity located at the origin of the coordinate system and that its boundary coordinate $\bar{\psi}$ is small (say $< 10^{-1}$). This corresponds to a physical distance of $U = S\bar{\psi} \sim 10^{27}$ cm. Now the smallness condition on η_ψ places a weaker smallness condition on $\eta_{\psi\psi}$. That is to say, we need only require that $S^{-1}\bar{\psi}N = \zeta \sim o(1)$, where N is the average value of $|\eta_{\psi\psi}|$ over the range $(0,\bar{\psi})$. If we define P to be the average of $|\rho_1/\rho_0|$ over the same range then from equation (12a) we have (for $\eta_\psi/\psi \sim \eta_{\psi\psi}$)

$$P = 4c^2S^{-3}\rho_0^{-1}N.$$

Using current estimates of $H^2 = 1\cdot 02\times 10^{-35}\,\mathrm{s}^{-2}$ and $\rho_0 = 7\times 10^{-31}\,\mathrm{g\,cm}^{-3}$, for which the corresponding value of S is $0\cdot 96\times 10^{28}$ cm, this equation becomes (Robertson and Noonan 1968)

$$P = 3\times 10^{57}\,NS^{-3}. \tag{13}$$

In terms of U, the smallness condition on N is $S^{-2}UN = \zeta$, and combining this with equation (13) we have

$$PU = 10^5\times\zeta\ \mathrm{Mpc}. \tag{14}$$

For $\zeta < 0\cdot 1$, the condition (14) is

$$PU < 10^4\ \mathrm{Mpc}. \tag{15}$$

We may now refer to recorded observations to check which stellar objects satisfy this condition. The average density and approximate radius of the densest members representative of a number of classes of stellar objects are presented by de Vaucouleurs (1970, Table 1). These range from neutron stars to superclusters, although it is only galaxies and larger objects with which this paper is concerned. Calculation of U and P for each of the entries in de Vaucouleurs's table shows that small spiral groups and all larger objects will satisfy (15). Alternatively, the density–radius relation in equation (4) of de Vaucouleurs could be used.

Essentially, de Vaucouleurs (1970) postulates (and is largely borne out by observation) that galaxies and all larger objects satisfy the relation

$$\log\rho_1 \leqslant -1\cdot 7\log U + 15$$

(where de Vaucouleurs uses ρ and R for ρ_1 and U respectively). Calculation shows that objects for which U is greater than 5×10^{-2} Mpc automatically satisfy both this condition and (15).

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Appendix

Since much of the foregoing depends on the adequacy of the expression given by equation (9) as an approximation to the solution of (8), it is as well to consider this point in more detail. If we write $\omega = \lambda + \nu$ then equation (8) becomes

$$\omega_{rr} - \omega_r/r - \tfrac{1}{2}\omega_r^2 = -\nu_r^2. \tag{A1}$$

We have already seen that, for small pressure gradients, $\nu_r^2 \ll 1$ and this suggests the approximation used in Section II to obtain equation (9). Strictly, however, we need to ensure the suitability of (9) by showing that ν_r^2 is small in comparison with the terms on the left-hand side of (A1), and this is considered in the following analysis.

If $\bar{\omega}$ is the particular solution to

$$\bar{\omega}_{rr} - \bar{\omega}_r/r - \tfrac{1}{2}\bar{\omega}_r^2 = 0 \tag{A2}$$

given by the equations (10), then for intermediate values of r in the range $(0, \infty)$ we have

$$\bar{\omega}_{rr} \approx \bar{\omega}_r/r \approx \bar{\omega}_r^2 \sim O(1),$$

which is obtained by differentiation of equations (10). For such values of r it is therefore sufficient that ν_r^2 be very much less than unity. From equation (10b) we obtain

$$\nu_r = -2S(t)\,\eta_r/\{S(t)+\eta(r)\},$$

and if we arbitrarily impose the condition $|\eta| \approx |\eta_r|$, which amounts to requiring only that the arbitrary constant obtained by integrating η_r be small, then the condition $\nu_r^2 \ll 1$ is equivalent to the condition $(\eta_r/S)^2 \approx (\eta/S)^2 \ll 1$.

Now, for large values of r, $\bar{\omega}_{rr} \sim O(r^{-2})$ and so we shall also require, in order to maintain the same relative magnitude between ν_r and the terms in ω, that at least

$$\eta_r/S \sim O(r^{-1}) \qquad \text{as} \qquad r \to \infty. \tag{A3}$$

This is not a particularly stringent condition, for when the transformation is made from r to ψ we have

$$\eta_\psi = (1+\tfrac{1}{4}kr^2)\eta_r,$$

so that the condition (A3) is satisfied if $\eta_\psi/S \sim o(1)$. In any case, as we are only interested in the properties of the model near the origin, we could for present purposes set $\eta_r = 0$ for large r.

Finally, for small values of r $(0 < r^2 \ll 1)$, we have that $|\bar{\omega}_{rr}|$ and $|\bar{\omega}_r/r|$ approach unity, but $\bar{\omega}_r \sim O(r)$ as $r \to 0$. Hence the $\bar{\omega}_r^2$ term on the left-hand side of (A2) becomes small compared with the other two terms. For these values of r we should therefore approximate ω by ξ, the solution of

$$\xi_{rr} - \xi_r/r = 0,$$

which is $\xi = q(t)\,r^2 + p(t)$, where $q(t)$ and $p(t)$ are arbitrary functions. We then have

$\exp \xi = \exp\{p(t) + q(t)\, r^2\}$, but since r is small

$$\exp \xi \simeq \exp\{p(t)\}\{1 + q(t)\, r^2\}.$$

If we choose $\exp\{p(t)\} = S^2(t)$ and $q(t) = -\frac{1}{2}k$ then

$$\exp \xi \simeq S^2(t)\,(1 - \tfrac{1}{2}kr^2).$$

Also, for small r, we may write

$$\exp \bar{\omega} \simeq S^2(t)\,(1 - \tfrac{1}{2}kr^2).$$

Hence $\bar{\omega}$ can be regarded as a suitable approximation to ω throughout the range of r.