SHORT COMMUNICATIONS

ON THE DRIFT VELOCITY OF ELECTRONS IN A GAS*

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In a recent paper, Crompton, Elford, and Robertson (1970; hereafter referred to as CER) considered certain questions concerning the steady-state distribution of electrons moving in a neutral gas under the influence of a uniform electric field $\boldsymbol{E}$. The present communication comments on some aspects of the error discussion in the Appendix of the paper by CER. The analysis will be restricted to the case of isotropic scattering. Two different definitions of the drift velocity are considered by CER, namely

$$\boldsymbol{W}_1 \equiv \int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v})\,\boldsymbol{v} \Big/ \int \mathrm{d}\boldsymbol{v}\, f(\boldsymbol{v}) \tag{1}$$

and

$$\boldsymbol{W}_2 \equiv \int \mathrm{d}\boldsymbol{v}\, g(v)\,\langle s\rangle^{+}(v) \Big/ \int \mathrm{d}\boldsymbol{v}\, g(v)\,\langle t\rangle^{+}(v)\,, \tag{2}$$

where $f(\boldsymbol{v})$ is the conventional distribution function, $g(v)$ is the distribution function immediately after collision, and

$$\begin{aligned}\langle s\rangle^{+}(v_0) &\equiv w_0(v_0)\,\langle t\rangle^{+}(v_0)\\ &\equiv \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu \int_0^{\infty}\mathrm{d}t\,\frac{\frac{1}{2}at^2+v_0 t\mu}{\tau(v(t))}\exp\!\left(-\int_0^t \frac{\mathrm{d}t'}{\tau(v(t'))}\right)\end{aligned} \tag{3}$$

and

$$\langle t\rangle^{+}(v_0) \equiv \frac{1}{2}\int_{-1}^{1}\mathrm{d}\mu \int_0^{\infty}\mathrm{d}t\,\frac{t}{\tau(v(t))}\exp\!\left(-\int_0^t \frac{\mathrm{d}t'}{\tau(v(t'))}\right) \tag{4}$$

are "path integral" expectation values. In equations (3) and (4)

$$\mu = -\boldsymbol{v}_0\cdot\boldsymbol{E}/v_0 E\,, \qquad \boldsymbol{a} = -|e/m_{\mathrm{e}}|\,\boldsymbol{E}\,, \qquad v(t) = (v_0^2+a^2t^2+2v_0\mu at)^{\frac{1}{2}}\,.$$

It should be noted that some changes in nomenclature have been made; in particular, the quantities f, f_0, T, λ_{m}/c, $\cos\theta$, W, and W_c of CER correspond to f_0, g, $\langle t\rangle^{+}$, τ, μ, W_1', and W_2 of the present paper respectively. It can be seen that the relations (A2) and (A3) of CER are equivalent to the above relations (2) and (3).

The corresponding Boltzmann equation (see e.g. Paveri-Fontana 1970) is

$$\boldsymbol{a}\cdot\partial f/\partial\boldsymbol{v} = g(v)/T - f(\boldsymbol{v})/\tau(v)\,. \tag{5}$$

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One should note the functional dependence of the initial distribution function $g(v)$ on $f(\boldsymbol{v})$:

$$g/T = g(v,\{f\})/T = \int \mathrm{d}\boldsymbol{v}'\, f(\boldsymbol{v}')P(v' \to v)/\tau(v')\,. \tag{6}$$

A well-known method of approximation is to expand the function $f(\boldsymbol{v})$ in terms of Legendre polynomials and then truncate the series after the second term. This leads to the equations

$$\frac{a}{3v^2}\frac{\mathrm{d}(v^2 f_1)}{\mathrm{d}v} = \frac{g(v,\{f\})}{T} - \frac{f_0(v)}{\tau(v)}, \qquad a\frac{\mathrm{d}f_0}{\mathrm{d}v} = -\frac{f_1(v)}{\tau(v)}, \tag{7}$$

$$f(\boldsymbol{v}) \approx f_{1\mathrm{eg}}(\boldsymbol{v}) \equiv f_0(v) + \mu f_1(v)\,. \tag{8}$$

In order to solve these equations, one *must* assign an explicit form to $g(v)$ as a functional of $f(\boldsymbol{v})$. For instance, in order to derive the Davydov distribution that is employed in Section IV of CER, one replaces equation (5) by equations (7) and (8) and then assigns (see e.g. Bernstein 1969; Wannier 1971)

$$\frac{g(v,\{f\})}{T} = \frac{f_0(v)}{\tau(v)} + \frac{\epsilon}{v^2}\frac{\mathrm{d}}{\mathrm{d}v}\left\{\frac{v^3}{\tau(v)}\left(1 + \frac{kT}{m_e v}\frac{\mathrm{d}}{\mathrm{d}v}\right) f_0(v)\right\}, \tag{9}$$

where ϵ is the ratio of the electron mass to the molecular mass. After evaluating $f_0(v)$ and $f_1(v)$, substitution back into equation (9) gives $g(v,\{f_{1\mathrm{eg}}\})$ and then further substitution into equations (1) and (2) provides *approximate* expressions for the drift velocity. Let these approximations be W_1' and W_2'. It should be noted that W_1' is the same as W defined by relation (A1) of CER whenever $v^2\lambda(v)\, f_0(v) \to 0$ for $v \to 0$, while W_2' is the approximation for W_c given on page 681 of CER (remembering that f and f_0 of CER are equivalent to the present f_0 and g respectively).

Now it can be shown that in steady-state conditions $\boldsymbol{W} = \boldsymbol{W}_1 = \boldsymbol{W}_2$ if both $\tau(v)/v = O(v^{-\alpha})$ when $v \to \infty$ ($\alpha > 0$) and exact solutions of the Boltzmann equation (5) are used (Cavalleri and Paveri-Fontana 1972). However, one cannot expect that generally W_1' will be equal to W_2', as they are both approximate expressions. As a matter of fact, the quantity $(W_1' - W_2')/W_1'$ could possibly be used as a measure of how suitable a Legendre truncation technique is for a particular problem.

In their analysis, however, CER obtained another expression W_2'' for the drift velocity by inserting into equation (2) the approximate relation

$$T^{-1} g(v,\{f\})\, \langle t \rangle^{+}(v) \simeq f_0(v)\,. \tag{10}$$

They then concluded that this new expression should be better than W_1'. Such a claim appears questionable, as both W_1' and W_2' are obtained by applying one approximate technique (Legendre truncation) to the Boltzmann equation while W_2'' is obtained by applying a further approximation (relation (10)) to the results obtained by applying the previous approximation. It does not seem reasonable to expect that the second approximation will somehow cancel the effect of the first. Also note that, if the approximate values of $f_0(v)$ and $f_1(v)$ (based on approximation (7)) approach their exact values, then both W_1' and W_2' approach the exact W but W_2'' generally does not.

For these reasons, generally one should expect W_2'' *to provide less reliable results than* W_1' and W_2' unless the contrary can be proved for the specific case considered.

To the author's knowledge, only the value of W_1' is known for a large number of cases, and there are no published computations of W_2', W_2'', or the exact W. For this reason comparisons based on numerical results are not possible at present. The case $\tau(v) = \text{const.}$ is well known, but one finds $W_1' = W_2' = W_2'' = W = a\tau$ and no general conclusion can be drawn. Another special case is considered below.

An Example

The example introduced here is used to illustrate the previous considerations. Among the cases studied by the author, it is the only one that allows easy analytical manipulations.

Let $f_{\text{leg}}(\boldsymbol{v})$ be the approximate solution obtained by replacing the Boltzmann equation (5) by the Legendre truncated relations (7). Then W_1' is obtained by substituting $f_{\text{leg}}(\boldsymbol{v})$ into relation (1); the resulting expression is equal to relation (A1) of CER whenever $v^3\tau(v) f_0(v) \to 0$ for $v \to 0$. Also, W_2' and W_2'' are obtained by substituting $g(v, \{f_{\text{leg}}\})$ and $f_0(v)\, T/\langle t\rangle^+(v)$ respectively into equation (2). Now consider the relations

$$\boldsymbol{a} \,.\, \partial f'/\partial \boldsymbol{v} + f'/\tau(v) = g(v, \{f_{\text{leg}}\})/T\,, \tag{11}$$

$$\boldsymbol{a} \,.\, \partial f''/\partial \boldsymbol{v} + f''/\tau(v) = f_0(v)/\langle t\rangle^+(v)\,, \tag{12}$$

remembering that $f_0(v)$ and $f_{\text{leg}}(\boldsymbol{v})$ are the approximate quantities given by the relations (7) and (8). By a technique similar to that of Cavalleri and Paveri-Fontana (1972) it can be shown that W_2' and W_2'' are equal to the value that W_1 takes when f' and f'' are substituted into equation (1). This is valid whenever $\tau(v)/v = O(v^{-\alpha})$ for $v \to \infty$ $(\alpha > 0)$.

Now consider the special case $1/\tau(v) = H(v - v^*)/\tau_0$, where H is the Heaviside step function. It can be considered a fairly good approximation for certain cases where the electron–molecule cross section presents a significant Ramsauer dip. Also assume that the Davydov representation (9) of the collision source is valid. Multiplying equations (5), (11), and (12) by $\boldsymbol{v}\,\mathrm{d}\boldsymbol{v}$ and integrating leads to a relation of the form

$$\int_{v \geq v^*} \mathrm{d}\boldsymbol{v}\; \boldsymbol{v}\, y(\boldsymbol{v}) = \boldsymbol{a}\tau_0 \int \mathrm{d}\boldsymbol{v}\; y(\boldsymbol{v})\,, \tag{13}$$

where $y(\boldsymbol{v})$ can equal any of the quantities $f(\boldsymbol{v})$, $f'(\boldsymbol{v})$, or $f''(\boldsymbol{v})$. The absence of a collision source for $v < v^*$ in equations (5) and (11) guarantees that, for reasons of symmetry, both $f(v, \mu) = f(v, -\mu)$ and $f'(v, \mu) = f'(v, -\mu)$ for $v < v^*$. However, it is easy to show that in equation (12)

$$f''(v, \mu) > f''(v, -\mu) \qquad \text{for} \qquad 0 < \mu \leqslant 1\,, \quad 0 < v < v^*\,.$$

For this reason, from the form of the relation (13), one finds that the exact electron drift velocity is given by

$$\boldsymbol{W} = \int \mathrm{d}\boldsymbol{v}\; \boldsymbol{v}\, f(\boldsymbol{v}) \Big/ \int \mathrm{d}\boldsymbol{v}\; f(\boldsymbol{v}) = \boldsymbol{a}\tau_0$$

and that also

$$\boldsymbol{W}_2' = \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f'(\boldsymbol{v}) \Big/ \int \mathrm{d}\boldsymbol{v}\, f'(\boldsymbol{v}) = \boldsymbol{a}\tau_0 = \boldsymbol{W},$$

while

$$|\boldsymbol{W}_2''| = \left| \int \mathrm{d}\boldsymbol{v}\, \boldsymbol{v} f''(\boldsymbol{v}) \Big/ \int \mathrm{d}\boldsymbol{v}\, f''(\boldsymbol{v}) \right| > a\tau_0 .$$

Let us now consider W_1'. Substituting $1/\tau(v) = H(v-v^*)/\tau_0$ into the well-known Davydov distribution function (see e.g. equation (1) of CER) gives

$$\begin{aligned} f_0(v) &= C && \text{for} \quad v < v^*, \\ &= C \exp\left(- \int_{v^*}^{v} \frac{c\,\mathrm{d}c}{kT/m_\mathrm{e} + (a\tau_0)^2/3\epsilon} \right) && v > v^*. \end{aligned}$$

Then, using the relations (1) and (7), we once again find

$$\boldsymbol{W}_1' = \frac{\boldsymbol{a}}{3a} \int \mathrm{d}v\, v^3 f_1(v) \Big/ \int \mathrm{d}v\, v^2 f_0(v) = \boldsymbol{a}\tau_0 .$$

For one specific case we have thus computed the exact W and the approximations W_1', W_2', and W_2''. In this example W_1' and W_2' yield the exact value while W_2'' does not. This result would seem to confirm the earlier conclusions.

Actually, in this problem, the approximation (10) is an extremely poor representation of $g(v, \{f\})$ as it yields a nonzero g even for $v < v^*$. Indeed the exact result $\boldsymbol{W}_2'' = \boldsymbol{a}\tau_0$ could be obtained by postulating

$$g(v)/T = \{f_0(v)/\langle t \rangle^+(v)\} H(v-v^*) \tag{14}$$

instead of (10). However, this choice would have other shortcomings. Whereas equation (11) conserves the number of collision events in the sense that

$$\int \mathrm{d}\boldsymbol{v}\, f'(\boldsymbol{v})/\tau(v) = \int \mathrm{d}\boldsymbol{v}\, f_{\mathrm{leg}}(\boldsymbol{v})/\tau(v),$$

and equation (12), based on (10), conserves the number of electrons in the sense that

$$\int \mathrm{d}\boldsymbol{v}\, f''(\boldsymbol{v}) = \int \mathrm{d}\boldsymbol{v}\, f_{\mathrm{leg}}(\boldsymbol{v}),$$

it can be shown that the representation (14) fails to satisfy either conservation property. Furthermore, it seems impossible to generalize (14) to a general $\tau(v)$.

Now, it could be claimed that the present example violates the condition of validity for the approximation (10), which, as shown by Cavalleri (1969, relation (13)) and Braglia (1970, relation (31)), can be stated as

$$a\tau(v)/v \ll 1 . \tag{15}$$

This is indeed true. However, it should be noted that the experimental example

considered by CER also violates the condition (15), since, for $v \to 0$, λ approaches a finite value and therefore $a\tau(v)/v = a\lambda/v^2 \to \infty$.

Thus, on the basis of the example presented here and the preceding discussion, we may conclude that:

(1) After a Legendre truncation has been applied, it would probably be better to follow the procedure that leads to the evaluation of W_1' and W_2' instead of W_1' and W_2''.

(2) The discrepancy found by Crompton, Elford, and Robertson (1970) in the region $v \to 0$ between the integrands appearing in the evaluations of W_1' and W_2'' most probably arises more from the shortcomings of the assumption (10) than from the inadequacy of the Legendre truncation technique.

(3) In the region $v \to 0$, the condition of validity of approximation (10) (namely condition (15)) is not met.

Further clarification of this subject could be obtained, for instance, by calculating W_2' numerically from the data of Crompton, Elford, and Robertson. The resulting values of W_1' and W_2'' could then be compared. Further comparisons could also be made between the values of W_1', W_2', and W_2'' obtained from the formulae given here with computations of W_2 based on the numerical solution of Cavalleri and Sesta's (1968, 1969) integral equation for $g(v)$.

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