DEPENDENCE OF HELICON WAVE RADIAL STRUCTURE ON ELECTRON INERTIA

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Abstract

An analytic expression for the dispersion of travelling helicon waves in a uniform non-resistive plasma is used to show the effect of electron inertia on the radial structure of the wave in a cylindrical plasma bounded by perfectly conducting walls.

Introduction

The use of helicon waves as a diagnostic tool was initially suggested by Gallet *et al.* (1960) and later by a number of authors (Blevin and Christiansen 1966; Blevin, Christiansen, and Davies 1968; Jolly, Martelli, and Troughton 1969). The importance of the electron inertial term in the generalized Ohm's law (Spitzer 1962) has been shown by Davies (1970) to be important in low density laboratory plasmas even if the wave frequency $\omega \ll \Omega_{\mathrm{e}}$, the electron cyclotron frequency. The purpose of this paper is to show that electron inertia must be accounted for when interpreting the radial wave profiles in cylindrical geometry, due to the possible importance of higher order radial modes. Approximations appropriate to $\Omega_{\mathrm{e}} > \omega \gg \nu,\ \Omega_{\mathrm{i}}$ are made. Although the inclusion of the collision frequency ν increases the complexity of the equations, its main effect on the dispersion relation is to introduce an imaginary part for the wavenumber, while the real part of the wavenumber is affected only to second order in ν/Ω_{e}.

Dispersion Relation

Using the generalized Ohm's law and assuming immobile ions, a uniform electron density N, and negligible resistivity, the equation to be solved becomes (in Gaussian CGS units)

$$(M/Ne^2)\,\partial \boldsymbol{j}/\partial t = \boldsymbol{E} - (Nec)^{-1}\boldsymbol{j} \times \boldsymbol{B}_0\,, \tag{1}$$

where M is the electron mass and B_0 the uniform static magnetic field. Comparing the left-hand side of equation (1) with the $\boldsymbol{j} \times \boldsymbol{B}_0$ term, the ratio of the magnitudes of these two terms is seen to be $\sim \omega/\Omega_{\mathrm{e}}$ and it would appear that the inertial term could be neglected when $\omega \ll \Omega_{\mathrm{e}}$. However, the following analysis shows that this is not necessarily the case.

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Using Maxwell's equations, and seeking solutions of the kind

$$f(r)\exp\{\mathrm{i}(\omega t - kz - m\theta)\}$$

for perturbing quantities in a cylindrical plasma, some simple algebra yields

$$\omega\nabla\times\nabla\times\boldsymbol{b} - k\Omega_{\mathrm{e}}\nabla\times\boldsymbol{b} + (4\pi Ne^2\omega/Mc^2)\boldsymbol{b} = 0\,, \tag{2}$$

where $\boldsymbol{b}$ is the magnetic field of the wave. Following the method of Klosenberg, McNamara, and Thonemann (1965), the solution of (2) is therefore the sum of solutions of

$$\nabla\times\boldsymbol{b} = q_1\boldsymbol{b} \qquad \text{and} \qquad \nabla\times\boldsymbol{b} = q_2\boldsymbol{b}\,, \tag{3}$$

where q_1 and q_2 satisfy

$$\omega q^2 - k\Omega_{\mathrm{e}}q + (4\pi Ne^2\omega/Mc^2) = 0\,. \tag{4}$$

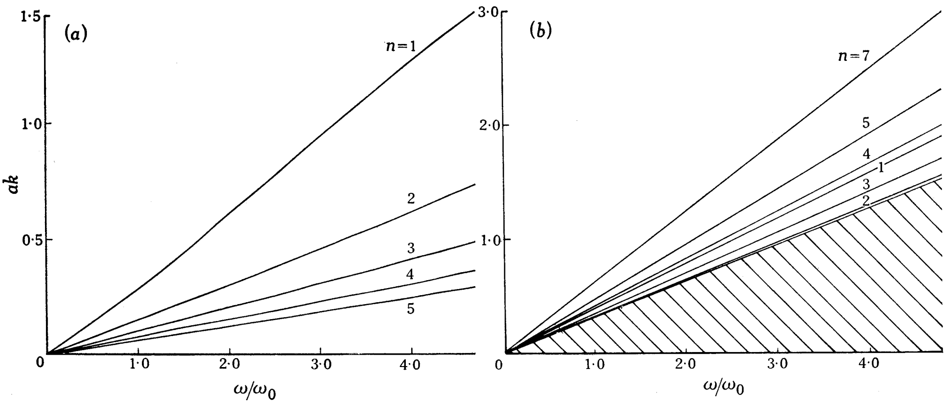


Fig. 1.—Dispersion curves for the $m = 1$ azimuthal mode showing the radial mode variation for (a) equation (7) and (b) equation (5).

With a conducting wall at $r = a$, the tangential components of the electric field must be zero and the dispersion relation is

$$\gamma_1 \mathrm{J}_m(\gamma_1 a)\left(\frac{m}{\gamma_2 a}\mathrm{J}_m(\gamma_2 a) + \frac{k}{q_2}\mathrm{J}'_m(\gamma_2 a)\right) - \gamma_2 \mathrm{J}_m(\gamma_2 a)\left(\frac{m}{\gamma_1 a}\mathrm{J}_m(\gamma_1 a) + \frac{k}{q_1}\mathrm{J}'_m(\gamma_1 a)\right) = 0\,, \tag{5}$$

with

$$q_{1,2} = \{\Omega_{\mathrm{e}}k \pm (\Omega_{\mathrm{e}}^2 k^2 - 4\omega^2\omega_{\mathrm{p}}^2/c^2)^{\frac{1}{2}}\}/2\omega\,,$$

where the plasma frequency $\omega_{\mathrm{p}} = (4\pi Ne^2/M)^{\frac{1}{2}}$, and

$$\gamma_{1,2} = (q_{1,2}^2 - k^2)^{\frac{1}{2}}\,. \tag{6}$$

Since resistivity has been neglected, $q_{1,2}$ must be purely real and hence

$$k \geqslant 2\omega\,\omega_{\mathrm{p}}/\Omega_{\mathrm{e}}c\,.$$

To compute the roots of the dispersion relation, dimensionless parameters are used and a characteristic frequency $\omega_0 = B_0 c/4\pi Nea^2$ is introduced, where a is the plasma radius.

For high electron densities $(\Omega_e/\omega_0 \rightarrow \infty)$ it can be shown that the dispersion relation (5) reduces to the result obtained by omitting the electron inertial term (Davies 1970), namely

$$(m/ak\gamma_2)\mathrm{J}_m(\gamma_2 a)+q_2^{-1}\,\mathrm{J}'_m(\gamma_2 a) = 0\,. \tag{7}$$

This dispersion relation is plotted in Figure 1(a) for a number of radial modes and for the $m = 1$ azimuthal mode.

The effect of the electron inertial term is shown in Figure 1(b), the curves representing computed roots of the dispersion relation (5) for a number of radial modes. The shaded region represents values of $k < 2\omega\,\omega_p/\Omega_e c$ for which solutions are not allowed. A comparison of Figures 1(a) and 1(b) shows that the effect of the electron inertial term is similar to a wedge being pushed into the curves from below, the first to be affected being the higher radial modes. The importance of the inertial term is increased by decreasing Ω_e/ω_0.

Asymptotic Limits

For $k \gg 2\omega\,\omega_p/\Omega_e c$ we can find asymptotic solutions. In this limit

$$aq_1 \simeq \Omega_e ak/\omega \qquad \text{and} \qquad aq_2 \simeq \omega/\omega_0 ak\,. \tag{8}$$

From equations (6) and (8) we can derive a simple form for the dispersion relation for large ak and when ω approaches Ω_e. As ak becomes large, $a\gamma_2 \sim iak$ and the Bessel function $\mathrm{J}_m(\gamma_2 a)$ can be replaced by its asymptotic form in equation (5) yielding, for $m = 1$,

$$\frac{1}{(ak)(\gamma_2 a)} - \frac{1}{(ak)(\gamma_1 a)^2} + \frac{1}{(a\gamma)(aq_2)} - \frac{1}{(aq_1)(a\gamma_1)}\frac{\mathrm{J}'_1(\gamma_1 a)}{\mathrm{J}_1(\gamma_1 a)} = 0\,. \tag{9}$$

The first two terms on the left-hand side of (9) are small for large ak whereas the third term, being independent of ak, remains finite. The last term must therefore remain finite for large ak. The coefficient of this term varies as $(ak)^{-2}$ and hence $\mathrm{J}'_1(\gamma_1 a)/\mathrm{J}_1(\gamma_1 a)$ is large for large ak. Since the function $(a\gamma_1)$ is large for large ak, the Bessel functions can be replaced by their asymptotic expressions giving

$$\mathrm{J}'_1(\gamma_1 a)/\mathrm{J}_1(\gamma_1 a) \simeq \tan(\gamma_1 a - \tfrac{1}{4}\pi)\,.$$

We therefore require $(\gamma_1 a - \frac{1}{4}\pi) \simeq (2n-1)\pi/2$, which for large n reduces to $\gamma_1 a \simeq n\pi$. Using equations (6) and (8)

$$ak \simeq (\omega/\Omega_e)n\pi\,, \tag{10}$$

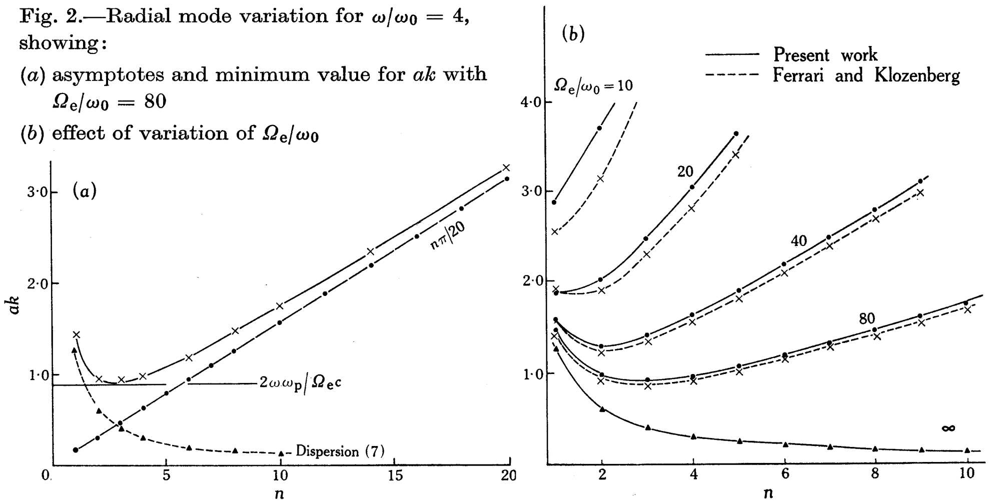
where n determines the radial mode number. Figure 2(a) shows how the higher order modes are approximated by this asymptote, whereas the low order modes are approximated by the dispersion curve neglecting electron inertial effects. The effect of the cyclotron term can be clearly seen even though the chosen values of ω/ω_0 and

Ω_e/ω_0 represent a value of $\omega/\Omega_e = 1/20$. In Figure 2(*b*) similar curves are plotted for a number of values of Ω_e/ω_0 to show the effect of this parameter on the dispersion of the waves.

Fig. 2.—Radial mode variation for $\omega/\omega_0 = 4$, showing:

(*a*) asymptotes and minimum value for ak with $\Omega_e/\omega_0 = 80$

(*b*) effect of variation of Ω_e/ω_0



Discussion

It can be seen from the curves that, over quite a large range of Ω_e/ω_0 (which includes values for typical experimental conditions), two or more differing radial modes can propagate with approximately the same wavelength. Experimental difficulties will therefore be expected to arise in trying to launch pure modes in a low density plasma with small damping. However, when appreciable resistivity is included, computed results have shown that the higher order radial modes suffer greater attenuation than the fundamental mode. The pure fundamental would therefore be expected to be observed after the wave has travelled a few wavelengths down the plasma cylinder.

It is interesting to compare the results derived from the analytic expression for the dispersion relation with the approximate form of Ferrari and Klosenberg (1968). Although their work applies to helicon waves in the range $\Omega_e \gg \nu \gg \omega$, their expansion of ak as a power series in $\xi = (\omega + i\nu)/\Omega_e$ to only the first order in ξ is remarkably accurate for ω quite close to Ω_e.

Their power series is

$$ak = K_0 + K_1 \xi + K_2 \xi^2 + \ldots,$$

where K_0 is the value of ak neglecting electron inertia and collisions and K_1 and K_2 are the first- and second-order corrections to the dispersion relation. Examining Figure 5 in their paper, the values of K_1 for the first three radial modes can be seen to be increasing as $\sim n\pi$ over a large range of ω/ω_0. To the first order we can therefore write (for negligible resistivity)

$$ak = K_0 + n\pi(\omega/\Omega_e)\,. \tag{11}$$

The second term is the same as the asymptote defined earlier in equation (10). The values of ak obtained from (11) are also plotted in Figure 2(*b*). This approximation is accurate to within a few per cent for $\Omega_e/\omega \gtrsim 20$ but for $\Omega_e/\omega \lesssim 20$ higher order terms in their expansion are required.

Since Ω_e/ω_0 is proportional to the electron density N, the effect of mode mixing will be most obvious in low density ($\sim 10^{13}$ electrons cm^{-3}) plasmas and will tend toward the case discussed by Klosenberg, McNamara, and Thonemann (1965) for high density ($\sim 10^{15}$ electrons cm^{-3}) plasmas where electron inertia plays a negligible role. This can be seen from Figure 2(*b*) for the higher values of Ω_e/ω_0.

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