GENERAL RELATIVITY IN THE EQUAL PROPER TIME FORMALISM

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Abstract

The general theory of relativity is discussed within the framework of the concept of surfaces of equal proper time as outlined in a previous paper. The three main tests of general relativity, namely the precession of the perihelion of Mercury, the gravitational shift of spectral lines, and the gravitational deflection of light rays by massive sources, are considered and it is shown that, though the modified equations are Lorentz-invariant with respect to distant observers, the deviations from conventional results are so minute as to be undetectable.

I. Introduction

In a previous paper, Cook (1972; hereinafter referred to as Paper I) gave models of special relativistic dynamics which permit the evaluation of trajectories for two interacting bodies, and, in particular, investigated the motion of planets within the context of these models. The special relativistic effects upon orbits were found to be very small. It is well known (Einstein 1915, 1916) that the effects of general relativistic considerations are detectable in the form of the advance of the perihelion of Mercury, the gravitational shift of spectral lines, and the gravitational deflection of electromagnetic waves. The effects predicted in Paper I are much smaller than these and the question naturally arises as to whether the modified theory can be made compatible with general relativity. In the present paper, the precession of the perihelion of Mercury is discussed using a combination of the theory in Paper I and general relativity.

If one considers the Schwarzschild (1916) exterior solution and considers the metric as a function of radius and angle, it can be seen that these parameters are defined relative to the centre of a massive Sun which has no motion. Should an observer A be viewing the planetary system from a distance very much greater than the radii of the planetary orbits, he would have to be at rest relative to the Sun in order to test the predictions of general relativity. On the other hand, if one wishes to know how an observer B who is moving rapidly with respect to the Sun would describe the system, the logical thing to do would be to regard the Schwarzschild coordinates as relative coordinates and carry out a Lorentz transformation upon the metric. As a simple illustration, observer A would determine a metric given by (Møller 1952)

$$\mathrm{d}\sigma^2 = (1-\alpha/r)^{-1}\,\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\phi^2 - (1-\alpha/r)\mathrm{d}t^2\,, \tag{1}$$

where α is the Schwarzschild constant, (r, θ, ϕ) are configurational polar coordinates, and t is the coordinate time. Suppose B is moving rapidly with a velocity V relative

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to A towards the system in a direction momentarily along the radius vector $\boldsymbol{r}$, at a point where θ is zero. The coordinates defined by B have the values relative to those defined by A of (Møller 1952)

$$r' = \beta(r - Vt), \qquad ct' = \beta(t - Vr/c^2), \tag{2}$$

and the metric (1) is not invariant under the Lorentz transformation (2). How then can one describe the planetary system in a way which leaves the metric unaltered by (2)? We assume that the observers A and B are so distant that the gravitational influence of the planetary system upon them is negligible, and they can be considered to be in frames of reference that are approximately inertial.

Such a model is given here. Before considering this model, however, it is necessary to discuss further the concept of equal proper time surfaces as proposed in Paper I.

II. Surfaces of Equal Proper Time

It is usually difficult to visualize the physical significance of equal proper time surfaces because of the lack of a simple reference model. Let us consider its meaning in general relativistic terms. Imagine that an observer A who wishes to investigate a particular region in space can send a shower of test particles through it and can determine the trajectory of each. A gravitational field may be present both due to the mutual interaction of the particles, which we shall ignore, or more likely due to some strong sources which may be present. To define the coordinate time t, we suppose that in a particular reference frame the space through which the particles pass contains a grid of clocks, which have all been calibrated with the aid of a slowly moving master clock, to read coordinate time t. As an illustration, though not essential to the model, let us choose a particular surface of equal coordinate time as determined by observer A to be a "calibration surface" over which he sets all test particle proper times τ_i to zero.

Let γ_j and χ be the usual vector and scalar gravitational potentials. The increment in proper time for each particle is related to the increment in coordinate time by (Møller 1952)

$$\mathrm{d}\tau_i = \mathrm{d}t_i[\{(1+2\chi/c^2)^{\frac{1}{2}} - \gamma_j v_i^j/c\}^2 - v_i^2/c^2]^{\frac{1}{2}}, \tag{3}$$

where the v_i^j are the 3-velocity components of particle i. The arc length in 4-space travelled by each particle from the calibration surface is

$$c\tau_i = c\int_0^{t_i} [\{(1+2\chi/c^2)^{\frac{1}{2}} - \gamma_j v_i^j/c\}^2 - v_i^2/c^2]^{\frac{1}{2}}\, \mathrm{d}t_i'. \tag{4}$$

If we take the points where all particles have travelled an equal distance along their particular trajectory in 4-space then, provided the distribution of velocities is known at the calibration surface and is assumed to be a given continuous function of particle velocities, the points can be taken to lie upon a "surface of equal proper time".

As an illustration of the two-dimensional case, in Figure 1 the extremities of the equal length arcs traced out by the test particles lie on a line of equal proper time. Without the specification of an initial velocity distribution, this line would not be unique. Once such a distribution $V(x)$ is given, one can idealize the situation to the results of observing many such showers, and the line of equal proper time is not only unique but is determined for all positive values of the τ_i. It can be seen from equation (4) that, by setting all of the τ_i equal over each such surface, the corresponding coordinate times t_i will in general be different. Therefore, relative time differences

$$T_{ij}(\tau) = t_i(\tau) - t_j(\tau) \tag{5}$$

arise over these surfaces that are functions of $\tau = \tau_i$.

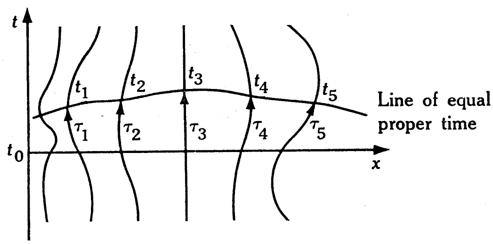

Fig. 1.—Line of equal proper time connecting the extremities of equal length arcs traced out by the test particles in two dimensions.

In relation to the two-body problem being considered by observers A and B, the notation of Paper I is used here and we adopt the relative coordinates

$$\mathbf{R} = (\boldsymbol{R}, -\mathrm{i}cT)\,, \quad S = (R^2 - c^2T^2)^{\frac{1}{2}}\,, \quad \mathbf{R} = \mathbf{r}_1 - \mathbf{r}_2\,, \quad T = t_1 - t_2\,, \tag{6a}$$

where all coordinates refer to an equal proper time surface. We also use as in Paper I, the variable γ defined by

$$S\cosh\gamma = R\,, \qquad S\sinh\gamma = cT\,. \tag{6b}$$

The principal concept outlined here, the use of surfaces of equal proper time to describe dynamical interactions, is not really practicable for distant observers such as A and B and therefore, although of some benefit in visualizing these surfaces, the concept has to be modified when one examines the motion of two distant bodies by means of light signals. Thus, instead of introducing the shower of test particles, we suppose that both observers can view many such systems and can therefore obtain some idea as to the effect of varying the initial chosen conditions. For example, if each observer finds that the motion is determined by four second-order differential equations in the four relative coordinates and proper time, then eight initial conditions per particle, or per pair of particles after centre-of-mass motion is removed, are required. These conditions can be chosen to be the initial position in 4-space and the initial velocities in 4-space. By varying these initial conditions, through examining different planetary systems, one is in effect using the masses themselves as test particles. Therefore, in discussing the motion of planet and Sun, we are referring to the coordinates which relate the position and motion of the Sun at a time when a clock there appears to read proper time τ to A and B, to the planet when it is at a position where a clock upon it also appears to read proper time τ to A and B. Naturally, the above events local to planet and Sun will not occur at the same coordinate time t in general. This idea is difficult to grasp unless one realizes that the relative interval between planet and Sun is actually

$$\mathbf{R}(\tau) = \mathbf{r}_1(t_1) - \mathbf{r}_2(t_2), \qquad t_1 = t_1(\tau), \quad t_2 = t_2(\tau),$$

and $t_1 \neq t_2$ over the surface. Specification of a continuous distribution of initial 4-velocities and a set of initial coordinates $\mathbf{R}$ then allows one to locate the surfaces of equal proper time.

III. General Relativity Considerations

To investigate general relativistic effects, it is necessary to obtain the analogue of the Schwarzschild exterior solution. As pointed out above, this solution is applicable to a static gravitational field around a massive stationary source. We require a set of two-body dynamical field equations and their solution which preserve Lorentz invariance between observers A and B. The gravitational field equations according to Einstein (1915, 1922) can be written

$$M_{ik} = -KT_{ik}, \tag{7}$$

where M_{ik} represents a system of second-order differential equations and T_{ik} is the energy–momentum tensor. The metric for the problem is obtained by assuming the existence of lengths of arc in the 5-space defined by the coordinates (R_1, R_2, R_3, T, τ) for which an element of arc is

$$\mathrm{d}\sigma^2 = b(S)(-c^2\,\mathrm{d}\tau^2) + a(S)\,G_{\mu\nu}\,\mathrm{d}R^\mu\,\mathrm{d}R^\nu, \tag{8}$$

where $G_{\mu\nu}$ is the appropriate metric tensor in 4-space. Using pseudospherical coordinates as in Paper I, we assume, in analogy with Schwarzschild (1916),

$$\begin{aligned}\mathrm{d}\sigma^2 &= -c^2 b(S)\,\mathrm{d}\tau^2 + a(S)\,\mathrm{d}S^2 - S^2\,\mathrm{d}\gamma^2 + R^2\,\mathrm{d}\theta^2 + R^2\sin^2\theta\,\mathrm{d}\phi^2 \\ &= g_{\lambda\rho}\,\mathrm{d}R^\lambda\,\mathrm{d}R^\rho, \qquad \lambda, \rho = 0,1,2,3,4,\end{aligned} \tag{9}$$

with

$$R^0 = S, \qquad R^1 = \gamma, \qquad R^2 = \theta, \qquad R^3 = \phi, \qquad R^4 = c\tau.$$

In what follows, Roman indices will indicate 4-space and Greek indices 5-space.

By comparison with the case of weak fields dealt with in Paper I, we see that the equations (7) must reduce for weak fields to

$$\Box^2\chi = 4\pi k\rho, \tag{10}$$

where ρ is the gravitational source density and k a constant. Now it can be shown from standard methods (Møller 1952) that

$$\chi = \tfrac{1}{2}c^2(-1-g_{44}), \tag{11}$$

and we can put $T_{\lambda\sigma}$ as the 5-space equivalent to T_{ik} in equations (7) with

$$T_{\lambda\sigma} = g_{\lambda\mu}\,U^\mu U^\sigma, \qquad M_{\lambda\sigma} = -KT_{\lambda\sigma}, \tag{12a, b}$$

U^μ being the 5-velocity $\mathrm{d}R^\mu/\mathrm{d}\tau$.

Equations (7) express the assumption that a differential operator acting on g_{44} is proportional to T_{44}. Therefore, following Einstein (1915, 1922), we assume

$$M_{\lambda\mu} = \mathscr{R}_{\lambda\mu} + c_1 \mathscr{R} g_{\lambda\mu} + c_2 g_{\lambda\mu}, \tag{13}$$

where the curvature tensor $\mathscr{R}_{\lambda\mu}$ is given by

$$\mathscr{R}_{\lambda\mu} = \frac{\partial \Gamma^{\sigma}_{\lambda\sigma}}{\partial R^{\mu}} - \frac{\partial \Gamma^{\sigma}_{\lambda\sigma}}{\partial R^{\sigma}} + \Gamma^{\rho}_{\lambda\sigma}\Gamma^{\sigma}_{\mu\rho} - \Gamma^{\rho}_{\lambda\mu}\Gamma^{\sigma}_{\sigma\rho}. \tag{14}$$

In terms of mixed components, we find

$$M^{\mu}_{\lambda} = \mathscr{R}^{\mu}_{\lambda} - \tfrac{1}{2}\mathscr{R}\delta^{\mu}_{\lambda} - \lambda\delta^{\mu}_{\lambda}, \tag{15}$$

where

$$\mathscr{R} = \mathscr{R}^{\lambda}_{\lambda}, \qquad \mathscr{R}^{\mu}_{\lambda} = g^{\mu\sigma}\mathscr{R}_{\lambda\sigma}, \tag{16}$$

λ is a constant of cosmological significance, and the usual Christoffel symbols are defined as

$$\Gamma_{\lambda,\mu\sigma} = \tfrac{1}{2}\left(\frac{\partial g_{\lambda\mu}}{\partial R^{\sigma}} + \frac{\partial g_{\lambda\sigma}}{\partial R^{\mu}} - \frac{\partial g_{\mu\sigma}}{\partial R^{\lambda}}\right) = g_{\lambda\rho}\Gamma^{\rho}_{\mu\sigma} \tag{17a}$$

with

$$g_{\lambda\rho}g^{\rho\sigma} = \delta^{\sigma}_{\lambda}. \tag{17b}$$

Using the relations (17) together with the metric (9), we find

$$\left.\begin{aligned} &\Gamma^{0}_{00} = a'/2a, \qquad \Gamma^{0}_{11} = S/a, \qquad \Gamma^{0}_{22} = -S\cosh^2\gamma/a, \\ &\Gamma^{0}_{33} = -S\cosh^2\gamma\sin^2\theta/a, \qquad \Gamma^{0}_{44} = b'/2a, \end{aligned}\right\} \tag{18}$$

where the prime denotes differentiation with respect to S. The components that are independent of $a(S)$ and $b(S)$ are

$$\left.\begin{aligned} &\Gamma^{1}_{01} = \Gamma^{1}_{10} = \Gamma^{2}_{02} = \Gamma^{2}_{20} = \Gamma^{3}_{03} = \Gamma^{3}_{30} = S^{-1}, \\ &\Gamma^{1}_{22} = \sinh\gamma\cosh\gamma, \quad \Gamma^{2}_{12} = \Gamma^{2}_{21} = \Gamma^{3}_{13} = \Gamma^{3}_{31} = \tanh\gamma, \\ &\Gamma^{1}_{33} = \sinh\gamma\cosh\gamma\sin^2\theta, \quad \Gamma^{2}_{33} = -\sin\theta\cos\theta, \quad \Gamma^{3}_{23} = \Gamma^{3}_{32} = \cot\theta. \end{aligned}\right\} \tag{19}$$

The remaining components are zero.

Employing the metric (9) and equations (16) and (17), we obtain

$$\mathscr{R}^{0}_{0} = \frac{b''}{2ab} - \frac{b'^2}{4ab^2} - \frac{a'b'}{4a^2b} - \frac{3a'}{2a^2S}, \tag{20a}$$

$$\mathscr{R}^{1}_{1} = \frac{2}{S^2}\left(\frac{1}{a} - 1\right) + \frac{1}{aS}\left(\frac{a'}{2a} + \frac{b'}{2b}\right) - \frac{a'}{a^2S} = \mathscr{R}^{2}_{2} = \mathscr{R}^{3}_{3}, \tag{20b}$$

$$\mathscr{R}^{4}_{4} = \frac{b''}{2ab} - \frac{b'^2}{4ab^2} - \frac{a'b'}{4a^2b} + \frac{3b'}{2abS}, \tag{20c}$$

and from (16)

$$\mathscr{R} = \frac{b''}{ab} - \frac{b'^2}{2ab^2} - \frac{a'b'}{2a^2b} - \frac{3a'}{a^2S} + \frac{3b'}{abS} + \frac{6}{S^2}\left(\frac{1}{a} - 1\right). \tag{21}$$

Thus from (12b) we have

$$M_0^0 = -\frac{3b'}{2abS} + \frac{3}{S^2}\left(1 - \frac{1}{a}\right) - \lambda = -KT_0^0, \tag{22a}$$

$$M_1^1 = M_2^2 = M_3^3$$

$$= -\frac{b''}{2ab} + \frac{b'^2}{4ab^2} + \frac{a'b'}{4a^2b} + \frac{a'}{a^2S} + \frac{2b'}{abS} - \frac{1}{S^2}\left(\frac{1}{a} - 1\right) - \lambda = -KT_1^1, \tag{22b}$$

$$M_4^4 = \frac{3a'}{2a^2S} + \frac{3}{S^2}\left(1 - \frac{1}{a}\right) - \lambda = -KT_4^4. \tag{22c}$$

It was shown in Paper I that, in order to obtain the exact analogue of the 3-space potential from the electromagnetic field equations, it appears necessary to postulate that the "vacuum" specified by letting the masses tend to zero is such that when either mass is nonzero a source term is present. To obtain the correct S^{-1} behaviour for the potential, it is necessary to make this source

$$T_0^0 = T_4^4 = H/S^3, \qquad T_1^1 = T_2^2 = T_3^3 = 0, \tag{23}$$

where H is a constant to be determined. Such a source obeys the continuity conditions (Møller 1952)

$$|g|^{-\frac{1}{2}}\partial(|g|^{\frac{1}{2}}T_\lambda^\mu)/\partial R^\mu = 0, \tag{24}$$

where $g = \det \mathbf{g} = abS^6 \cosh^4\gamma \sin^2\theta$ and we make use of the fact that ab is constant. The latter can be proved by subtracting (22c) from (22a) to obtain

$$ab' + a'b = 0.$$

We choose $ab = 1$. However, from equations (22) we also find that for any values of a and b

$$\tfrac{1}{3}SM_4'^4 + M_4^4 = M_1^1 = M_2^2 = M_3^3 = 0. \tag{25}$$

The three vanishing source terms therefore follow from the values of T_0^0 and T_4^4 chosen to keep equation (12b) consistent.

To solve equations (22), we put $y = a^{-1}$ and obtain

$$M_4^4 = -\tfrac{3}{2}S^{-1}y' + 3S^{-2}(1-y) - \lambda = -KT_4^4, \tag{26}$$

which, upon substituting the relations (23) and integrating, yields

$$b = a^{-1} = 1 - \tfrac{1}{6}\lambda S^2 + \tfrac{2}{3}KHS^{-1} + BS^{-2}, \tag{27}$$

where B is a constant of integration. Since no S^{-3} force is observed, we put $B = 0$.

In the weak field approximation, using the reduced mass μ of the system, we must obtain the Newtonian-like potential of Paper I. Therefore, using the Newtonian gravitational constant G, we have

$$-\tfrac{2}{3}\mu KH = Gm_1 m_2/c^2 . \tag{28}$$

However, from Einstein (1916), we assume that the equations should yield

$$K \approx G/8\pi c^2 \tag{29}$$

and we have

$$-H = 12\pi(m_1+m_2) . \tag{30}$$

It therefore follows that $|H|$ is 12π times the total rest mass of the particles. If both masses tend to zero, this term vanishes.

We note that τ here has the significance of being the proper time as determined by the observer A or B. Since A is at rest relative to the Sun, τ is approximately equal to t, the usual coordinate time, in his frame of reference. The observer B will not, however, observe this relation if he moves rapidly relative to A. Observer A can then make the approximations $\gamma \approx 0$, $S \approx r$, and $\mathrm{d}\gamma^2 \approx 0$, and will find that Einstein's model is correct.

IV. Comparison with Results of General Relativity

(a) *Calculation of Precession*

Putting

$$\alpha = 2G(m_1+m_2)/c^2 , \tag{31}$$

we obtain the equations of motion, as demonstrated by Møller (1952), of

$$(S^2 \cosh^2\gamma)\dot{\phi}^2/(1-\alpha S) = \text{const.}, \qquad (\Lambda/A)\tan\gamma = \cos\phi , \tag{32}$$

where Λ and A are the constants of motion defined in Paper I, the motion is confined to the plane $\theta = \frac{1}{2}\pi$, and the angular velocity

$$\dot{\omega} = (\cosh^2\gamma)\dot{\phi} = \dot{\phi}/\{1-(A^2/\Lambda^2)\cos^2\phi\} \tag{33}$$

is used throughout the derivation. We then arrive at the additional precession for one rotation in terms of ω as

$$\Delta\omega = \tfrac{3}{2}\pi\alpha(S_1^{-1}+S_2^{-1}) , \tag{34}$$

where S_1 and S_2 are the maximum and minimum values of S respectively. It follows that the observed precession is

$$\Delta\phi = \Delta\omega\{1-(A^2/\Lambda^2)\cos^2\phi\} . \tag{35}$$

This leads to a correction at maximum deviation of 1 part in 4×10^8 for Mercury. Such a small discrepancy is unlikely to be observable, and therefore the above theory agrees closely with standard general relativity. It has the added feature, however, that $\Delta\omega$ in equation (34) is a Lorentz-invariant quantity, and the observers A and B discussed in the previous sections would determine the same equations of motion and the same value for $\Delta\omega$, which, for example, is $42 \cdot 9$ min of arc per century in the case of Mercury.

(*b*) *Other General Relativistic Effects*

From the theory outlined in Paper I, it is straightforward to show that the gravitational shift of spectral lines and the gravitational deflection of light signals are almost identical with the conventional results. For example, in the former case we obtain (Møller 1952) for observer A, situated distant from a light source m_1 and observing light emitted from it,

$$\frac{\Delta\nu}{\nu_0} = -\frac{Gm_1}{c^2}\left(\frac{1}{S_1} - \frac{1}{S_{1\mathrm{A}}}\right), \qquad m_2 \ll m_1, \quad S_{1\mathrm{A}} \gg S_1, \tag{36}$$

where $\Delta\nu$ is the frequency shift from the vacuum value due to the presence of the source, ν_0 the vacuum spectral line frequency of the gravitational source, S_1 the covariant interval from the centre of mass of m_1 to its surface, and $S_{1\mathrm{A}}$ the covariant interval from the centre of mass of m_1 to observer A. Observer A would therefore find corrections only of the order of 10^{-8} to the conventional result. Observer B would, of course, see conventional special relativistic Doppler line shifts, because of his motion relative to A.

To consider the gravitational deflection of light, we assume that m_2 is zero and that the light signal obeys Fermat's principle (Møller 1952) in the relative 4-space. We then obtain

$$\frac{\mathrm{d}(g_{ik}\dot{R}^k)}{\mathrm{d}\lambda} - \frac{1}{2}\frac{\partial g_{kl}}{\partial R^l}\dot{R}^k\dot{R}^l = -\frac{1}{w}\frac{\partial w}{\partial R^l}, \tag{37}$$

where $w = c(1-\alpha/S)^{\frac{1}{2}}$ is the proper velocity of light in the medium. For $i,k = 2,3$ equation (37) just gives the conventional result for an angular deflection $\Delta\psi$ of

$$\Delta\psi = 4Gm_1/c^2S_\mathrm{m}, \tag{38}$$

where S_m is the minimum proper distance between the light ray and mass m_1. There appears to be a correction of the order of 10^{-8} to the result (38) for observer A since S_m is not quite equal to the usual radius $r_\mathrm{m} = |\,\boldsymbol{r}_1(t) - \boldsymbol{r}_2(t)\,|$. Observer B would see a different deflection owing to the fact that the incident and deflected light waves undergo special relativistic aberration (McCrea 1954). The deflection for observer B can easily be calculated from standard formulae.

V. Conclusions

It has been shown that the theory presented by Cook (1972) is not so much inconsistent with general relativity but forms distinct models that are valid within the framework of special relativity. General relativity can be encompassed by such models, but it must be emphasized that the corrections found are so small that at present the theory remains only a formalism which cannot do more to conventional models than make them Lorentz-invariant.

VI. References

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