COMPOUND NUCLEUS FORMULATION OF REACTION MATRIX THEORY

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Abstract

It is shown that multilevel resonance parameters for each element of the reaction matrix cannot be determined from available data. However, additional constraints may be introduced without affecting agreement with experiment. The Bohr compound nucleus hypothesis, which states that the modes of formation and decay of a compound nucleus are independent, is applied to the T-matrix and it is found, as in Newton's model, that the channel matrix can be inverted analytically to provide simple formulae for cross sections, for both the real Wigner–Eisenbud reaction matrix and Moldauer's complex reaction matrix. Wigner–Eisenbud theory leads directly to Newton's strong correlation model and its unacceptable consequences. Moldauer's theory does not, however, and can explain cross section behaviour adequately while being consistent with Bohr's hypothesis. Cross sections can be written as a sum of single level contributions, as in the Adler–Adler formulation. Finally, Moldauer's statistical theory is shown to be applicable, and expressions are derived for the averaged cross sections as functions of the complex Moldauer resonance parameters.

I. Introduction

The concept of a compound nucleus was introduced by Bohr (1936), and for some time it was regarded as the principal nuclear reaction mechanism. At about the same time Breit and Wigner (1936) proposed their celebrated single level formula, which gave excellent agreement with isolated levels in cross sections and satisfied the Bohr compound nucleus postulate. Much later, Wigner and Eisenbud (1947) presented a more rigorous multilevel theory which was capable of explaining the strong interference between levels observed in fission cross sections. This theory did not employ the Bohr postulate concerning the resonant cross sections, except to assume the existence of a single internal compound state wavefunction and its associated Hamiltonian. There occurred in this theory a troublesome channel matrix which had to be inverted, and Wigner and Eisenbud gave an expansion for this inversion. Later refinements of the multilevel formalism by Vogt (1958) and Reich and Moore (1958) used the concept of replacing the channel matrix to be inverted by a level matrix and then employing approximations to evaluate the inverse. None of these multilevel theories used the Bohr assumption.

A multilevel theory which does lead directly to the Bohr assumption is the theory employing strong correlation conditions put forward by Teichmann (1950) and Newton (1952). This theory was discarded by later authors because the assumed

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correlations were not observed in experimentally determined widths, and because all cross sections were predicted to vanish between resonances. Lane and Thomas (1958) concluded from this that the Bohr postulate must be incorrect as an exact relation, and instead holds for averaged cross sections over a large number of resonances. It is pointed out in the present paper that this is not the only alternative. If future experiments should reveal that the Bohr postulate is in fact an exact relationship, one can accommodate this conclusion within the framework of reaction matrix theory by using the complex R-matrix defined by Moldauer (1964). This does not imply the strong correlations assumed by Newton and the only difficulty encountered is that the single level approximation no longer satisfies unitarity, which requires compensating multilevel effects.

From a pragmatic point of view, the theories of Vogt (1958) and Reich and Moore (1958) suffer from the defect that the formulae obtained are so complicated that they are generally not useful for analytical calculations of resonance absorption in reactors, and analytic expressions for the Doppler-broadened cross sections do not exist. However, the formalism proposed by Adler and Adler (1963) is equivalent to the above theories, does allow analytic formulae to be written for Doppler-broadened cross sections, and lends itself readily to calculations of resonance absorption in reactors since it represents cross sections as sums over single level contributions. We show here that, if the Bohr hypothesis is employed, the theory of Adler and Adler remains exact over restricted energy ranges.

Most authors would agree that in neutron resonance reactions the available measurements are insufficient to determine every element of the reaction matrix. This point is discussed in detail in Section III and a transformation between equivalent reaction matrices is given. Furthermore, with the Bohr hypothesis, and unitarity, we can construct a reduced transition matrix of rank one in which all channels have the same phase, called the compound phase. It follows that the Wigner–Eisenbud (1947) or Moldauer (1964) reaction matrix is of rank one. Using this property in Section IV, the channel matrix is inverted to yield an expression for the scattering matrix in which the compound phase is given as a function of the reaction matrix. Formulae for cross sections are then derived which are manifestly consistent with Bohr's hypothesis.

Conditions for the general reaction matrix which ensure that the collision matrix is symmetric and unitary are derived in Section VI. The Wigner–Eisenbud (1947) boundary conditions lead inevitably to the strong correlation model of Newton with its unsatisfactory features, but Moldauer's (1964) complex boundary conditions permit the absence of cross section zeros and do not imply the strong correlation conditions.

The poles and residues of the S-matrix are evaluated in Section VII and multilevel effects are found to alter the positions of complex poles from their values in the single level approximation. This approximation violates unitarity in the Moldauer theory. It is demonstrated in Section VIII that the cross sections can be parameterized exactly as a sum of single level contributions with asymmetric correction terms.

Finally, in Section IX we show that the formal statistical theory given by Moldauer (1964) is applicable to our representation of cross sections. His equations are reviewed and presented in our terminology.

II. Standard Theory

In terms of the **S** matrix, the cross section for a reaction from an incident channel c to a final channel c' is given by (Preston 1962)

$$\sigma_{cc'} = (\pi/k_c^2) \sum_{ljJl'j'} g_J \, | \, \delta_{clj,c'l'j'} - S^J_{clj,c'l'j'} \, |^2 , \tag{1}$$

where

$$k_c^2 = 2\mu_c E_c , \qquad g_J = (2J+1)/2(2I+1) ,$$

E_c is the energy in channel c, μ_c the reduced mass in channel c, l the orbital angular momentum, j the channel spin, I the spin of the target nucleus, and J the total angular momentum. The total cross section may be expressed as

$$\sigma_{c\mathrm{T}} = (2\pi/k_c^2) \sum_{ljJ} g_J \{1 - \mathrm{Re}(S^J_{clj,clj})\} . \tag{2}$$

The transition matrix **T** is defined as

$$\mathbf{T} = (\mathbf{S} - \mathbf{I})/2\mathrm{i} . \tag{3}$$

Since the **S** matrix is unitary and symmetric, that is,

$$\mathbf{S}\dagger\,\mathbf{S} = \mathbf{S}\,\mathbf{S}\dagger = \mathbf{I} , \tag{4}$$

it follows that

$$\mathrm{Im}\,\mathbf{T} = \mathbf{T}\dagger\,\mathbf{T} . \tag{5}$$

The **S** matrix is related to the reaction matrix **R** by the equation

$$\mathbf{S} = \boldsymbol{\Omega}[\mathbf{I} + 2\mathrm{i}\,\mathbf{P}^{\frac{1}{2}}(\mathbf{I} - \mathbf{R}\,\mathbf{L}^0)^{-1}\,\mathbf{R}\,\mathbf{P}^{\frac{1}{2}}]\boldsymbol{\Omega} , \tag{6}$$

where

$$R_{cc'} = \sum_{\lambda} \gamma_{\lambda c}\gamma_{\lambda c'}/(E_\lambda - E) , \tag{7}$$

the constants $\gamma_{\lambda c}$ and E_λ, the energy at the poles in **R**, being real in the Wigner–Eisenbud (1947) theory and complex in the Moldauer (1964) theory. In equation (6)

$$\mathbf{L}^0 = \mathbf{L} - \mathbf{B} \qquad \text{with} \qquad \mathbf{L} = \mathbf{s} + \mathrm{i}\mathbf{P} ;$$

$\boldsymbol{\Omega}$, **P**, and **s** are diagonal matrices with elements $\Omega_c = \exp(-\mathrm{i}\phi_c)$, P_c the penetration factor, and s_c the level shift respectively; and $B_{cc'}$ are elements of the boundary condition matrix **B**. In the above two theories

$$B_{cc'} = B_c \, \delta_{cc'} ,$$

where B_c is real in the Wigner–Eisenbud theory and complex in the Moldauer theory. The evaluation of these quantities has been reported by many authors (e.g. Lane and Thomas 1958; Preston 1962; Lynn 1968).

The transition matrix is then given by

$$\mathbf{T} = (\boldsymbol{\Omega}^2 - \mathbf{I})/2\mathrm{i} + \boldsymbol{\Omega}\,\mathbf{P}^{\frac{1}{2}}(\mathbf{I} - \mathbf{R}\,\mathbf{L}^0)^{-1}\,\mathbf{R}\,\mathbf{P}^{\frac{1}{2}}\boldsymbol{\Omega} . \tag{8}$$

It is also convenient to define the reduced transition matrix $\mathbf{T}'$ through the relation

$$\mathbf{T} = (\mathbf{\Omega}^2 - \mathbf{I})/2i + \mathbf{\Omega}\,\mathbf{T}'\,\mathbf{\Omega}, \tag{9}$$

and then

$$\mathbf{T}' = \mathbf{P}^{\frac{1}{2}}(\mathbf{I} - \mathbf{R}\,\mathbf{L}^0)^{-1}\,\mathbf{R}\,\mathbf{P}^{\frac{1}{2}}. \tag{10}$$

The "reduced $\mathbf{S}$ matrix" $\mathbf{V}$ obeys the relations

$$\mathbf{V} = \mathbf{\Omega}^{-1}\,\mathbf{S}\,\mathbf{\Omega}^{-1}, \qquad V_{\lambda\mu} = V_{\mu\lambda}, \tag{11}$$

$$= \mathbf{I} + 2i\mathbf{T}' \tag{12}$$

and is unitary. Therefore $\mathbf{T}'$ satisfies an equation similar to (5), namely

$$\operatorname{Im}\mathbf{T}' = \mathbf{T}'\dagger\,\mathbf{T}'. \tag{13}$$

The complexity of multichannel expressions for cross sections arose from the difficulty in inverting the channel matrix $(\mathbf{I} - \mathbf{R}\,\mathbf{L}^0)^{-1}$.

Let

$$\mathbf{Z}_1 = \mathbf{P}^{\frac{1}{2}}\,\mathbf{R}(\mathbf{L} - \mathbf{B})\mathbf{P}^{-\frac{1}{2}}, \qquad \mathbf{Z}_2 = \mathbf{P}^{\frac{1}{2}}\,\mathbf{R}(\mathbf{L}^* - \mathbf{B})\mathbf{P}^{-\frac{1}{2}}, \tag{14}$$

where the asterisk denotes complex conjugation. It then follows from equations (8) and (11) that

$$\mathbf{V} = (\mathbf{I} - \mathbf{Z}_1)^{-1}(\mathbf{I} - \mathbf{Z}_2). \tag{15}$$

Applying the unitary condition

$$\mathbf{V}\dagger\,\mathbf{V} = \mathbf{I}, \tag{16}$$

one finds after some manipulation that the conditions under which (16) should hold are

$$(\mathbf{I} - \mathbf{Z}_1)(\mathbf{I} - \mathbf{Z}_1\dagger) = (\mathbf{I} - \mathbf{Z}_2)(\mathbf{I} - \mathbf{Z}_2\dagger),$$

which yields

$$\mathbf{R} - \mathbf{R}\dagger = \mathbf{R}(\mathbf{B} - \mathbf{B}\dagger)\mathbf{R}\dagger. \tag{17}$$

Similarly, the conditions under which $\mathbf{V}$ should be a symmetric matrix, that is, $\mathbf{V} = \mathbf{V}^{\mathrm{t}}$ where the superscript t denotes the operation of transposition, are

$$(\mathbf{I} - \mathbf{Z}_1)(\mathbf{I} - \mathbf{Z}_1^{\mathrm{t}}) = (\mathbf{I} - \mathbf{Z}_2)(\mathbf{I} - \mathbf{Z}_2^{\mathrm{t}}),$$

which gives the condition

$$\mathbf{R} - \mathbf{R}^{\mathrm{t}} = \mathbf{R}(\mathbf{B} - \mathbf{B}^{\mathrm{t}})\mathbf{R}^{\mathrm{t}}. \tag{18}$$

It follows immediately from equations (17) and (18) that if the Wigner–Eisenbud (1947) boundary conditions are used then

$$\mathbf{B} = \mathbf{B}\dagger, \quad \mathbf{B}^{\mathrm{t}} = \mathbf{B}, \qquad \mathbf{R} = \mathbf{R}\dagger, \quad \mathbf{R}^{\mathrm{t}} = \mathbf{R}, \tag{19}$$

and the Wigner–Eisenbud $\mathbf{R}$ matrix is real and symmetric. In Moldauer's (1964) theory, $\mathbf{B}$ is no longer self-adjoint but is still symmetric so that the Moldauer $\mathbf{R}$ matrix is a complex symmetric matrix obeying the relations

$$\mathbf{R} - \mathbf{R}^* = \mathbf{R}(\mathbf{B} - \mathbf{B}^*)\mathbf{R}^*, \qquad \mathbf{R} = \mathbf{R}^{\mathrm{t}}. \tag{20}$$

III. Uniqueness Question

The main value of multilevel theories is that they show how to parameterize the collision matrix as a function of energy. Though it is not possible in neutron reactions to determine every element of the reaction matrix, Vogt (1958) and Reich and Moore (1958) have shown that, within the framework of the level matrix expansion method, sums of reduced widths over channels arise which can be used as parameters. It is clear that in being forced to use these sums as parameters rather than individual reduced widths we do not have the necessary information from experiment to determine each element of the reaction matrix.

To discuss this in more detail, let us consider the analysis of experimental cross sections whose purpose is to calculate the reaction matrix in some multilevel scheme. For such an analysis, the differential cross section σ_{11} for elastic scattering and the various total reaction cross sections $\sum_c \sigma_{1c}$ are usually available. Initially, there are no restrictions on the elements of the **S** matrix other than those imposed by symmetry and unitarity. If we consider the transition matrix **T**, this means that at best from the given data we can determine only the quantities $\sum_c |T_{1c}|^2$ or, if in the extreme case each cross section σ_{1c} is available, we know only $|T_{1c}|$. For the partial wave amplitude T_{11}, each phase shift as an eigenstate of (l,J) is known except for an arbitrary common phase which is usually set equal to zero. Therefore we shall assume that we can determine the phase of T_{11} but not of any other element. Thus, if a reaction matrix **R** has been obtained from the given data and if **T** is the corresponding transition matrix with elements $T_{cc'} = t_{cc'}\exp(i\theta_{cc'})$, then in general it is possible to find another reaction matrix $\tilde{\mathbf{R}}$, which leads to a transition matrix $\tilde{\mathbf{T}}$ with elements $\tilde{T}_{cc'} = \tilde{t}_{cc'}\exp(i\theta_{cc'})$, such that this too will reproduce the given cross sections. The matrices **R** and $\tilde{\mathbf{R}}$ may then be related by introducing a symmetric matrix **A** with elements

$$A_{cc'} = a_{cc'}\exp(i\phi_{cc'})$$

such that

$$\tilde{\mathbf{T}} = \mathbf{T}+\mathbf{A}. \tag{21}$$

Since **S** and $\tilde{\mathbf{S}}$ must both be unitary, **A** must satisfy

$$(\mathbf{S}\dagger\mathbf{A})\dagger(\mathbf{S}\dagger\mathbf{A}) = \mathrm{Im}(\mathbf{S}\dagger\mathbf{A}).$$

The matrix $\tilde{\mathbf{T}}$ will reproduce the original data if

$$a_{11} = 0\,; \qquad a_{1c} = t_{1c}\{2-2\cos(\tilde{\theta}_{1c}-\theta_{1c})\}^{\frac{1}{2}}, \quad c \neq 1\,;$$

and

$$\tan\phi_{1c} = (\cos\tilde{\theta}_{1c}-\cos\theta_{1c})/(\sin\tilde{\theta}_{1c}-\sin\theta_{1c}).$$

The matrices **R** and $\tilde{\mathbf{R}}$ can then be related through equations (8) and (21).

It therefore follows that, even in the extreme case where every reaction cross section is known, the reaction matrix is not uniquely determined by the available data. However, the Wigner–Eisenbud (1947) form for **R**, given by equation (7), does introduce additional constraints upon the **S** matrix. In this matrix, there are n channels and $2n^2$ functions. Unitarity and symmetry imply that only $\frac{1}{2}n(n+1)$ of

these functions are independent. Without further constraints, the Wigner–Eisenbud **R** matrix would be real and symmetric and therefore would consist of $\frac{1}{2}n(n+1)$ independent functions. However, the special form for **R** in equation (7) states that if we specify n components as a function of energy, say along a row, then all of the Wigner–Eisenbud resonance parameters are determined and thereby all elements of **R** are determined. In this case, there are only n independent functions which specify the **S** matrix. This is also the number of cross sections and it follows that the n independent functions σ_{1c} determine the **S** matrix completely. The Wigner–Eisenbud derivation of equation (7) imposes $\frac{1}{2}n(n-1)$ constraints in addition to unitarity and symmetry. In this sense, the Wigner–Eisenbud theory is not a general one, as such a theory would not have these constraints. The generalized Wigner–Eisenbud reaction matrix would be of the form

$$R_{cc'} = \sum_{\lambda} a_{\lambda cc'}/(E_{\lambda cc'}-E), \qquad a_{\lambda cc'} = a_{\lambda c'c}, \quad E_{\lambda cc'} = E_{\lambda c'c}, \tag{22}$$

where $a_{\lambda cc'}$ and $E_{\lambda cc'}$ are real constants.

The above constraints result from the definition of internal states by real boundary conditions applied at some appropriate radius, not necessarily the nuclear surface. Such questions as to how one allows for the variation in nuclear density near the surface can be overcome by choosing the radius to be the minimum value where the nuclear density is negligible. Cook (1968) has shown that the form of equation (22) can be produced by consistent models more general than the above.

The additional constraints of the Wigner–Eisenbud theory are that the residues $a_{\lambda cc'}$ form a matrix of rank one in channel space and that the eigenvalues $E_{\lambda cc'}$ are independent of c and c'. If we consider the case where these assumptions are valid, even then in actual measurements we can only determine some number p of sums over the cross sections σ_{1c}, where $p < n$. Therefore, $n-p$ functions remain undetermined in the **S** matrix. For radiative capture processes, usually n is of the order of 100 and $p = 2$ or 3, and this leads to a large uncertainty in the phases of the **T** matrix. For Moldauer's (1964) theory, there are $2n$ functions to be determined, and only n functions σ_{1c}. We would therefore require both the magnitude and phase of each T_{1c} to determine $R_{cc'}$, and this is clearly impossible. We can therefore conclude that to date, and in the foreseeable future, limitations in the data will never allow the determination of a unique **R** matrix for low energy neutron collisions with nuclei. Instead, one should use multilevel theories as a guide to ascertain the functional form of cross sections, and this form should be simplified as much as possible.

IV. Compound Nucleus Assumption

Applying Bohr's compound nucleus hypothesis to the resonant part $\sigma^{\mathrm{r}}_{cc'}$ of all cross sections, we can write (Blatt and Weisskopf 1952)

$$\sigma^{\mathrm{r}}_{cc'} = \sigma_c(c)\,\mathscr{P}_{c'}(c), \tag{23}$$

where $\sigma_c(c)$ is the cross section for formation of the compound nucleus in channel c and $\mathscr{P}_c(c)$ is the probability of decay of the compound nucleus via channel c, with

$\sum_c \mathscr{P}_c(c) = 1$. From reciprocity it follows that

$$\mathscr{P}_c(c) = k_c^2 \sigma_c(c) \Big/ \left(\sum_{c'} k_{c'}^2 \sigma_{c'}(c) \right).$$

Writing

$$G^2 = (1/4\pi) \sum_{c'} k_{c'}^2 \sigma_{c'}(c),$$

equation (23) may be expressed in the form

$$\sigma_{cc'}^{\mathrm{r}} = (4\pi/k_c^2) G^2 \mathscr{P}_c \mathscr{P}_{c'}. \tag{24}$$

We define the resonant part of the cross section to be the contribution from the reduced transition matrix $\mathbf{T}'$:

$$\sigma_{cc'}^{\mathrm{r}} = (4\pi/k_c^2) |\, T'_{cc'} \,|^2. \tag{25}$$

Thus comparing (24) and (25) we find that

$$T'_{cc'} = \rho_c \rho_{c'} \exp(\mathrm{i}\theta_{cc'}), \qquad \rho_c = (G\mathscr{P}_c)^{\frac{1}{2}}. \tag{26a,b}$$

For the phases $\theta_{cc'}$, the unitarity relation (13) yields the equations

$$\sum_{c''} \rho_{c''}^2 \cos(\theta_{cc''} - \theta_{c''c'}) = \sin\theta_{cc'}, \qquad \sum_{c''} \rho_{c''}^2 \sin(\theta_{cc''} - \theta_{c''c'}) = 0; \tag{27a, b}$$

when $c = c'$, equation (27a) immediately yields

$$\sin\theta_{cc} = \sum_{c''} \rho_{c''}^2.$$

Following Newton (1952), we define the compound phase by the equations

$$\sin\delta = G = \sum_c \rho_c^2, \qquad \cos\delta = (1 - G^2)^{\frac{1}{2}}.$$

The solutions of equations (27a) and (27b) for the phases of the diagonal elements of $\mathbf{T}'$ then are

$$\theta_{cc} = \delta \quad \text{or} \quad \pi - \delta.$$

Newton maintained that only the first solution exists. This is not so, as can be illustrated by the counter example for $n = 2$ where the equations (27) can be solved quite easily to yield two distinct solutions, namely

$$\theta_{11} = \theta_{12} = \theta_{22} = \delta,$$

which is Newton's solution, and

$$\theta_{11} = \delta, \qquad \theta_{22} = \pi - \delta, \qquad \tan\theta_{12} = (\rho_1^2 - \rho_2^2)/\cos\delta.$$

As far as calculating cross sections is concerned, the above two solutions are completely equivalent as both will give the same value for the resonant part of the reaction cross section $\sigma_{cc'}^{\mathrm{r}}$. In the general case, when the number of channels is

arbitrary, equations (27) will have a large but nevertheless finite number of solutions, all of which, as far as the analysis of cross sections is concerned, are completely equivalent. Newton's (1952) solution with all $\theta_{cc'} = \delta$ is a particularly interesting one as it leads to a simple R-matrix theory. The choice of this solution in fact constitutes the equiphase principle investigated by Cook (1967*b*). This principle, which has been applied to the photoproduction of pions in relation to scattering (Bethe and de Hoffman 1955) has also often been applied, although perhaps unwittingly, to low energy nuclear reactions. This is because the most widely used form of R-matrix theory, the single level approximation, embodies this same principle.

Thus we have that the reduced transition matrix is of the form

$$T'_{cc'} = \rho_c \rho_{c'} \exp(i\delta) = (\mathscr{P}_c \mathscr{P}_{c'})^{\frac{1}{2}} \exp(i\delta) \sin\delta \tag{28}$$

and is therefore of rank one. It can now be shown that $\mathbf{R}$, whether real or complex, must necessarily be of unit rank also.

From equation (10) we find

$$\mathbf{R} = \mathbf{P}^{-\frac{1}{2}} \mathbf{T}' \mathbf{P}^{-\frac{1}{2}} [\mathbf{I} + \mathbf{L}^0 \mathbf{P}^{-\frac{1}{2}} \mathbf{T}' \mathbf{P}^{-\frac{1}{2}}]^{-1} = \mathbf{A}[\mathbf{I}+\mathbf{C}]^{-1},$$

where

$$\mathbf{A} = \mathbf{P}^{-\frac{1}{2}} \mathbf{T}' \mathbf{P}^{-\frac{1}{2}}, \qquad \mathbf{C} = \mathbf{L}^0 \mathbf{A}. \tag{29}$$

Since the matrices $\mathbf{L}^0$ and $\mathbf{P}$ are diagonal, we find that because $\mathbf{A}$ and $\mathbf{C}$ are both of unit rank

$$\mathbf{R} = \mathbf{A}[\mathbf{I}-\mathbf{C}+\mathbf{C}^2-\mathbf{C}^3+\ldots] = \mathbf{A}[\mathbf{I}-\mathbf{C}(1+\mathrm{trace}\,\mathbf{C})^{-1}]$$

$$= (\mathbf{L}^0)^{-1} \mathbf{C}(1+\mathrm{trace}\,\mathbf{C})^{-1}.$$

It follows that $\mathbf{R}$ must also be of unit rank. This leads to the results

$$(\mathbf{R}\mathbf{L}^0)^n = \{\mathrm{trace}(\mathbf{R}\mathbf{L}^0)\}^{n-1} \mathbf{R}\mathbf{L}^0, \tag{30a}$$

$$(\mathbf{I}-\mathbf{R}\mathbf{L}^0)^{-1} = \mathbf{I}+\mathbf{R}\mathbf{L}^0\{1-\mathrm{trace}(\mathbf{R}\mathbf{L}^0)\}^{-1}, \tag{30b}$$

$$(\mathbf{I}-\mathbf{R}\mathbf{L}^0)^{-1}\mathbf{R} = \mathbf{R}\{1-\mathrm{trace}(\mathbf{R}\mathbf{L}^0)\}^{-1}. \tag{30c}$$

We see from the Bohr postulate (23) that the previously difficult inversion of the channel matrix in equation (30c) has become a simple analytical result. The $\mathbf{S}$ matrix then becomes

$$\mathbf{S} = \boldsymbol{\Omega}[\mathbf{I}+2i\,\mathbf{P}^{\frac{1}{2}}\mathbf{R}\,\mathbf{P}^{\frac{1}{2}}\{1-\mathrm{trace}(\mathbf{R}\mathbf{L}^0)\}^{-1}]\boldsymbol{\Omega}, \tag{31}$$

or

$$S_{cc'} = \Omega_c \Omega_{c'} \delta_{cc'} + 2i\Omega_c \Omega_{c'} (P_c P_{c'})^{\frac{1}{2}} R_{cc'} \{1-\mathrm{trace}(\mathbf{R}\mathbf{L}^0)\}^{-1}$$

$$= \Omega_c \Omega_{c'} \delta_{cc'} + 2i\Omega_c \Omega_{c'} (P_c |R_{cc}|)^{\frac{1}{2}} (P_{c'} |R_{c'c'}|)^{\frac{1}{2}} \exp(i\alpha_{cc'}) \{1-\mathrm{trace}(\mathbf{R}\mathbf{L}^0)\}^{-1}, \tag{32}$$

where

$$R_{cc'} = |R_{cc'}| \exp(i\alpha_{cc'}), \qquad |R_{cc}||R_{c'c'}| = |R_{cc'}|^2, \tag{33}$$

since the matrix $\mathbf{R}$ is of rank one. In the Wigner–Eisenbud theory, $\alpha_{cc'} = 0$. For the Moldauer theory, we can simplify the model by writing the boundary condition

matrix as

$$B_{cc'} = B_c \delta_{cc'} + i\epsilon\delta_{cc'}, \tag{34}$$

where B_c and ϵ are real constants. We then note from the unitarity requirement (20a) that the equiphase solution for $\mathbf{R}$ is

$$R_{cc'} = (|R_{cc}||R_{c'c'}|)^{\frac{1}{2}} \exp(i\xi), \tag{35}$$

where

$$\alpha_{cc'} = \xi, \qquad \sin\xi = \epsilon \sum_c |R_{cc}|.$$

This leads immediately to the equiphase solution for (32) of

$$S_{cc'} = \Omega_c \Omega_{c'} \delta_{cc'} + 2i\Omega_c \Omega_{c'} (P_c |R_{cc}|)^{\frac{1}{2}} (P_{c'} |R_{c'c'}|)^{\frac{1}{2}} \exp(i\xi)/\{1 - \text{trace}(\mathbf{R}\mathbf{L}^0)\}. \tag{36}$$

V. Cross Sections

From the expression (7) for the matrix elements $R_{cc'}$ we have

$$1 - \text{trace}(\mathbf{R}\mathbf{L}^0) = 1 - \sum_c \sum_\lambda \gamma_{\lambda c}^2 L_c^0/(E_\lambda - E). \tag{37}$$

Using the conventional definitions (Lane and Thomas 1958) for quantities which are now complex, we obtain for the partial level shift $\Delta_{\lambda c}$ the relation

$$\Delta_{\lambda c} = -\gamma_{\lambda c}^2 (S_c - B_c) = -(\alpha_{\lambda c}^2 + i\beta_{\lambda c}^2)(S_c - B_c),$$

and for the partial width $\Gamma_{\lambda c}$

$$\Gamma_{\lambda c} = 2P_c \gamma_{\lambda c}^2 = 2P_c(\alpha_{\lambda c}^2 + i\beta_{\lambda c}^2),$$

with also

$$L_c^0 = S_c - B_c + iP_c. \tag{38}$$

Introduction of the total level shift and total width, given respectively by

$$\Delta_\lambda = \sum_c \Delta_{\lambda c} \qquad \text{and} \qquad \Gamma_\lambda = \sum_c \Gamma_{\lambda c}, \tag{39}$$

leads to the expression

$$1 - \text{trace}(\mathbf{R}\mathbf{L}^0) = 1 + \sum_\lambda \Delta_\lambda/(E_\lambda - E) - \tfrac{1}{2}i \sum_\lambda \Gamma_\lambda/(E_\lambda - E). \tag{40}$$

If the real part of $B_{cc'}$ is chosen so that $S_c \approx B_c$ then $\Delta_{\lambda c} \approx 0$ and

$$1 - \text{trace}(\mathbf{R}\mathbf{L}^0) \approx 1 - \tfrac{1}{2}i \sum_\lambda \Gamma_\lambda/(E_\lambda - E) = Z \exp(-i\eta), \tag{41}$$

where

$$Z = \left(1 + \tfrac{1}{4}\left|\sum_\lambda \Gamma_\lambda/(E_\lambda - E)\right|^2\right)^{\frac{1}{2}} \sec\eta.$$

The reduced $\mathbf{T}$ matrix is then

$$\mathbf{T}' = \boldsymbol{\rho} \exp\{i(\eta + \xi)\} = \boldsymbol{\rho} \exp(i\delta),$$

where $\delta = \eta + \xi$ and

$$\rho_{cc'} = (P_c P_{c'})^{\frac{1}{2}} |R_{cc'}|/Z = (P_c |R_{cc}|)^{\frac{1}{2}} (P_{c'} |R_{c'c'}|)^{\frac{1}{2}}/Z. \tag{42}$$

The elements of the **S** matrix can then by written as

$$S_{cc'} = \exp\{-i(\phi_c+\phi_{c'})\}[\delta_{cc'}+2i\,\rho_{cc'}\exp\{i(\eta+\xi)\}] \tag{43}$$

and the probabilities of decay through channels c become, from equation (26b),

$$\mathscr{P}_c = P_c\,|\,R_{cc}\,|/Z\sin\delta\,, \qquad \rho_{cc'} \equiv \rho_c\,\rho_{c'}\,. \tag{44}$$

The total cross section is

$$\begin{aligned}\sigma_{cT} &= \frac{2\pi}{k_c^2}\sum_{ljJ} g_J\{1-\mathrm{Re}(S_{clj,clj})\}\\ &= \frac{2\pi}{k_c^2}\sum_{ljJ} g_J\{(1-\cos\phi_{cl})-2\sin(2\phi_{cl}-\delta^J)\rho^J_{clj,clj}\}\\ &= \frac{4\pi}{k_c^2}\sum_{l}(2l+1)\sin^2\phi_{cl} - \frac{2\pi}{k_c^2}\sum_{ljJ} g_J\sin(2\phi_{cl}-\delta^J)(2P_{cl}\,|\,R_{clj,clj}\,|/Z^J)\,.\end{aligned} \tag{45}$$

The elastic scattering cross section is

$$\begin{aligned}\sigma_{cc} &= \frac{\pi}{k_c^2}\sum_{ljJ} g_J\,|\,1-S^J_{clj,clj}\,|^2\\ &= \frac{\pi}{k_c^2}\sum_{ljJ} g_J[2(1-\cos 2\phi_{cl})+4(\rho_{clj,clj})^2-4\{\sin(\delta^J)+\sin(2\phi_{cl}-\delta^J)\}\rho_{clj,clj}]\\ &= \frac{4\pi}{k_c^2}\sum_{l}(2l+1)\sin^2\phi_{cl} + \frac{\pi}{k_c^2}\sum_{ljJ} g_J(2P_{cl}\,|\,R_{clj,clj}\,|/Z^J)^2\\ &\quad - \frac{2\pi}{k_c^2}\sum_{ljJ} g_J\{\sin(\delta^J)+\sin(2\phi_{cl}-\delta^J)\}(2P_{cl}\,|\,R_{clj,clj}\,|/Z^J)\,.\end{aligned} \tag{46}$$

The reaction cross section is

$$\begin{aligned}\sigma_{cr} &= \frac{\pi}{k_c^2}\sum_{ljJl'j'} g_J\,|\,S^J_{clj,c'l'j'}\,|^2 = \frac{4\pi}{k_c^2}\sum_{ljJl'j'} g_J(\rho_{clj,c'l'j'})^2\\ &= \frac{\pi}{k_c^2}\sum_{ljJl'j'} g_J(2P_{cl}\,|\,R_{clj,clj}\,|)(2P_{c'l'}\,|\,R_{c'l'j',c'l'j'}\,|)/(Z^J)^2\,.\end{aligned} \tag{47}$$

Making use of the probabilities in equation (24) we find that the above cross sections can be written

$$\sigma_{cc} = \sigma^{\mathrm{p}}_{cc}+\sigma^{\mathrm{i}}_{cc}+\sigma^{\mathrm{r}}_{cc}\,,$$

where

$$\sigma^{\mathrm{p}}_{cc} = \frac{4\pi}{k_c^2}\sum_{l}(2l+1)\sin^2\phi_{cl} \tag{48a}$$

is the hard sphere cross section,

$$\sigma^{\mathrm{i}}_{cc} = -\frac{4\pi}{k_c^2}\sum_{ljJ} g_J\{\sin\delta^J+\sin(2\phi_{cl}-\delta^J)\}\mathscr{P}^J_{clj}\sin\delta^J \tag{48b}$$

is the interference scattering cross section,

$$\sigma_{cc}^{\mathrm{r}} = \frac{4\pi}{k_c^2} \sum_{ljJ} g_J (\mathscr{P}_{clj}^J)^2 \sin^2 \delta^J , \tag{48c}$$

and

$$\sigma_{cc'} = \frac{4\pi}{k_c^2} \sum_{ljJl'j'} g_J \mathscr{P}_{clj}^J \mathscr{P}_{c'l'j'}^J \sin^2 \delta^J . \tag{48d}$$

In the form of equations (48) the compound nucleus hypothesis (23) is manifestly exhibited.

In special cases, the above expressions for the cross sections are identical with those of conventional theory. For example, in the single level approximation we we have $\epsilon \approx 0$, $\xi \approx 0$, $E_\lambda = E_0$ which is real, Γ_λ is real, and

$$Z = (E_0 - E)^{-1} \{(E_0 - E)^2 + \tfrac{1}{4}(\Gamma_\lambda)^2\}^{\frac{1}{2}} , \tag{49}$$

and in this case equations (45), (46), and (47) become

$$\sigma_{c\mathrm{T}} = \frac{4\pi}{k_c^2} \sum_l (2l+1) \sin^2 \phi_{cl} + \frac{\pi}{k_c^2} \sum_{lj} \frac{g_J \{\Gamma^J \Gamma_{clj}^J \cos 2\phi_{cl} - 2(E_0 - E)\Gamma_{clj}^J \sin 2\phi_{cl}\}}{(E_0 - E)^2 + \tfrac{1}{4}(\Gamma^J)^2} , \tag{50a}$$

$$\sigma_{cc} = \frac{4\pi}{k_c^2} \sum_l (2l+1) \sin^2 \phi_{cl} + \frac{\pi}{k_c^2} \sum_{lj} \frac{g_J (\Gamma_{clj}^J)^2}{(E_0 - E)^2 + \tfrac{1}{4}(\Gamma^J)^2}$$

$$- \frac{2\pi}{k_c^2} \sum_{lj} \frac{g_J \{\Gamma_{clj}^J \Gamma^J \sin^2 \phi_{cl} + (E_0 - E)\Gamma_{clj}^J \sin 2\phi_{cl}\}}{(E_0 - E)^2 + \tfrac{1}{4}(\Gamma^J)^2} , \tag{50b}$$

and

$$\sigma_{cc'} = \frac{\pi}{k_c^2} \sum_{ljl'j'} \frac{g_J \Gamma_{clj}^J \Gamma_{c'l'j'}^J}{(E_0 - E)^2 + \tfrac{1}{4}(\Gamma^J)^2} , \tag{50c}$$

which are the usual expressions for an isolated level (Schmidt 1966).

VI. Correlations between Reduced Widths

Newton (1952) assumed that the E_λ and γ_λ^2 are real quantities, as in the Wigner–Eisenbud theory, and postulated the strong correlation condition

$$\gamma_{\lambda c}/\gamma_{\lambda c'} = \gamma_{\mu c}/\gamma_{\mu c'} , \tag{51}$$

which ensures that the reaction matrix (7), without the background term, is of rank one. Now, for a complex equiphase reaction matrix we have

$$R_{cc} R_{c'c'} = R_{cc'}^2 , \tag{52}$$

and after substituting equation (7) into (52) and eliminating common terms on each side we find

$$\sum_\lambda (E_\lambda - E)^{-1} \sum_{\mu \neq \lambda} (\gamma_{\lambda c} \gamma_{\mu c'} - \gamma_{\lambda c'} \gamma_{\mu c})^2 / (E_\mu - E_\lambda) = 0 . \tag{53}$$

It follows that the residue at each pole must vanish in order that equation (53) should hold for all E, and hence

$$\sum_{\mu \neq \lambda} (\gamma_{\lambda c}\gamma_{\mu c'} - \gamma_{\lambda c'}\gamma_{\mu c})^2/(E_\mu - E_\lambda) = 0\,. \tag{54}$$

Should the $\gamma_{\lambda c}$ and E_λ be real, and there exist a lower bound to the values of E_λ, then at this lower bound all of the weights $E_\mu - E_\lambda$ are positive. Since the numerators are all perfect squares, the sum of these squares with positive weights cannot vanish unless each term vanishes. Newton's (1952) correlation (51) follows immediately. If there is no lower bound to E_λ, the condition (51) does not follow from the rank one condition (54), since for any finite E_λ the positive terms can cancel the negative terms. Newton's condition also follows if there is an upper bound to E_λ.

The probability of decay in channel c can be evaluated using equations (51) and (54) with $\epsilon = 0$, to give

$$\gamma_{\lambda c}/\gamma_{\lambda c'} = \gamma_{\mu c}/\gamma_{\mu c'} = A_c/A_{c'} = R_{cc}/R_{c'c'}\,,$$

where the A_c's are constants, and hence

$$\mathscr{P}_c = \frac{P_c R_{cc}}{\sum_{c'} P_{c'} R_{c'c'}} = \frac{A_c P_c}{\sum_{c'} A_{c'} P_{c'}}\,. \tag{55}$$

The $\mathscr{P}_c$'s are therefore independent of γ_λ^2 and E_λ. For radiative capture and fission processes, $\mathscr{P}_c$ would be approximately constant with energy. This is not what is observed.

Furthermore, Newton (1952) pointed out that in the expressions (45), (46), and (47), the reaction matrix has zeros between each pole at the same energy in all channels. Therefore, for a given spin state, the reaction cross sections should vanish at some energy between each resonance. This is not observed. Lane and Thomas (1958) consequently rejected Newton's model, and with it the Bohr hypothesis, as not being in accord with experiment. However, this reasoning is faulty since, if the γ_λ^2 and E_λ's are complex, all that equation (54) implies is that a set of equations exist which correlate the imaginary parts of γ_λ^2 and E_λ with the real parts. For example, an acceptable special solution to (54) would be, as in Newton's example,

$$\frac{\gamma_{\lambda c}}{\gamma_{\lambda c'}} = \frac{\gamma_{\mu c}}{\gamma_{\mu c'}}, \qquad \frac{\gamma_{\lambda c}^2}{\gamma_{\lambda c'}^2} = \frac{\gamma_{\mu c}^2}{\gamma_{\mu c'}^2} = \frac{A_c + \mathrm{i}D_c}{A_{c'} + \mathrm{i}D_{c'}}, \tag{56}$$

where the A_c and D_c are constants independent of λ, and putting

$$(\alpha_{\lambda c} + \mathrm{i}\beta_{\lambda c})/(\alpha_{\lambda c'} + \mathrm{i}\beta_{\lambda c'}) = (A_c + \mathrm{i}D_c)/(A_{c'} + \mathrm{i}D_{c'})$$

we can evaluate the imaginary parts to obtain

$$\beta_{\lambda c} = \frac{\alpha_{\lambda c'}(A_c^2 + D_c^2) - \alpha_{\lambda c}(A_c A_{c'} + D_c D_{c'})}{A_c D_{c'} + A_{c'} D_c}, \quad \beta_{\lambda c'} = \frac{\alpha_{\lambda c}(A_{c'}^2 + D_{c'}^2) - \alpha_{\lambda c'}(A_c A_{c'} + D_c D_{c'})}{A_c D_{c'} + A_{c'} D_c}. \tag{57}$$

Equations (57) represent a solution where the $\beta_{\lambda c}$ are correlated to the corresponding $\alpha_{\lambda c}$ through constants depending only on the channels present. The $\alpha_{\lambda c}$ remain arbitrary. Therefore Newton's correlation condition is a much weaker constraint for a complex **R** matrix.

Furthermore, the cross sections (47) can vanish for a given set of quantum numbers (ljJ) only if the real and imaginary parts of $R_{cc'}$ vanish at the same energy. Such a restriction implies further strong correlations between the complex resonance parameters which are unnecessary in general. There are sufficient parameters in (47) to fit resonant cross sections adequately without restriction on the minimum value between resonances. Therefore, we can encompass the Bohr postulate within the framework of Moldauer's (1964) theory without contradicting observed features of cross sections. Finally, we note from equations (41) and (44) that the probability of decay through channel c is

$$\mathscr{P}_c = \frac{P_c\,|\,R_{cc}\,|}{\sin\delta\sec\eta} = \frac{P_c\,|R_{cc}\,|}{\sin\xi+\cos\xi\tan\eta} = \frac{P_c\,|\,R_{cc}\,|}{\sin\xi+(\cos\xi)Z/(1-Z^2)^{\frac{1}{2}}}, \tag{58}$$

where Z is given by equation (41) and ξ is the argument of $R_{cc'}$. Clearly equation (58) can be a strong function of energy, dependent upon both the real and imaginary parts of the E_λ and γ_λ^2.

If, as an alternative, the real reaction matrix (22) is used, it is found that equations (45), (46), and (47) remain valid and the cross section must still vanish between resonances. However, the strong correlation condition (51) is no longer meaningful. It appears that Moldauer's (1964) theory is necessary to explain Bohr's hypothesis satisfactorily, and implies that the effective potential to use in the Schrödinger equation for the channel wavefunction must be complex. This may be deduced as a manifestation of channel–channel coupling not present in Wigner–Eisenbud (1947) theory.

VII. Poles and Residues of S Matrix

It is clear from equations (6) and (8) that the poles in the reduced **T** matrix are also contained in the full **S** matrix. From equations (10) and (36) we see that

$$T'_{cc'} = (P_c\,|\,R_{cc}\,|)^{\frac{1}{2}}(P_{c'}\,|\,R_{c'c'}\,|)^{\frac{1}{2}}\exp(\mathrm{i}\xi)/\{1-\mathrm{trace}(\mathbf{R}\,\mathbf{L}^0)\}\,. \tag{59}$$

The positions of the complex poles are obtained as the values of the complex energy where

$$\mathrm{trace}(\mathbf{R}\,\mathbf{L}^0) = 1\,. \tag{60}$$

We may formally represent the spectrum of poles by a vector such that at each pole

$$E = \mathscr{E}_\mu - \tfrac{1}{2}\mathrm{i}\,Y_\mu\,. \tag{61}$$

Neglecting the level shift contribution with the factor (S_c-B_c) and setting

$$E_\lambda = U_\lambda+\tfrac{1}{2}\mathrm{i}V_\lambda\,, \qquad \Gamma_\lambda = \Gamma_\lambda^{\mathrm{r}}+\mathrm{i}\Gamma_\lambda^{\mathrm{i}} = 2P_c\gamma_{\lambda c}^2\,, \tag{62}$$

we obtain the two equations from equation (60) of the form

$$\sum_{\lambda} \frac{\Gamma_{\lambda}^{r}(U_{\lambda}-\mathscr{E}_{\mu})-\frac{1}{2}\Gamma_{\lambda}^{i}(V_{\lambda}-Y_{\mu})}{(U_{\lambda}-\mathscr{E}_{\mu})^{2}+\frac{1}{4}(V_{\lambda}-Y_{\mu})^{2}}=0, \qquad \sum_{\lambda} \frac{\Gamma_{\lambda}^{i}(U_{\lambda}-\mathscr{E}_{\mu})+\frac{1}{2}\Gamma_{\lambda}^{r}(V_{\lambda}-Y_{\mu})}{(U_{\lambda}-\mathscr{E}_{\mu})^{2}+\frac{1}{4}(V_{\lambda}-Y_{\mu})^{2}}=-2. \quad (63)$$

These equations can be summarized in the form

$$\sum_{\lambda} \Gamma_{\lambda} f_{\lambda\mu} = Y_{\mu}, \quad (64)$$

where

$$f_{\lambda\mu} = -\tfrac{1}{2}\mathrm{i}Y_{\mu}/\{(U_{\lambda}-\mathscr{E}_{\mu})+\tfrac{1}{2}\mathrm{i}(V_{\lambda}-Y_{\mu})\}$$

is a complex "mixing matrix" which is such that equation (64) states that a linear superposition of the complex widths gives the equivalent real width in the **S** matrix. When V_{λ} and Y_{μ} are very much less than $U_{\lambda}-\mathscr{E}_{\mu}$ for $\lambda \neq \mu$ the off-diagonal elements of $f_{\lambda\mu}$ are approximately zero and, for $V_{\lambda} \ll Y_{\mu}$, $f_{\lambda\mu}$ becomes the identity matrix.

To get good approximations to the positions of the poles it is more convenient to let

$$\tau(E) = \sum_{\lambda} \Gamma_{\lambda}/(E_{\lambda}-E) = 2\mathrm{i}\,\mathrm{trace}(\mathbf{R}\mathbf{L}^{0}) \quad (65)$$

and to define

$$\tau_{\mu}(E) = \sum_{\lambda\neq\mu} \Gamma_{\lambda}/(E_{\lambda}-E) = \tau(E) - \Gamma_{\mu}/(E_{\mu}-E). \quad (66)$$

To find $\mathscr{E}_{\mu}$ and Y_{μ} we must solve the equation

$$\tau(\mathscr{E}_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu}) = -2\mathrm{i}. \quad (67)$$

Let $\mathscr{E}_{\mu} = E_{\mu}+\Delta_{\mu}$ where Δ_{μ} is complex. Then

$$\tau(E_{\mu}+\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu}) = -\Gamma_{\mu}/(\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu}) + \tau_{\mu}(E_{\mu}+\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu})$$

and, since $\tau_{\mu}(E)$ has no singularities at $E = E_{\mu}$, we can use the Taylor expansion

$$\tau_{\mu}(E_{\mu}+\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu}) = \tau_{\mu}(E_{\mu}) + (\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu})\tau'_{\mu}(E_{\mu}) + \ldots.$$

The nth derivative of τ_{μ} is just

$$\tau_{\mu}^{(n)}(E) = n! \sum_{\lambda\neq\mu} \Gamma_{\lambda}/(E_{\lambda}-E)^{n+1}. \quad (68)$$

Equation (67) then becomes

$$-2\mathrm{i} = -\Gamma_{\mu}/(\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu}) + \tau_{\mu}(E_{\mu}) + (\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu})\tau'_{\mu}(E_{\mu}) + \ldots, \quad (69)$$

that is,

$$Y_{\mu}+2\mathrm{i}\Delta_{\mu}-\Gamma_{\mu}+(\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu})\tau_{\mu}(E_{\mu}) + (\Delta_{\mu}-\tfrac{1}{2}\mathrm{i}Y_{\mu})^{2}\tau'_{\mu}(E_{\mu}) + \ldots = 0. \quad (70)$$

Equation (70) can be solved for the real and imaginary parts to any desired order of accuracy. For an isolated level, $\tau_{\mu} \approx 0$ and we have

$$\Gamma_{\mu} \approx Y_{\mu}+2\mathrm{i}\Delta_{\mu},$$

that is,

$$Y_{\mu} \approx \Gamma_{\mu}^{r}+V_{\mu}, \qquad \mathscr{E}_{\mu} \approx U_{\mu}+\tfrac{1}{2}\Gamma_{\mu}^{i}. \quad (71)$$

In the next approximation, we assume that the derivatives of τ_μ may be neglected. This yields the simultaneous equations

$$Y_\mu(1+\tfrac{1}{2}\operatorname{Im}\tau)+(U_\mu-\mathscr{E}_\mu)\operatorname{Re}\tau = \Gamma_\mu^{\mathrm{r}}+V_\mu(1+\tfrac{1}{2}\operatorname{Im}\tau), \tag{72a}$$

$$Y_\mu(-\tfrac{1}{2}\operatorname{Re}\tau)+(U_\mu-\mathscr{E}_\mu)(2+\operatorname{Im}\tau) = \Gamma_\mu^{\mathrm{i}}, \tag{72b}$$

which have solutions

$$Y_\mu = \frac{(1+\tfrac{1}{2}\operatorname{Im}\tau)\{\Gamma_\mu^{\mathrm{r}}+V_\mu(1+\tfrac{1}{2}\operatorname{Im}\tau)\}-\tfrac{1}{2}(\operatorname{Re}\tau)\Gamma_\mu^{\mathrm{i}}}{(1+\tfrac{1}{2}\operatorname{Im}\tau)^2+\tfrac{1}{4}(\operatorname{Re}\tau)^2}, \tag{73a}$$

$$\mathscr{E}_\mu = U_\mu-\frac{\tfrac{1}{2}\Gamma_\mu^{\mathrm{i}}(1+\tfrac{1}{2}\operatorname{Im}\tau)+\tfrac{1}{4}(\operatorname{Re}\tau)\{\Gamma_\mu^{\mathrm{r}}+V_\mu(1+\tfrac{1}{2}\operatorname{Im}\tau)\}}{(1+\tfrac{1}{2}\operatorname{Im}\tau)^2+\tfrac{1}{4}(\operatorname{Re}\tau)^2}. \tag{73b}$$

The residue of the reduced **T** matrix at each pole is obtained by using the fact that, if $G(E)$ is a function with a simple pole at $E=z$, the residue at that pole is given by

$$\{\operatorname{Res}G(z)\}^{-1} = \operatorname*{Lim}_{E\to z}\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{1}{G(E)}\right). \tag{74}$$

Let

$$G(E)^{-1} = 1-\operatorname{trace}(\mathbf{R}\mathbf{L}^0) = 1-\tfrac{1}{2}\mathrm{i}\sum_\lambda \Gamma_\lambda/(E_\lambda-E). \tag{75}$$

Equation (75) tells us that if we put

$$\frac{\mathrm{d}}{\mathrm{d}E}\left(\frac{1}{G(E)}\right) = -\tfrac{1}{2}\mathrm{i}\sum_\lambda \Gamma_\lambda/(E_\lambda-E)^2 \tag{76}$$

then by neglecting the local energy variation of each Γ_λ the residue of $T'_{cc'}$ at the pole $\mathscr{E}_\mu-\tfrac{1}{2}\mathrm{i}Y_\mu$ is $\tfrac{1}{2}a_{\mu cc'}$ where

$$a_{\mu cc'} = \frac{4\mathrm{i}(P_c P_{c'})^{\frac{1}{2}}R_{cc'}(z_\mu)}{\sum_\lambda \Gamma_\lambda/(E_\lambda-z_\mu)^2} = \frac{4\mathrm{i}\{P_c\,|\,R_{cc}(z_\mu)\,|\}^{\frac{1}{2}}\{P_{c'}\,|\,R_{c'c'}(z_\mu)\,|\}^{\frac{1}{2}}\exp\{\mathrm{i}\xi(z_\mu)\}}{\sum_\lambda \Gamma_\lambda/(E_\lambda-z_\mu)^2}, \tag{77}$$

with $z_\mu = \mathscr{E}_\mu-\tfrac{1}{2}\mathrm{i}Y_\mu$.

From equations (59) and (75) we see that

$$T'_{cc'} = \left(\tfrac{1}{2}\left|\sum_\lambda(\Gamma_{\lambda c}\Gamma_{\lambda c'})^{\frac{1}{2}}/(E_\lambda-E)\right|\exp(\mathrm{i}\xi)\right)\Big/\left(1-\tfrac{1}{2}\mathrm{i}\sum_\lambda \Gamma_\lambda/(E_\lambda-E)\right). \tag{78}$$

We can approximate equation (78) at a given energy to an arbitrary degree of accuracy by considering the sums over λ to extend to a finite number N of terms. By taking the sums over λ over a common denominator we see that $T'_{cc'}$ can be expressed in the form

$$T'_{cc'} = p_{N-1}(E)/q_N(E), \tag{79}$$

where p_N and q_N are polynomials in E of order N with complex coefficients. The

polynomial $q_N(E)$ in equation (79) can be factorized into a product of the form

$$q_N(E) = \prod_{\lambda} (\mathscr{E}_\lambda - E - \tfrac{1}{2}\mathrm{i}Y_\lambda). \tag{80}$$

Hence equation (79) can be reduced by partial fractions to a sum of simple pole terms such that

$$T'_{cc'} = \tfrac{1}{2}\sum_{\lambda} a_{\lambda cc'}/(\mathscr{E}_\lambda - E - \tfrac{1}{2}\mathrm{i}Y_\lambda), \tag{81}$$

where the $a_{\lambda cc'}$ are the complex residues given by (77). Equation (81) states that the reduced **T** matrix can be expressed as a coherent sum of effective single level terms. Sailor (1955) used this property as an assumption in fitting fission cross sections, and it was first conjectured by Feshback, Porter, and Weisskopf (1954). The **S** matrix can therefore be written as

$$S_{cc'} = \Omega_c \Omega_{c'}\left(\delta_{cc'} + \mathrm{i}\sum_{\lambda} a_{\lambda cc'}/(\mathscr{E}_\lambda - E - \tfrac{1}{2}\mathrm{i}Y_\lambda)\right). \tag{82}$$

VIII. Cross Sections as Sums of Single Level Terms

It was shown by Cook (1967*a*) that the multilevel expressions for the fission cross section used by Sailor (1955) could be written as a sum of single level contributions. Clayton (1970) showed that the same result holds for the scattering cross section. Using these formulae we obtain results of the type derived by Adler and Adler (1963). In the present case, where we have complex residues, the **S** matrix can be written as

$$S_{cc'} = \exp\{-\mathrm{i}(\phi_c + \phi_{c'})\}\left(\delta_{cc'} + \mathrm{i}\sum_{\lambda} \frac{a_{\lambda cc'}(\mathscr{E}_\lambda - E + \tfrac{1}{2}\mathrm{i}Y_\lambda)}{(\mathscr{E}_\lambda - E)^2 + \tfrac{1}{4}Y_\lambda^2}\right), \tag{83}$$

which gives for $c \neq c'$

$$\begin{aligned}\sigma_{cc'} &= (\pi/k_c^2) S^*_{cc'} S_{cc'} \\ &= \frac{\pi}{k_c^2}\sum_{\lambda} \frac{|a_{\lambda cc'}|^2}{(\mathscr{E}_\lambda - E)^2 + \tfrac{1}{4}Y_\lambda^2} \\ &\quad + \frac{\pi}{k_c^2}\sum_{\substack{\lambda,\mu \\ \lambda\neq\mu}} \frac{a_{\lambda cc'} a^*_{\lambda cc'}(\mathscr{E}_\lambda - E + \mathrm{i}Y_\lambda)(\mathscr{E}_\mu - E - \tfrac{1}{2}\mathrm{i}Y_\mu)}{\{(\mathscr{E}_\lambda - E)^2 + \tfrac{1}{4}Y_\lambda^2\}\{(\mathscr{E}_\mu - E)^2 + \tfrac{1}{4}Y_\mu^2\}}.\end{aligned} \tag{84}$$

The last term in equation (84) can be separated into single level contributions by partial fractions to yield

$$\sigma_{cc'} = \frac{\pi}{k_c^2}\sum_{\lambda} \frac{\sigma_{\lambda cc'}}{(\mathscr{E}_\lambda - E)^2 + \tfrac{1}{4}Y_\lambda^2} - \frac{2\pi}{k_c^2}\sum_{\lambda} \frac{(\mathscr{E}_\lambda - E)\eta_{\lambda cc'}}{(\mathscr{E}_\lambda - E)^2 + \tfrac{1}{4}Y_\lambda^2}, \tag{85a}$$

where

$$\sigma_{\lambda cc'} = |a_{\lambda cc'}|^2 + \sum_{\lambda\neq\mu} \frac{Y_\lambda \beta_{\lambda\mu cc'}}{(\mathscr{E}_\lambda - \mathscr{E}_\mu)^2 + \tfrac{1}{4}(Y_\lambda + Y_\mu)^2}, \tag{85b}$$

$$\eta_{\lambda cc'} = \sum_{\mu\neq\lambda} \frac{\alpha_{\lambda\mu cc'}}{(\mathscr{E}_\lambda - \mathscr{E}_\mu)^2 + \tfrac{1}{4}(Y_\lambda + Y_\mu)^2}, \tag{85c}$$

and

$$a_{\lambda cc'}a^{*}_{\mu cc'}\{(\mathscr{E}_\lambda-\mathscr{E}_\mu)+\tfrac{1}{2}\mathrm{i}(Y_\lambda+Y_\mu)\}=\alpha_{\lambda\mu cc'}+\mathrm{i}\beta_{\lambda\mu cc'}\,. \tag{85d}$$

Equation (85a) has the form of a sum of single level terms, where the first sum is over the terms symmetric about the resonance energy $\mathscr{E}_\lambda$ and the second sum is over corresponding asymmetric contributions.

Similarly, the elastic scattering cross section can be expressed as

$$\begin{aligned}\sigma_{cc}&=(\pi/k_c^2)|\,1-S_{cc}\,|^2\\&=\frac{4\pi}{k_c^2}\sin^2\phi_c+\frac{\pi}{k_c^2}\sum_\lambda\frac{\sigma_{\lambda cc}}{(\mathscr{E}_\lambda-E)^2+\frac{1}{4}Y_\lambda^2}-\frac{2\pi}{k_c^2}\sum_\lambda\frac{(\mathscr{E}_\lambda-E)\eta_{\lambda cc}}{(\mathscr{E}_\lambda-E)^2+\frac{1}{4}Y_\lambda^2}\\&\quad-\frac{2\pi}{k_c^2}\sum_\lambda\frac{\sin(2\phi_c)\,\mathrm{Re}\{a_{\lambda cc}(\mathscr{E}_\lambda-E+\frac{1}{2}\mathrm{i}Y_\lambda)\}-\cos(2\phi_c)\,\mathrm{Im}\{a_{\lambda cc}(\mathscr{E}_\lambda-E+\frac{1}{2}\mathrm{i}Y_\lambda)\}}{(\mathscr{E}_\lambda-E)^2+\frac{1}{4}Y_\lambda^2},\end{aligned} \tag{86}$$

where $\sigma_{\lambda cc}$ and $\eta_{\lambda cc}$ are as defined by equations (85b) and (85c). The total cross section is found from equation (2) by evaluating

$$\begin{aligned}\sigma_{cT}&=(2\pi/k_c^2)\{1-\mathrm{Re}(S_{cc})\}\\&=\frac{4\pi}{k_c^2}\sin^2\phi_c+\frac{2\pi}{k_c^2}\,\mathrm{Im}\left(\exp(-2\mathrm{i}\phi_c)\sum_\lambda\frac{a_{\lambda cc'}(\mathscr{E}_\lambda-E+\frac{1}{2}\mathrm{i}Y_\lambda)}{(\mathscr{E}_\lambda-E)^2+\frac{1}{4}Y_\lambda^2}\right)\\&=\frac{4\pi}{k_c^2}\sin^2\phi_c\\&\quad-\frac{2\pi}{k_c^2}\sum_\lambda\frac{\sin(2\phi_c)\,\mathrm{Re}\{a_{\lambda cc'}(\mathscr{E}_\lambda-E+\frac{1}{2}\mathrm{i}Y_\lambda)\}-\cos(2\phi_c)\,\mathrm{Im}\{a_{\lambda cc'}(\mathscr{E}_\lambda-E+\frac{1}{2}\mathrm{i}Y_\lambda)\}}{(\mathscr{E}_\lambda-E)^2+\frac{1}{4}Y_\lambda^2}.\end{aligned} \tag{87}$$

IX. Statistical Theory

From equation (82) we see that the **S** matrix can be written as

$$S_{cc'}=S^0_{cc'}-\mathrm{i}\sum_\lambda g_{\lambda c}g_{\lambda c'}/(\mathscr{E}_\lambda-E-\tfrac{1}{2}\mathrm{i}Y_\lambda) \tag{88}$$

for an equiphase **R** matrix, where

$$S^0_{cc'}=\Omega_c\Omega_{c'}\,\delta_{cc'}=\exp\{-\mathrm{i}(\phi_c+\phi_{c'})\}\,\delta_{cc'} \tag{89a}$$

and

$$g_{\lambda c}=2\mathrm{i}\left(\frac{\mathrm{i}P_c\,R_{cc}(z_\mu)}{\sum_\mu\Gamma_\mu/(E_\mu-z_\mu)^2}\right)^{\frac{1}{2}}\Omega_c\,. \tag{89b}$$

Equation (88) is precisely the form postulated by Moldauer (1964) to calculate average cross sections. If we assume that the apparent widths Y_λ and apparent resonance energies $\mathscr{E}_\lambda$ have the same statistical properties as the widths $|\,\Gamma_\lambda\,|$ and

eigenvalues $\mathrm{Re}(E_\lambda)$, which we assume to have the usual statistical properties, then we may employ the relations derived by Moldauer to calculate the following average cross sections:

$$\langle\sigma_{c\mathrm{T}}\rangle = (2\pi/k_c^2)(1-\mathrm{Re}\langle S_{cc}\rangle)\,, \tag{90a}$$

$$\langle\sigma_{cc'}\rangle = (\pi/k_c^2)\langle|\,\delta_{cc'}-S_{cc'}\,|^2\rangle\,, \tag{90b}$$

the direct reaction cross section

$$\langle\sigma_{cc'}^{\mathrm{D}}\rangle = (\pi/k_c^2)|\,\delta_{cc'}-\langle S_{cc'}\rangle\,|^2\,, \tag{90c}$$

the absorption cross section

$$\langle\sigma_c^{\mathrm{A}}\rangle = (\pi/k_c^2)(1-|\langle S_{cc}\rangle|^2) = (\pi/k_c^2)\mathscr{T}_c\,, \tag{90d}$$

where $\mathscr{T}_c$ is the average transmission coefficient, and the fluctuation cross section

$$\sigma_{cc'}^{\mathrm{F}} = (\pi/k_c^2)(\langle|\,S_{cc'}\,|^2\rangle-|\langle S_{cc'}\rangle|^2)\,. \tag{90e}$$

These cross sections can be evaluated by using the relationships given by Moldauer

$$\langle S_{cc'}\rangle = S_{cc'}^0-\pi\langle g_{\lambda c}g_{\lambda c'}\rangle_\lambda/D \tag{91a}$$

and

$$\langle|\,S_{cc'}\,|^2\rangle = |\langle S_{cc'}\rangle|^2+(2\pi/D)\langle|\,g_{\lambda c}\,|^2|\,g_{\lambda c'}\,|^2/Y_\lambda\rangle_\lambda-M_{cc'}\,, \tag{91b}$$

where D is the average level spacing, and

$$M_{cc'} = \frac{2\pi^2}{D^2}\left\{|\langle g_{\lambda c}g_{\lambda c'}\rangle_\lambda|^2-\left\langle g_{\lambda c}g_{\lambda c'}g_{\mu c}g_{\mu c'}\Phi_0\left(\frac{Y_\lambda+Y_\mu}{D}\right)\right\rangle_{\lambda\neq\mu}\right\} \tag{91c}$$

is a function defined by Moldauer.

Assuming that the $g_{\mu c}$ for different channels are uncorrelated, we find that

$$\langle g_{\lambda c}g_{\lambda c'}\rangle_\lambda = \delta_{cc'}\langle g_{\lambda c}^2\rangle_\lambda \tag{92}$$

and, since Φ_0 is a slowly varying function of its argument, we obtain

$$M_{cc'} \approx \delta_{cc'}(2\pi^2/D^2)|\langle g_{\lambda c}^2\rangle_\lambda|^2(1-\Phi_0) \tag{93}$$

and

$$\langle S_{cc'}\rangle \approx \{\exp(-2\mathrm{i}\phi_c)-(\pi/D)\langle g_{\lambda c}^2\rangle_\lambda\}\delta_{cc'}\,, \tag{94a}$$

$$\langle|\,S_{cc'}\,|^2\rangle \approx \delta_{cc'}[1-(2\pi/D)\mathrm{Re}\{\exp(2\mathrm{i}\phi_c)\langle g_{\lambda c}^2\rangle_\lambda\}+(\pi^2/D^2)|\langle g_{\lambda c}^2\rangle_\lambda|^2]$$

$$+(2\pi/D)\langle|\,g_{\lambda c}\,|^2|\,g_{\lambda c'}\,|^2/Y_\lambda\rangle_\lambda-M_{cc'}\,. \tag{94b}$$

When $\langle|\,\Gamma_\lambda\,|\rangle$ is very much less than D, the approximate solutions to equation (88) are found to be

$$g_{\lambda c} \approx (\Gamma_{\lambda c})^{\frac{1}{2}}\exp(-\mathrm{i}\phi_c)\,, \qquad Y_\lambda \approx \Gamma_\lambda^{\mathrm{r}}+V_\lambda\,. \tag{95}$$

It follows from equations (90), (91), and (94) that

$$\langle\sigma_{cc}\rangle = \frac{4\pi}{k_c^2}\sin^2\phi_c$$

$$-\frac{2\pi^2}{k_c^2 D}\Big((1-\cos 2\phi_c)\mathrm{Re}\langle\Gamma_{\lambda c}\exp(-2\mathrm{i}\phi_c)\rangle_\lambda+(\sin 2\phi_c)\mathrm{Im}\langle\Gamma_\lambda\exp(-2\mathrm{i}\phi_c)\rangle_\lambda\Big)$$

$$+\frac{\pi^2}{D^2}|\langle\Gamma_{\lambda c}\rangle_\lambda|^2+\frac{2\pi}{D}\left\langle\frac{|(\Gamma_{\lambda c})^{\frac{1}{2}}|^4}{Y_\lambda}\right\rangle_\lambda, \tag{96a}$$

$$\langle\sigma_{cc'}\rangle = \frac{2\pi^2}{k_c^2 D}\left\langle\frac{|\Gamma_{\lambda c}||\Gamma_{\lambda c'}|}{Y_\lambda}\right\rangle_\lambda, \tag{96b}$$

and

$$\langle\sigma_{c\mathrm{T}}\rangle = \frac{2\pi}{k_c^2}\Big(1-\cos 2\phi_c+\frac{\pi}{D}\mathrm{Re}\langle(\Gamma_{\lambda c}\Gamma_{\lambda c'})^{\frac{1}{2}}\exp(-2\mathrm{i}\phi_c)\rangle_\lambda\Big), \tag{96c}$$

which are generalizations of the equations in conventional statistical theory.

X. Conclusions

It has been demonstrated that existing multilevel theories, when applied to present experiments, cannot yield a unique set of reaction matrix parameters. When the Bohr (1936) compound nucleus hypothesis is applied, the Wigner–Eisenbud (1947) theory cannot give agreement with experiment if any appreciable resonance overlap occurs but Moldauer's (1964) reaction matrix theory can. The expressions derived for multilevel cross sections which are consistent with the Bohr hypothesis and Moldauer's reaction matrix theory have been shown to be equivalent to the assumption that the **S** matrix consists of a sum of simple poles times hard sphere phase factors. Although the pole representation is not manifestly in accordance with the Bohr assumption, and thus with the equations derived here, it is necessary that a subtle connection exists. It has been pointed out by Cook (1967*a*) that the single level form for cross sections can be modified readily to allow for Doppler broadening.

XI. References

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