MATHEMATICAL MODEL FOR A GROUP OF ELECTRONS DRIFTING BETWEEN PLANE PARALLEL ELECTRODES

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Abstract

A mathematical model for a group of electrons drifting in a gas in a uniform electric field is proposed. The relevance of this model for the assessment of the errors that arise from forward and backward diffusion to anode and cathode in time of flight measurements of drift velocity is discussed. The theory of steady streams from pole and dipole sources is related to that of the travelling group.

I. Travelling Groups

The theoretical analysis of observations of the drift velocity W of electrons in gases in uniform electric fields is usually made in terms of an ideal travelling and diffusing group within which the time and spatial dependences of the number density $n(x,y,z,t) \equiv n$ of the electrons are assumed to be such that n is a solution of the equation of continuity (e.g. Huxley 1972)

$$-\frac{\mathrm{d}n}{\mathrm{d}t}+D\left(\frac{\partial^2 n}{\partial x^2}+\frac{\partial^2 n}{\partial y^2}\right)+D_{\mathrm{L}}\frac{\partial^2 n}{\partial z^2}-W\frac{\partial n}{\partial z}=0\,, \tag{1}$$

in which W is the drift velocity, D the lateral (or isotropic) coefficient of diffusion, and D_{L} the longitudinal coefficient of diffusion.

The simplest solution of equation (1) that represents an isolated travelling group and has hitherto been used in the analysis of time of flight measurements of W is

$$n=\frac{n_0}{(4\pi Dt)(4\pi D_{\mathrm{L}}t)^{\frac{1}{2}}}\exp\left(-\frac{\rho^2}{4Dt}\right)\exp\left(-\frac{(z-Wt)^2}{4D_{\mathrm{L}}t}\right), \tag{2}$$

where $\rho^2 = x^2+y^2$. This expression represents a diffusing group of n_0 electrons whose centroid moves at uniform speed W along the $+0z$ axis of coordinates. At time $t=0$ the group is concentrated within an infinitesimal volume at the origin of co-ordinates. The symmetry of the spatial distribution of n shows that the centroid is coincident in position with the maximum value n_{max} of n at any specified time t. In earlier discussions, before the existence of anisotropic diffusion had been recognized, D_{L} was considered to be the same as D.

In the earliest application of equation (2) to the interpretation of experimental determinations of W based on an electrical shutter technique, it was assumed that the time at which the maximum current was transmitted through the anode shutter,

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when briefly open, corresponded to the time t of arrival of the centroid of the group after the departure of the group from the cathode. If the separation of the electrodes was h it was then supposed that $W = h/t$. However, as techniques of measurement improved it was seen that errors greater than experimental errors were introduced through imperfect mathematical analysis of the physical conditions of the experiment. In particular, errors were introduced through neglect of forward diffusion to the anode and backward diffusion to the cathode which modified the actual distribution of the number density so that $n(x, y, z, t)$ was not strictly represented by equation (2) (Duncan 1957; Lowke 1962). It is not the intention here to discuss these sources of error in detail, and it suffices to remark that the simple representation of the group given in equation (2) does not faithfully describe a group that travels between plane metal electrodes in the planes $z = 0$ and h since the boundary condition $n = 0$ over these planes is not met by this solution of equation (1).

The next step was to improve equation (2) by adding an appropriate "image" term such that the combined solution gave $n = 0$ everywhere over the anode $z = h$ (Lowke 1962). Equation (2) was then replaced by

$$n = \frac{n_0}{(4\pi Dt)(4\pi D_{\mathrm{L}} t)^{\frac{1}{2}}} \exp\left(-\frac{\rho^2}{4Dt}\right)\left\{\exp\left(-\frac{(z-Wt)^2}{4D_{\mathrm{L}} t}\right) - \exp\left(\frac{hW}{D_{\mathrm{L}}} - \frac{(z-2h-Wt)^2}{4D_{\mathrm{L}} t}\right)\right\}. \tag{3}$$

This expression is a solution of equation (1) and gives $n = 0$ when $z = h$ for all values of x, y, and t.

Although equation (3) is an improvement on equation (2) and has strengthened the analysis of the errors referred to above, it is still imperfect because it does not give $n = 0$ at the cathode $z = 0$ at all times t and does not therefore explicitly account for errors arising from back diffusion to the cathode. It is the purpose of this paper to remedy this defect by formulating a solution of equation (1) that represents a travelling group moving between a pair of plane electrodes and gives $n = 0$ over both planes $z = 0$ and h at all times t.

We consider first a solution of equation (1) appropriate to the case where h is large and the anode exerts no influence on the number density within the travelling group when it is in the vicinity of the cathode. In order to ensure that $n = 0$ when $z = 0$ at all t we adopt a distribution for $n(x, y, z, t)$ that is the superposition of the distribution of equation (2) upon a "doublet" (dipole) distribution of appropriate relative strength that is also a solution of equation (1). We therefore adopt the solution

$$n = \frac{n_0}{(4\pi Dt)(4\pi D_{\mathrm{L}} t)^{\frac{1}{2}}} \exp\left(-\frac{\rho^2}{4Dt}\right)\left[\exp\left(-\frac{(z-Wt)^2}{4D_{\mathrm{L}} t}\right) - \frac{2D_{\mathrm{L}}}{W}\frac{\partial}{\partial z}\left\{\exp\left(-\frac{(z-Wt)^2}{4D_{\mathrm{L}} t}\right)\right\}\right]. \tag{4}$$

Equation (4) is equivalent to

$$n = \frac{n_0 \exp(-\rho^2/4Dt)}{(4\pi Dt)(4\pi D_{\mathrm{L}} t)^{\frac{1}{2}}} \frac{z}{Wt} \exp\left(-\frac{(z-Wt)^2}{4D_{\mathrm{L}} t}\right). \tag{5}$$

It can be seen from equation (5) that $n = 0$ over the plane $z = 0$ at all t.

Finally, to ensure that $n = 0$ also over the anode $z = h$ an "image" solution is associated with the solution (4) or (5) of appropriate strength to give $n = 0$ at $z = h$. This image group, like that in equation (3), is located at $t = 0$ at the point $(0,0,2h)$. The required solution is therefore

$$n = \frac{n_0 \exp(-\rho^2/4Dt)}{(4\pi Dt)(4\pi D_L t)^{\frac{1}{2}}} \frac{1}{Wt}\left\{z \exp\left(-\frac{(z-Wt)^2}{4D_L t}\right) + (z-2h)\exp\left(-\frac{(z-Wt)^2+4h(h-z)}{4D_L t}\right)\right\}, \tag{6}$$

which gives $n = 0$ when $z = h$. However, the introduction of the image term has disturbed the condition $n = 0$ when $z = 0$ and to ensure that $n = 0$ at both $z = 0$ and $z = h$ it would be necessary to invoke a hierarchy of images. In practice, however, the error in adopting equation (6) can be made undetectable by avoiding the use of small values of h. Equation (6) has already been found to be an effective analytical tool for correcting in a quantitative manner for the errors that arise from forward and backward diffusion to the electrodes.

II. Steady Diffusing Streams

It has been shown (Huxley 1968, 1972) that the number density $n(x,y,z,t)$ in a steady stream of electrons in a gas from a steady point source of electrons is given by the superposition of the number densities in a sequence of identical groups emitted regularly over an interval extending from $t = 0$ to infinity. The demonstration is based on the formula (Watson 1944)

$$\int_0^\infty \tau^{-(\nu+1)} \exp(-\tau - s^2/4\tau)\, \mathrm{d}\tau = 2(2/s)^\nu \mathrm{K}_\nu(s)\,, \tag{7}$$

where $\mathrm{K}_\nu(s)$ is a modified Bessel function of the second kind and order ν.

(a) *Stream from Isolated Point Source*

Let the current from the point source be i. It can therefore be represented as the emission of a sequence of groups $n_0 = i\,\mathrm{d}t/e$ at times ranging from $t = 0$ to infinity; here e is the electronic charge. Since we suppose the source to be isolated and the influence of electrodes on the stream to be unimportant, we assume that the elementary groups are described by equation (2) with $n_0 = i\,\mathrm{d}t/e$. The number density $n(x,y,z,t)$ in the stream is then, by superposition,

$$n(x,y,z) = \frac{i}{e(4\pi D)(4\pi D_L)^{\frac{1}{2}}} \int_0^\infty t^{-3/2} \exp\left(-\frac{\rho^2}{4Dt} - \frac{(z-Wt)^2}{4D_L t}\right) \mathrm{d}t\,. \tag{8}$$

With $2\lambda_L = W/D_L$ and $\tau = tW^2/4D_L = \frac{1}{2}\lambda_L Wt$, equation (8) can be transformed to

$$n(x,y,z) = \frac{i\lambda_L \exp(\lambda_L z)}{e(4\pi D)(4\pi)^{\frac{1}{2}}} \int_0^\infty \tau^{-3/2} \exp\left(-\tau - \frac{\lambda_L^2\{z^2+(D_L/D)\rho^2\}}{4\tau}\right) \mathrm{d}\tau\,. \tag{9}$$

Comparison of equations (9) and (7) shows that

$$n(x,y,z) = \frac{i\,\lambda_L \exp(\lambda_L z)}{e(4\pi D)}\left(\frac{2}{\pi\lambda_L\{z^2+(D_L/D)\rho^2\}^{\frac{1}{2}}}\right)^{\frac{1}{2}} K_{\frac{1}{2}}(\lambda_L\{z^2+(D_L/D)\rho^2\}^{\frac{1}{2}}),$$

which from the properties of the K_ν functions gives

$$n(x,y,z) = \frac{i}{e(4\pi D)}\frac{\exp\{-\lambda_L(r'-z)\}}{r'}, \tag{10}$$

where $r'^2 = z^2+(D_L/D)\rho^2$.

(b) Stream from Point Source located on Cathode

We assume that this case includes a point source which is a small hole in the cathode through which the stream of electrons enters the diffusion chamber. Alternatively the stream may be maintained by photoelectric emission from a point (the origin) on the cathode. If we let $n_0 = i\,\mathrm{d}t/e$ in equation (5) and integrate over all times from $t = 0$ to infinity, the number density in the stream is given by

$$n(x,y,z) = \frac{i}{e(4\pi D)(4\pi D_L)^{\frac{1}{2}}}\frac{z}{W}\int_0^\infty t^{-5/2}\exp\left(-\frac{\rho^2}{4Dt}-\frac{(z-Wt)^2}{4D_L t}\right)\mathrm{d}t. \tag{11}$$

With the same change of variable as that used to transform equation (8), the above equation becomes

$$n(x,y,z) = \frac{i\,\lambda_L^2 \exp(\lambda_L z)\,z}{e(4\pi D)\pi^{\frac{1}{2}}\;\;4}\int_0^\infty \tau^{-5/2}\exp\left(-\tau-\frac{(\lambda_L r')^2}{4\tau}\right)\mathrm{d}\tau,$$

which, from equation (7), is equivalent to

$$n(x,y,z) = \frac{i\,\lambda_L^2}{e(4\pi D)}\exp(\lambda_L z)\,z\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{K_{3/2}(\lambda_L r')}{(\lambda_L r')^{3/2}}$$

$$= -\frac{i\,\lambda_L^2}{e(4\pi D)}\exp(\lambda_L z)\frac{z}{\lambda_L r'}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}\frac{\mathrm{d}}{\mathrm{d}(\lambda_L r')}\left(\frac{K_{\frac{1}{2}}(\lambda_L r')}{(\lambda_L r')^{\frac{1}{2}}}\right)$$

$$= -\frac{i}{e(4\pi D)}\frac{\exp(\lambda_L z)}{\lambda_L}\frac{z}{r'}\frac{\mathrm{d}}{\mathrm{d}r'}\left(\frac{\exp(-\lambda_L r')}{r'}\right)$$

$$= -\frac{i\exp(\lambda_L z)}{e(4\pi D)\lambda_L}\frac{\partial}{\partial z}\left(\frac{\exp(-\lambda_L r')}{r'}\right). \tag{12}$$

The expressions for $n(x,y,z)$ in equations (10) and (12) each satisfy the special form of equation (1) in which $\mathrm{d}n/\mathrm{d}t = 0$. Equation (10) is the "pole" solution and equation (12) the "dipole" solution when diffusion is anisotropic, i.e. $D_L \neq D$. It is found in practice that the properties of a diffusing stream issuing from a small hole in a cathode are well described by equation (12) (Lowke 1971; Huxley 1972).

III. References

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