SECOND-ORDER SOLUTIONS FOR STEADY MAGNETOHYDRODYNAMIC CHANNEL FLOW WITH ANISOTROPIC CONDUCTIVITY*

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Abstract

Further investigation of steady magnetohydrodynamic flow through a straight channel of arbitrary cross section with nonconducting walls is considered, in the presence of anisotropic conductivity due to the Hall effect, where no restriction is made on the Reynolds number or magnetic Reynolds number. An approximate solution is provided by a perturbation expansion in terms of the Hall parameter, assumed small. Corrections are made to the first-order solutions established by Panton and Hosking (1971) and the solutions are then extended to the second order for a square channel. It is found that both the Reynolds number and magnetic Reynolds number terms have a significant influence on the mass transport, the former far outweighing the contribution to the flow established by Tani (1962) for the values of the flow parameters assumed.

I. Introduction

In the previous paper by Panton and Hosking (1971; hereinafter referred to as Paper I) an investigation of magnetohydrodynamic (MHD) channel flow with anisotropic conductivity was carried out, where the anisotropy was considered to be due to a significant Hall effect. An error in the boundary condition on $\boldsymbol{B}$ used in that paper necessitated the re-evaluation of the first-order stream function ψ_1 (the y component of the vector potential) and hence v_{1y} and B_{1y} as given in Paper I. The condition for ψ_1 on the boundary was found to be incorrect and the problem had to be extended over all space using a condition on ψ_1 at infinity. Since the solution of ψ_1 over all space was continuous across the boundary of the duct, that part of the solution applying within the duct could then be used for the subsequent calculations involving v_{1y} and B_{1y}. Results obtained by the method of Rayleigh and Ritz showed only slight difference, to first order, from those obtained in Paper I.

The significance of the previous calculations compared with those of Tani (1962) was in the inclusion of two parameters, namely the Reynolds number R and the magnetic Reynolds number R_{m}, which he had assumed to be small ($R \ll 1, R_{\mathrm{m}} \ll 1$). The analysis was performed by constructing minimum principles corresponding to the resulting MHD equations and boundary conditions. These principles were valid for arbitrary channel cross sections but were subsequently solved for a channel of square cross section by the method of Rayleigh and Ritz. The approximate solution was based on an expansion in terms of the Hall parameter k (considered small) to first

* First-order solutions were established in the paper by D. M. Panton and R. J. Hosking which appeared in *Aust. J. Phys.*, 1971, **24**, 61–9.

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order only, with no restriction on R and R_m. In this case, no modification to the mass transport was found but a net axial current was produced by the anisotropy in the conductivity. At the same time a secondary cross flow was established, identical with that obtained by Tani (1962) when R and R_m were very much less than unity.

In the present paper, the expansion is taken to order k^2, and now significant modifications in the mass transport are determined. One of these modifications corresponds to that found by Tani (1962), whilst two others arise from the R and R_m terms. The results show that whilst the R_m terms may be omitted with some justification, the contribution to the flow due to the presence of the R term cannot be ignored, and is more significant than the contribution obtained by Tani (1962). Furthermore, the R contribution is found to decrease the flow, that is, opposite in effect to the modification found by Tani (1962).

Whereas contributions from terms of order k^3 may have some influence on the cross-velocity profiles, these are not of principal interest here and, furthermore, the contribution to the axial velocity from these terms is asymmetric and hence provides no net change in the mass transport.

We consider a steady MHD flow in a straight channel of arbitrary cross section with nonconducting walls. An incompressible fluid with anisotropic conductivity flows in the presence of a uniform transverse magnetic field. A right-handed cartesian axis system is used such that the z axis is parallel to the imposed uniform magnetic field B_{0z} while the centre line of the channel is along the y axis. All physical quantities except pressure are assumed to be independent of y.

II. MHD Equations

The steady flow considered is governed by the following MHD equations: continuity equation for an incompressible fluid

$$\nabla . \boldsymbol{v} = 0 ; \tag{1}$$

equation of motion

$$\rho \boldsymbol{v} . \nabla \boldsymbol{v} + \nabla P = \mu^{-1}(\nabla \times \boldsymbol{B}) \times \boldsymbol{B} + \rho\nu\, \nabla^2 \boldsymbol{v} ; \tag{2}$$

Ohm's law in the form

$$\boldsymbol{j} = \sigma(\boldsymbol{E} + \boldsymbol{v} \times \boldsymbol{B}) - (\kappa/B)\boldsymbol{j} \times \boldsymbol{B} , \tag{3}$$

under the assumption of conditions where the ions in the fluid have no spiral paths, whereupon $\omega_i \tau_i \ll 1$, ω_i being the ion cyclotron frequency and τ_i the ion–neutral collision time, and thus the gradients of the electron pressure P_e can be neglected (Liubimov 1962); the electromagnetic equations

$$\nabla . \boldsymbol{B} = 0, \qquad \mu \boldsymbol{j} = \nabla \times \boldsymbol{B}, \qquad \nabla \times \boldsymbol{E} = 0 ; \tag{4}$$

and the equation of magnetic induction

$$\eta^{-1}\, \nabla \times (\boldsymbol{v} \times \boldsymbol{B}) - \nabla \times \nabla \times \boldsymbol{B} - \kappa B^{-1}\, \nabla \times \{(\nabla \times \boldsymbol{B}) \times \boldsymbol{B}\} = 0 , \tag{5}$$

where (5) may be obtained by taking the curl of (3) and using equations (4). In these equations $\boldsymbol{v}$ denotes the fluid velocity, $\boldsymbol{B}$ the magnetic field density, $\boldsymbol{E}$ the electric field, $\boldsymbol{j}$ the electric current density, ρ the fluid density, P the pressure, μ the magnetic permeability, and ν the kinematic viscosity. The resistivity $\eta = (\mu\sigma)^{-1}$, where σ is the fluid conductivity which is assumed to be constant. The anisotropy in the conductivity is due to the Hall term, which is characteristized by the parameter κ in equation (3). If the fluid is fully ionized we have $\kappa = \omega_e \tau_e = \sigma B/en_e$, where ω_e denotes the electron cyclotron frequency, τ_e the ion–electron collision time, e the electron charge, and n_e the electron number density. We find $\kappa \ll 1$ is a good assumption for many laboratory MHD flows, and consequently κ is a natural expansion parameter.

III. Perturbation Equations

We seek perturbation solutions of the MHD equations (1)–(5) by expanding the physical quantities in the form

$$f = f_0 + kf_1 + k^2 f_2 + \dots, \tag{6}$$

where $k = \kappa B_{0z}/B$ is very much less than unity. The zeroth-order solution corresponds to the primary isotropic conductivity flow (Shercliff 1953; Tani 1962). The equations relating to the zeroth order may be found in Paper I, but the first-order equations are now modified slightly and will be considered afresh, along with the second-order terms.

To first order, the MHD equations (1)–(5) may be expressed as

$$\nabla . \boldsymbol{v}_1 = 0\,; \tag{1a}$$

$$\rho(\boldsymbol{v}_0 . \nabla \boldsymbol{v}_1 + \boldsymbol{v}_1 . \nabla \boldsymbol{v}_0) + \nabla P_1 = \mu^{-1}\{(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_1 + (\nabla \times \boldsymbol{B}_1) \times \boldsymbol{B}_0\} + \rho\nu\, \nabla^2 \boldsymbol{v}_1\,; \tag{2a}$$

$$\boldsymbol{j}_1 = \sigma(\boldsymbol{E}_1 + \boldsymbol{v}_0 \times \boldsymbol{B}_1 + \boldsymbol{v}_1 \times \boldsymbol{B}_0) - B_{0z}^{-1}(\boldsymbol{j}_0 \times \boldsymbol{B}_0)\,; \tag{3a}$$

$$\nabla . \boldsymbol{B}_1 = 0\,, \qquad \mu \boldsymbol{j}_1 = \nabla \times \boldsymbol{B}_1\,, \qquad \nabla \times \boldsymbol{E}_1 = 0\,; \tag{4a}$$

and

$$\eta^{-1}\{\nabla \times (\boldsymbol{v}_1 \times \boldsymbol{B}_0) + \nabla \times (\boldsymbol{v}_0 \times \boldsymbol{B}_1)\} - \nabla \times \nabla \times \boldsymbol{B}_1 - B_{0z}^{-1}\, \nabla \times \{(\nabla \times \boldsymbol{B}_0) \times \boldsymbol{B}_0\} = 0\,; \tag{5a}$$

where

$$\boldsymbol{v}_0 = (0, v_{0y}(x,z), 0)\,, \qquad \boldsymbol{B}_0 = (0, B_{0y}(x,z), B_{0z})\,,$$

$$\boldsymbol{v}_1 = (v_{1x}, v_{1y}, v_{1z})\,, \qquad \boldsymbol{B}_1 = (B_{1x}, B_{1y}, B_{1z})\,,$$

$$\nabla = (\partial/\partial x\,, 0, \partial/\partial z)\,,$$

and all derivatives with respect to y vanish except the zeroth-order pressure gradient $\partial P_0/\partial y = -G$ (constant).

We now introduce the dimensionless quantities

$$x^* = x/a, \qquad y^* = y/a, \qquad z^* = z/a,$$

$$P^* = P/\rho V_0^2, \qquad \boldsymbol{v}^* = \boldsymbol{v}/V_0, \qquad B_{0y}^* = B_{0y}/B_{0z}R_{\mathrm{m}},$$

$$B_{0z}^* = 1 \qquad \boldsymbol{B}_1^* = \boldsymbol{B}_1/B_{0z}R_{\mathrm{m}}, \qquad \boldsymbol{B}_2^* = \boldsymbol{B}_2/B_{0z}R_{\mathrm{m}},$$

$$\boldsymbol{j}^* = \boldsymbol{j}/B_{0z}V_0\sigma,$$

where

$$R_{\mathrm{m}} = V_0 a/\eta, \qquad R = V_0 a/\nu, \qquad M = B_{0z}a(\sigma/\rho\nu)^{\frac{1}{2}},$$

in which a and V_0 denote the reference length and reference velocity and R, R_{m}, and M are the Reynolds number, the magnetic Reynolds number, and the Hartmann number respectively. The components of equations (2a) and (5a), with the asterisks omitted for convenience, are then given by

$$R\frac{\partial P_1}{\partial x} = -M^2R_{\mathrm{m}}\left(B_{1y}\frac{\partial B_{0y}}{\partial x}+B_{0y}\frac{\partial B_{1y}}{\partial x}\right)+M^2\nabla^2\psi_1+\frac{\partial(\nabla^2\chi_1)}{\partial z}, \tag{7}$$

$$R\left(\frac{\partial\chi_1}{\partial z}\frac{\partial v_{0y}}{\partial x}-\frac{\partial\chi_1}{\partial x}\frac{\partial v_{0y}}{\partial z}\right) = M^2R_{\mathrm{m}}\left(\frac{\partial\psi_1}{\partial z}\frac{\partial B_{0y}}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial B_{0y}}{\partial z}\right)+M^2\frac{\partial B_{1y}}{\partial z}+\nabla^2 v_{1y}, \tag{8}$$

$$R\frac{\partial P_1}{\partial z} = -M^2R_{\mathrm{m}}\left(B_{1y}\frac{\partial B_{0y}}{\partial z}+B_{0y}\frac{\partial B_{1y}}{\partial z}\right)-\frac{\partial(\nabla^2\chi_1)}{\partial x}, \tag{9}$$

$$-\frac{\partial(\nabla^2\psi_1)}{\partial z} = \frac{\partial^2\chi_1}{\partial z^2}+\frac{\partial^2 B_{0y}}{\partial z^2}, \tag{10}$$

$$-\nabla^2 B_{1y} = \frac{\partial v_{1y}}{\partial z}+R_{\mathrm{m}}\left(\frac{\partial\psi_1}{\partial z}\frac{\partial v_{0y}}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial v_{0y}}{\partial z}-\frac{\partial\chi_1}{\partial z}\frac{\partial B_{0y}}{\partial x}+\frac{\partial\chi_1}{\partial x}\frac{\partial B_{0y}}{\partial z}\right), \tag{11}$$

and

$$\frac{\partial(\nabla^2\psi_1)}{\partial x} = -\frac{\partial^2\chi_1}{\partial x\,\partial z}-\frac{\partial^2 B_{0y}}{\partial x\,\partial z}, \tag{12}$$

where the stream functions χ_1 and ψ_1 have been introduced such that

$$v_{1x} = \partial\chi_1/\partial z, \quad v_{1z} = -\partial\chi_1/\partial x, \qquad B_{1x} = \partial\psi_1/\partial z, \quad B_{1z} = -\partial\psi_1/\partial x. \tag{13}$$

The y component of equation (3a) is

$$j_{1y} = -\partial\chi_1/\partial z+j_{0x}, \tag{14}$$

since $E_y = 0$. (For $\nabla\times\boldsymbol{E} = 0$, E_y is constant and, since $\langle\boldsymbol{n}\times\boldsymbol{E}\rangle = 0$ across the fluid wall interface, E_y is continuous across this boundary. Assuming that E_y is zero at infinity, then $E_y = 0$ everywhere.)

Elimination between equations (7) and (9) yields

$$\nabla^2\nabla^2\chi_1 + M^2\,\partial(\nabla^2\psi_1)/\partial z = 0\,, \tag{15}$$

and hence with the help of equation (10)

$$\nabla^2\nabla^2\chi_1 = M^2(\partial^2\chi_1/\partial z^2 + \partial^2 B_{0y}/\partial z^2)\,. \tag{16}$$

Although equations (10)–(12) have been obtained by curling Ohm's Law, it is found preferable to use the y component of this equation for the evaluation of ψ_1, since it is of lower order in ψ_1 than either (10) or (12). Using the second of equations (4) in dimensionless form, to both zeroth and first order we obtain

$$j_{1y} = \partial B_{1x}/\partial z - \partial B_{1z}/\partial x = -\partial\chi_1/\partial z - \partial B_{0y}/\partial z\,,$$

which with the introduction of the stream function ψ_1 gives

$$\nabla^2\psi_1 = -\partial\chi_1/\partial z - \partial B_{0y}/\partial z\,. \tag{17}$$

Subject to the appropriate boundary conditions, equations (16) and (17) may be solved for χ_1 and ψ_1, and the solutions used in (8) and (11) for v_{1y} and B_{1y}.

The second-order perturbation equations derived from equations (1)–(5) are

$$\nabla\,.\,\boldsymbol{v}_2 = 0\,, \tag{1b}$$

$$\rho\boldsymbol{v}_1\,.\,\nabla\boldsymbol{v}_1 + \rho\boldsymbol{v}_2\,.\,\nabla\boldsymbol{v}_0 + \nabla P_2 = \mu^{-1}\{(\nabla\times\boldsymbol{B}_0)\times\boldsymbol{B}_2 + (\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_1 + (\nabla\times\boldsymbol{B}_2)\times\boldsymbol{B}_0\} + \rho\nu\,\nabla^2\boldsymbol{v}_2\,, \tag{2b}$$

$$j_{2y} = \sigma(v_{1z}B_{1x} - v_{1x}B_{1z} - B_{0z}v_{2x}) - B_{0z}^{-1}(j_{0z}B_{1x} - j_{0x}B_{1z} - B_{0z}j_{1x})\,, \tag{3b}$$

$$\nabla\,.\,\boldsymbol{B}_2 = 0\,, \qquad \mu\boldsymbol{j}_2 = \nabla\times\boldsymbol{B}_2\,, \qquad \nabla\times\boldsymbol{E}_2 = 0\,, \tag{4b}$$

$$\eta^{-1}\{\nabla\times(\boldsymbol{v}_0\times\boldsymbol{B}_2) + \nabla\times(\boldsymbol{v}_1\times\boldsymbol{B}_1) + \nabla\times(\boldsymbol{v}_2\times\boldsymbol{B}_0)\} - \nabla\times(\nabla\times\boldsymbol{B}_2) - B_{0z}^{-1}[\nabla\times\{(\nabla\times\boldsymbol{B}_0)\times\boldsymbol{B}_1\} - \nabla\times\{(\nabla\times\boldsymbol{B}_1)\times\boldsymbol{B}_0\}] = 0\,, \tag{5b}$$

where

$$\boldsymbol{v}_2 = (v_{2x}, v_{2y}, v_{2z})\,, \qquad \boldsymbol{B}_2 = (B_{2x}, B_{2y}, B_{2z})\,.$$

When dimensionless quantities are introduced, equations (2b) and (5b) in component form, with the asterisks again omitted for convenience, are

$$R\left(\frac{\partial\chi_1}{\partial z}\frac{\partial^2\chi_1}{\partial x\,\partial z} - \frac{\partial\chi_1}{\partial x}\frac{\partial^2\chi_1}{\partial z^2}\right) + R\frac{\partial P_2}{\partial x} = -M^2R_{\mathrm{m}}B_{2y}\frac{\partial B_{0y}}{\partial x} + M^2\,\nabla^2\psi_2 + M^2R_{\mathrm{m}}\left(-\frac{\partial\psi_1}{\partial x}(\nabla^2\psi_1) - B_{1y}\frac{\partial B_{1y}}{\partial x}\right) - M^2R_{\mathrm{m}}B_{0y}\frac{\partial B_{2y}}{\partial x} + \frac{\partial(\nabla^2\chi_2)}{\partial z}\,, \tag{18}$$

$$R\left(\frac{\partial\chi_1}{\partial z}\frac{\partial v_{1y}}{\partial x}-\frac{\partial\chi_1}{\partial x}\frac{\partial v_{1y}}{\partial z}\right)+R\left(\frac{\partial\chi_2}{\partial z}\frac{\partial v_{0y}}{\partial x}-\frac{\partial\chi_2}{\partial x}\frac{\partial v_{0y}}{\partial z}\right)$$

$$=M^2R_{\mathrm{m}}\left(\frac{\partial\psi_2}{\partial z}\frac{\partial B_{0y}}{\partial x}-\frac{\partial\psi_2}{\partial x}\frac{\partial B_{0y}}{\partial z}\right)+M^2R_{\mathrm{m}}\left(\frac{\partial\psi_1}{\partial z}\frac{\partial B_{1y}}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial B_{1y}}{\partial z}\right)+M^2\frac{\partial B_{2y}}{\partial z}+\nabla^2 v_{2y}, \quad (19)$$

$$R\left(\frac{\partial\chi_1}{\partial x}\frac{\partial^2\chi_1}{\partial x\,\partial z}-\frac{\partial\chi_1}{\partial z}\frac{\partial^2\chi_1}{\partial x^2}\right)+R\frac{\partial P_2}{\partial z}$$

$$=-M^2R_{\mathrm{m}}\left(B_{2y}\frac{\partial B_{0y}}{\partial z}+B_{1y}\frac{\partial B_{1y}}{\partial z}+\frac{\partial\psi_1}{\partial z}(\nabla^2\psi_1)+B_{0y}\frac{\partial B_{2y}}{\partial z}\right)-\frac{\partial(\nabla^2\chi_2)}{\partial x}, \quad (20)$$

$$-\frac{\partial(\nabla^2\psi_2)}{\partial z}=\frac{\partial^2\chi_2}{\partial z^2}+\frac{\partial^2 B_{1y}}{\partial z^2}$$

$$+R_{\mathrm{m}}\left(-\frac{\partial\chi_1}{\partial z}\frac{\partial^2\psi_1}{\partial x\,\partial z}-\frac{\partial\psi_1}{\partial x}\frac{\partial^2\chi_1}{\partial z^2}+\frac{\partial\chi_1}{\partial x}\frac{\partial^2\psi_1}{\partial z^2}+\frac{\partial\psi_1}{\partial z}\frac{\partial^2\chi_1}{\partial x\,\partial z}\right.$$

$$\left.+\frac{\partial\psi_1}{\partial z}\frac{\partial^2 B_{0y}}{\partial x\,\partial z}+\frac{\partial B_{0y}}{\partial x}\frac{\partial^2\psi_1}{\partial z^2}-\frac{\partial\psi_1}{\partial x}\frac{\partial^2 B_{0y}}{\partial z^2}-\frac{\partial B_{0y}}{\partial z}\frac{\partial^2\psi_1}{\partial x\,\partial z}\right), \quad (21)$$

$$-\nabla^2 B_{2y}=\frac{\partial v_{2y}}{\partial z}-\frac{\partial(\nabla^2\psi_1)}{\partial z}+R_{\mathrm{m}}\left(-\frac{\partial\psi_2}{\partial x}\frac{\partial v_{0y}}{\partial z}+\frac{\partial\psi_2}{\partial z}\frac{\partial v_{0y}}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial v_{1y}}{\partial z}\right.$$

$$\left.+\frac{\partial\chi_1}{\partial x}\frac{\partial B_{1y}}{\partial z}-\frac{\partial\chi_1}{\partial z}\frac{\partial B_{1y}}{\partial x}+\frac{\partial\psi_1}{\partial z}\frac{\partial v_{1y}}{\partial x}+\frac{\partial\chi_2}{\partial x}\frac{\partial B_{0y}}{\partial x}-\frac{\partial\chi_2}{\partial z}\frac{\partial B_{0y}}{\partial x}\right), \quad (22)$$

$$\frac{\partial(\nabla^2\psi_2)}{\partial x}=-\frac{\partial^2\chi_2}{\partial x\,\partial z}-\frac{\partial^2 B_{1y}}{\partial x\,\partial z}$$

$$+R_{\mathrm{m}}\left(-\frac{\partial\chi_1}{\partial x}\frac{\partial^2\psi_1}{\partial x\,\partial z}-\frac{\partial\psi_1}{\partial z}\frac{\partial^2\chi_1}{\partial x^2}+\frac{\partial\chi_1}{\partial z}\frac{\partial^2\psi_1}{\partial x^2}+\frac{\partial\psi_1}{\partial x}\frac{\partial^2\chi_1}{\partial x\,\partial z}\right.$$

$$\left.-\frac{\partial\psi_1}{\partial z}\frac{\partial^2 B_{0y}}{\partial x^2}-\frac{\partial B_{0y}}{\partial x}\frac{\partial^2\psi_1}{\partial x\,\partial z}+\frac{\partial\psi_1}{\partial x}\frac{\partial^2 B_{0y}}{\partial x\,\partial z}+\frac{\partial B_{0y}}{\partial z}\frac{\partial^2\psi_1}{\partial x^2}\right), \quad (23)$$

$$\nabla^2\psi_2=-\frac{\partial\chi_2}{\partial z}-\frac{\partial B_{1y}}{\partial z}+R_{\mathrm{m}}\left(-\frac{\partial\chi_1}{\partial x}\frac{\partial\psi_1}{\partial z}+\frac{\partial\chi_1}{\partial z}\frac{\partial\psi_1}{\partial x}-\frac{\partial B_{0y}}{\partial x}\frac{\partial\psi_1}{\partial z}+\frac{\partial B_{0y}}{\partial z}\frac{\partial\psi_1}{\partial x}\right), \quad (24)$$

where

$$v_{2x}=\partial\chi_2/\partial z, \quad v_{2z}=-\partial\chi_2/\partial x, \qquad B_{2x}=\partial\psi_2/\partial z, \quad B_{2z}=-\partial\psi_2/\partial x. \quad (25)$$

Elimination between equations (18) and (20) gives

$$R\left(\frac{\partial\chi_1}{\partial z}\frac{\partial(\nabla^2\chi_1)}{\partial x}-\frac{\partial\chi_1}{\partial x}\frac{\partial(\nabla^2\chi_1)}{\partial z}\right)=\nabla^4\chi_2+M^2R_{\mathrm{m}}\left(\frac{\partial\psi_1}{\partial z}\frac{\partial(\nabla^2\psi_1)}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial(\nabla^2\psi_1)}{\partial z}\right)+M^2\frac{\partial(\nabla^2\psi_2)}{\partial z}, \quad (26)$$

and hence from equation (21)

$$\nabla^2\nabla^2\chi_2 - M^2\frac{\partial^2\chi_2}{\partial z^2} = M^2\frac{\partial^2 B_{1y}}{\partial z^2} + R\left(\frac{\partial\chi_1}{\partial z}\frac{\partial(\nabla^2\chi_1)}{\partial x} - \frac{\partial\chi_1}{\partial x}\frac{\partial(\nabla^2\chi_1)}{\partial z}\right)$$
$$-M^2R_{\mathrm{m}}\left(\frac{\partial\psi_1}{\partial z}\frac{\partial(\nabla^2\psi_1)}{\partial x} - \frac{\partial\psi_1}{\partial x}\frac{\partial(\nabla^2\psi_1)}{\partial z} + \frac{\partial\psi_1}{\partial x}\frac{\partial^2(\chi_1+B_{0y})}{\partial z^2}\right.$$
$$\left. - \frac{\partial\psi_1}{\partial z}\frac{\partial^2(\chi_1+B_{0y})}{\partial x\,\partial z} - \frac{\partial^2\psi_1}{\partial z^2}\frac{\partial(\chi_1+B_{0y})}{\partial x} + \frac{\partial^2\psi_1}{\partial x\,\partial z}\frac{\partial(\chi_1+B_{0y})}{\partial z}\right). \quad (27)$$

The stream function ψ_2 may now be evaluated from equation (24), and equations (19) and (22) will then give v_{2y} and B_{2y} under appropriate boundary conditions.

We note that in the limits $R, R_{\mathrm{m}} \to 0$, equations (19) and (22) yield expressions that are equivalent to those used by Tani (1962).

IV. Boundary Conditions

The general boundary condition for the velocity $\boldsymbol{v}$ of a viscous fluid flowing past a solid surface applies on the boundary Γ, and is $\boldsymbol{v} = 0$. In terms of the stream function this implies

$$\chi = \text{const.}, \qquad \partial\chi/\partial n = 0, \quad (28)$$

on Γ, where $\partial/\partial n$ denotes the derivative in the direction normal to Γ.

The boundary condition on $\boldsymbol{B}$ as given in Paper I is incorrect and must be considered afresh. The condition $\boldsymbol{n}\,.\,\boldsymbol{j} = 0$ which must apply on the walls of an insulated duct gives $\partial B_y/\partial s = 0$, where s is a coordinate drawn along the boundary Γ. Thus B_y is constant on Γ and since the net current carried by the duct is zero this constant must also be zero (see e.g. Roberts 1967). Hence we have

$$B_y = 0 \qquad \text{on} \qquad \Gamma. \quad (29)$$

The components B_x and B_z, however, are subject only to the conditions $\langle\boldsymbol{n}\,.\,\boldsymbol{B}\rangle = 0$ and $\langle\boldsymbol{n}\times\boldsymbol{B}\rangle = 0$, where the angle brackets denote the jump across the fluid–wall interface, so that a knowledge of the external field is necessary to allow specification of these components on the boundary. In order to determine the solutions for B_x and B_z and hence ψ within the boundary Γ, a solution of the stream function ψ must be found over all space using the condition

$$\psi \to 0 \qquad \text{at} \qquad \text{infinity}. \quad (30)$$

The solution of ψ within the boundary Γ may then be retrieved as a part of the solution over all space which is continuous across the boundary Γ.

V. Variational Principles

The boundary value problems for the first- and second-order solutions, subject to the conditions (28), (29), and (30) are now replaced by corresponding functionals whose extremals give the required solutions. The variational principles relating to the first-order equations are identical with those in Paper I except for the principle involving ψ_1, which will now be examined.

Consider the integral

$$I_1(\psi_1) = \iint\limits_{\text{all space}} \left[\tfrac{1}{2}\left\{\left(\frac{\partial\psi_1}{\partial x}\right)^2+\left(\frac{\partial\psi_1}{\partial z}\right)^2\right\}-\psi_1\left(\frac{\partial\chi_1}{\partial z}+\frac{\partial B_{0y}}{\partial z}\right)\right]\mathrm{d}x\,\mathrm{d}z, \tag{31}$$

where χ_1 and B_{0y} are known functions of x and z. Suppose that $\psi_1(x,z)$ minimizes I_1 and that $\psi_1'(x,z)$ represents a variation in ψ_1 such that

$$\psi_1'(x,z) = \psi_1(x,z)+\epsilon\eta(x,z), \tag{32}$$

where ϵ is small and $\eta(x,z)$ is arbitrary. Then

$$\delta I_1(\psi_1) = \iint\limits_{\text{all space}} \left\{-\frac{\partial\chi_1}{\partial z}-\frac{\partial B_{0y}}{\partial z}-\frac{\partial}{\partial x}\left(\frac{\partial\psi_1}{\partial x}\right)-\frac{\partial}{\partial z}\left(\frac{\partial\psi_1}{\partial z}\right)\right\}\eta(x,z)\,\mathrm{d}x\,\mathrm{d}z$$

$$+\int_c \eta(x,z)\left(\frac{\partial\psi_1}{\partial x}\mathrm{d}z-\frac{\partial\psi_1}{\partial z}\mathrm{d}x\right) = 0 \tag{33}$$

(see e.g. Kantorovich and Krylov 1958), where c is the "contour at infinity". By the condition (30) η is zero on c and hence the second integral vanishes. Thus since $\eta(x,z)$ is arbitrary $\delta I_1 = 0$ if and only if equation (17) is valid over all space.

The structure of the second-order equations is essentially similar to the first-order perturbation equations for which variational procedures have been considered both in Paper I and above. Variational principles related to equations (27), (24), (19), and (22) will thus be stated only.

The integral

$$I_2(\chi_2) = \iint\limits_{A} \left[\tfrac{1}{2}(\nabla^2\chi_2)^2+\tfrac{1}{2}M^2\left(\frac{\partial\chi_2}{\partial z}\right)^2-\chi_2\left\{M^2\frac{\partial^2 B_{1y}}{\partial z^2}+R\left(\frac{\partial\chi_1}{\partial z}\frac{\partial(\nabla^2\chi_1)}{\partial x}-\frac{\partial\chi_1}{\partial x}\frac{\partial(\nabla^2\chi_1)}{\partial z}\right)\right.\right.$$

$$-M^2R_{\mathrm{m}}\left(\frac{\partial\psi_1}{\partial z}\frac{\partial(\nabla^2\psi_1)}{\partial x}-\frac{\partial\psi_1}{\partial x}\frac{\partial(\nabla^2\psi_1)}{\partial z}+\frac{\partial\psi_1}{\partial x}\frac{\partial^2\chi_1}{\partial z^2}+\frac{\partial\psi_1}{\partial x}\frac{\partial^2 B_{0y}}{\partial z^2}-\frac{\partial\psi_1}{\partial z}\frac{\partial^2\chi_1}{\partial x\,\partial z}\right.$$

$$\left.\left.\left.-\frac{\partial\psi_1}{\partial z}\frac{\partial^2 B_{0y}}{\partial x\,\partial z}-\frac{\partial^2\psi_1}{\partial z^2}\frac{\partial\chi_1}{\partial x}-\frac{\partial^2\psi_1}{\partial z^2}\frac{\partial B_{0y}}{\partial x}+\frac{\partial^2\psi_1}{\partial x\,\partial z}\frac{\partial\chi_1}{\partial z}+\frac{\partial^2\psi_1}{\partial x\,\partial z}\frac{\partial B_{0y}}{\partial z}\right)\right\}\right]\mathrm{d}x\,\mathrm{d}z \tag{34}$$

is stationary for χ_2, satisfying boundary condition (28) and equation (27).

Similarly, the integral

$$I_2(\psi_2) = \iint\limits_{\text{all space}} \left[\tfrac{1}{2}\left\{\left(\frac{\partial\psi_2}{\partial x}\right)^2+\left(\frac{\partial\psi_2}{\partial z}\right)^2\right\}-\psi_2\left\{\frac{\partial\chi_2}{\partial z}+\frac{\partial B_{1y}}{\partial z}+R_{\mathrm{m}}\left(\frac{\partial\chi_1}{\partial x}\frac{\partial\psi_1}{\partial z}-\frac{\partial\chi_1}{\partial z}\frac{\partial\psi_1}{\partial x}\right.\right.\right.$$

$$\left.\left.\left.+\frac{\partial B_{0y}}{\partial x}\frac{\partial\psi_1}{\partial z}-\frac{\partial B_{0y}}{\partial z}\frac{\partial\psi_1}{\partial x}\right)\right\}\right]\mathrm{d}x\,\mathrm{d}z \tag{35}$$

is stationary for values of ψ_2 satisfying (30) and (24).

Finally, the integral

$$
\begin{aligned}
I_4(v_{2y},B_{2y}) = \iint\limits_{\mathrm{A}} \Bigg[& -\tfrac{1}{2}\left\{\left(\frac{\partial v_{2y}}{\partial x}\right)^2+\left(\frac{\partial v_{2y}}{\partial z}\right)^2\right\}+\tfrac{1}{2}M^2\left\{v_{2y}\frac{\partial B_{2y}}{\partial z}-B_{2y}\frac{\partial v_{2y}}{\partial z}+\left(\frac{\partial B_{2y}}{\partial x}\right)^2\right. \\
& \left.+\left(\frac{\partial B_{2y}}{\partial z}\right)^2\right\}+v_{2y}\left\{-R\left(\frac{\partial \chi_2}{\partial z}\frac{\partial v_{0y}}{\partial x}-\frac{\partial \chi_2}{\partial x}\frac{\partial v_{0y}}{\partial z}+\frac{\partial \chi_1}{\partial z}\frac{\partial v_{1y}}{\partial x}-\frac{\partial \chi_1}{\partial x}\frac{\partial v_{1y}}{\partial z}\right)\right. \\
& \left.+M^2R_{\mathrm{m}}\left(\frac{\partial \psi_1}{\partial z}\frac{\partial B_{1y}}{\partial x}-\frac{\partial \psi_1}{\partial x}\frac{\partial B_{1y}}{\partial z}+\frac{\partial \psi_2}{\partial z}\frac{\partial B_{0y}}{\partial x}-\frac{\partial \psi_2}{\partial x}\frac{\partial B_{0y}}{\partial z}\right)\right\} \\
& +B_{2y}\left\{M^2\frac{\partial(\nabla^2\psi_1)}{\partial z}-M^2R_{\mathrm{m}}\left(-\frac{\partial \psi_2}{\partial x}\frac{\partial v_{0y}}{\partial z}+\frac{\partial \psi_2}{\partial z}\frac{\partial v_{0y}}{\partial x}-\frac{\partial \psi_1}{\partial x}\frac{\partial v_{1y}}{\partial z}\right.\right. \\
& \left.\left.+\frac{\partial \chi_1}{\partial x}\frac{\partial B_{1y}}{\partial z}-\frac{\partial \chi_1}{\partial z}\frac{\partial B_{1y}}{\partial x}+\frac{\partial \psi_1}{\partial z}\frac{\partial v_{1y}}{\partial x}+\frac{\partial \chi_2}{\partial x}\frac{\partial B_{0y}}{\partial z}-\frac{\partial \chi_2}{\partial z}\frac{\partial B_{0y}}{\partial x}\right)\right\}\Bigg]\,\mathrm{d}x\,\mathrm{d}z
\end{aligned}
\tag{36}
$$

is stationary for values of v_{2y} and B_{2y} which vanish on the boundary Γ of the region A and satisfy equations (19) and (22).

VI. Numerical Solution for a Square Channel

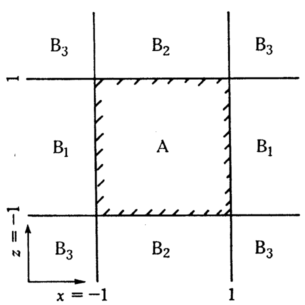


Fig. 1.—Regions of solution for ψ_1.

We now consider a numerical solution for a channel of square cross section, with $|x| \leqslant 1$ and $|z| \leqslant 1$ (region A in Fig. 1), using the direct method of Rayleigh and Ritz. The form of the solutions of the zeroth-order variables v_{0y} and B_{0y}, and the first-order terms χ_1, v_{1y}, and B_{1y} are identical with those used in the original calculation of Paper I. The solution for ψ_1 over all space will now be considered in detail.

We divide the whole space into four distinct regions of influence as shown in Figure 1. Within the duct (region A), we assume the trial form

$$
\psi_{1\mathrm{A}}(x,z) = \sum_{m,n} a_{mn}\, x^{2m} z^{2n} = P(x,z)\,, \tag{37}
$$

where the symmetry in x and z is determined from the trial forms of χ_1 and B_{0y} and equation (17).

Outside the duct we must assume forms for ψ_1 which vanish as x and z tend to infinity. In regions B_1, where $|z| \leqslant 1$, we choose

$$
\psi_{1\mathrm{B}_1}(x,z) = P(x,z)/x^{2r}\,, \tag{38}
$$

satisfying the condition (30) as $|x| \to \infty$, whilst in regions B_2, where $|x| \leqslant 1$, we choose

$$
\psi_{1\mathrm{B}_2}(x,z) = P(x,z)/z^{2s}\,, \tag{39}
$$

satisfying (30) as $|z| \to \infty$, and in regions B_3

$$\psi_{1B_3}(x,z) = P(x,z)/x^{2r} z^{2s}, \tag{40}$$

satisfying (30) as $|x|$ and $|z|$ both approach infinity.

The trial forms can now be substituted in equation (31), which may be written in the form

$$I_1(\psi_1) = \iint\limits_{A} F(x,z)\,dx\,dz + 2\iint\limits_{B_1} G(x,z)\,dx\,dz + 2\iint\limits_{B_2} G(x,z)\,dx\,dz$$

$$+4\iint\limits_{B_3} G(x,z)\,dx\,dz, \tag{41}$$

where

$$F(x,z) = \tfrac{1}{2}\left\{\left(\frac{\partial\psi_1}{\partial x}\right)^2 + \left(\frac{\partial\psi_1}{\partial z}\right)^2\right\} - \psi_1\left(\frac{\partial\chi_1}{\partial z} + \frac{\partial B_{0y}}{\partial z}\right), \tag{42}$$

$$G(x,z) = \tfrac{1}{2}\left\{\left(\frac{\partial\psi_1}{\partial x}\right)^2 + \left(\frac{\partial\psi_1}{\partial z}\right)^2\right\}, \tag{43}$$

and the coefficients a_{mn} are determined to minimize $I_1(\psi_1)$.

In region A, where values of the coefficients for χ_1 and B_{0y} have already been evaluated, substitution of the trial form (37) into the first part of equation (41) gives

$$I_{1A}(\psi_1) = \sum_{m,n=0}^{N} a_{mn}\left(\sum_{k,l=0}^{N} a_{kl} A^{(1)}_{m,n,k,l} - \sum_{k,l=0}^{3} \frac{4c_{kl}}{(2m+2k+1)(2n+2l+1)}\right), \tag{44}$$

where N is an integer to be determined, the c_{kl} are known coefficients relating to χ_1 and B_{0y}, and

$$A^{(1)}_{m,n,k,l} = \frac{8mk}{(2m+2k-1)(2n+2l+1)} + \frac{8nl}{(2m+2k+1)(2n+2l-1)}. \tag{45}$$

In the outer regions $\partial\chi_1/\partial z + \partial B_{0y}/\partial z$ vanishes, and we are left with the minimization of three integrals involving ψ_1 only. The trial form for region B_1 may be written

$$\psi_{1B_1}(x,z) = \sum_{m,n=0}^{N} a_{mn}\, x^{2m-2r} z^{2n},$$

where r is an integer $\geqslant N+1$ for convergence of the integral. Substitution into the second part of (41) gives

$$I_{1B_1}(\psi_1) = \sum_{m,n=0}^{N} \sum_{k,l=0}^{N} a_{mn} a_{kl} A^{(2)}_{m,n,k,l}, \tag{46}$$

where

$$A^{(2)}_{m,n,k,l} = -\frac{4(m-r)(k-r)}{(2m+2k-4r-1)(2n+2l+1)} - \frac{4nl}{(2m+2k-4r+1)(2n+2l-1)}. \tag{47}$$

For region B_2 we choose $s \geqslant N+1$ in the trial form (39) for convergence of the integral, giving

$$I_{1B_2}(\psi_1) = \sum_{m,n=0}^{N} \sum_{k,l=0}^{N} a_{mn} a_{kl} A^{(3)}_{m,n,k,l}, \tag{48}$$

where

$$A^{(3)}_{m,n,k,l} = -\frac{4mk}{(2m+2k-1)(2n+2l-4s+1)} - \frac{4(n-s)(l-s)}{(2m+2k+1)(2n+2l-4s-1)}. \quad (49)$$

Finally, in region B_3 for $r, s \geqslant N+1$, we obtain

$$I_{1B_3}(\psi_1) = \sum_{m,n=0}^{N} \sum_{k,l=0}^{N} a_{mn}\, a_{kl}\, A^{(4)}_{m,n,k,l}, \quad (50)$$

where

$$A^{(4)}_{m,n,k,l} = \frac{2(m-r)(k-r)}{(2m+2k-4r-1)(2n+2l-4s+1)} + \frac{2(n-s)(l-s)}{(2m+2k-4r+1)(2n+2l-4s-1)}. \quad (51)$$

Combination of the above expressions for the four regions then gives

$$I_1(\psi_1) = \sum_{m,n=0}^{N} a_{mn}\left(\sum_{k,l=0}^{N} a_{kl}\, A_{m,n,k,l} - \sum_{k,l=0}^{3} \frac{4c_{kl}}{(2m+2k+1)(2n+2l+1)}\right), \quad (52)$$

where

$$A_{m,n,k,l} = A^{(1)}_{m,n,k,l} + 2A^{(2)}_{m,n,k,l} + 2A^{(3)}_{m,n,k,l} + 4A^{(4)}_{m,n,k,l}. \quad (53)$$

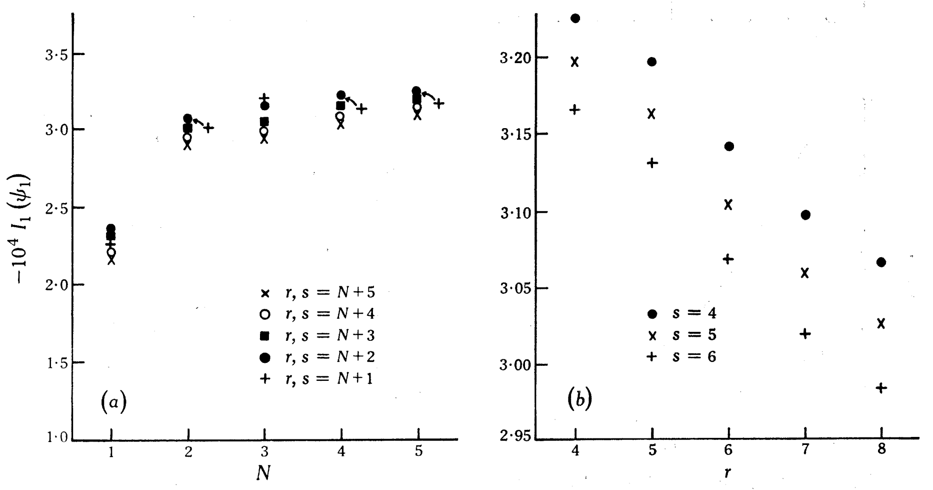


Fig. 2.—Values of the integral $I_1(\psi_1)$ at Hartmann number $M = 0\cdot5$ for (a) selected values of N, r, and s and (b) $N = 3$ and selected values of r and s.

The expression for $I_1(\psi_1)$ is now minimized with respect to the parameters a_{ij} such that $\partial I_1/\partial a_{ij}$ yields the (i,j) element of an array whose inverse provides the required solutions. Thus

$$\left[\frac{\partial I}{\partial a_{ij}}\right] = \left[\sum_{m,n=0}^{N} a_{mn}(A_{i,j,m,n} + A_{m,n,i,j}) - \sum_{k,l=0}^{3} \frac{4c_{kl}}{(2i+2k+1)(2j+2l+1)}\right] = 0, \quad (54)$$

for $0 \leqslant i, j \leqslant N$, is the matrix equation to be solved for the determination of the parameters a_{mn}.

In the numerical calculations that were carried out, the minimization of I_1 for a particular value of the Hartmann number depended on a suitable choice of N and consequently r and s. Parameters a_{mn} were evaluated for a range of Hartmann numbers and for values of N from 1 to 5. At the same time, r and s were allotted values within the range $N+1$ to $N+5$. Substitution of these solutions in the integral I_1 (equation (52)) then gave an indication of the correct choice of the parameters N, r, and s for the minimization of I_1.

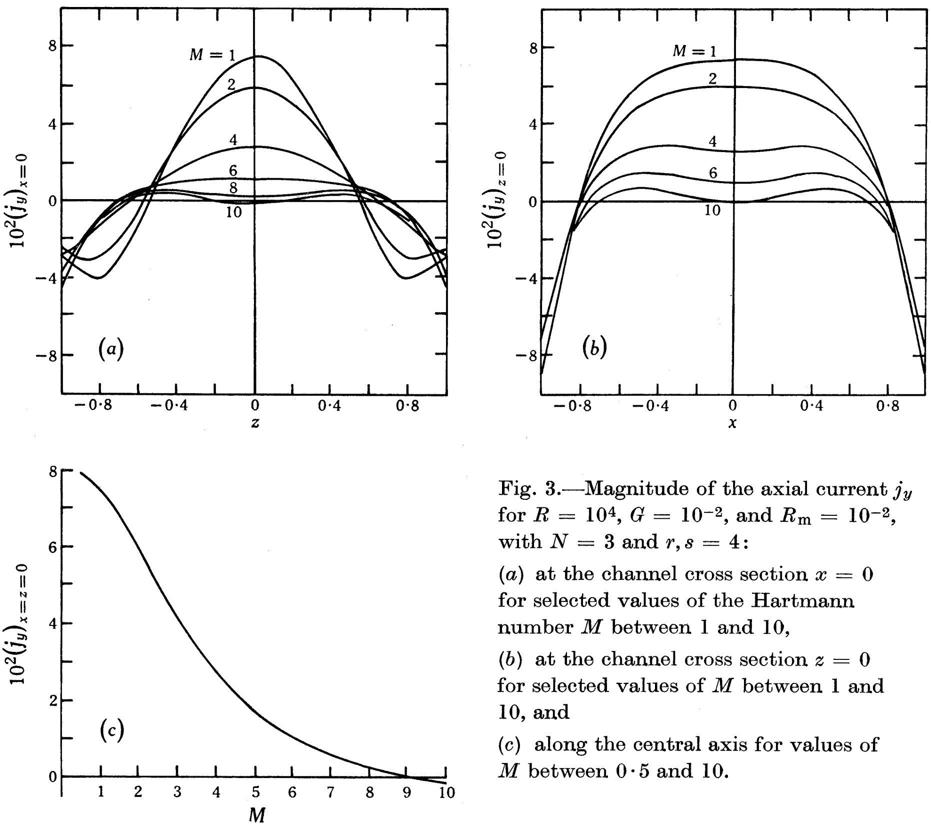


Fig. 3.—Magnitude of the axial current j_y for $R = 10^4$, $G = 10^{-2}$, and $R_m = 10^{-2}$, with $N = 3$ and $r, s = 4$:

(a) at the channel cross section $x = 0$ for selected values of the Hartmann number M between 1 and 10,

(b) at the channel cross section $z = 0$ for selected values of M between 1 and 10, and

(c) along the central axis for values of M between $0 \cdot 5$ and 10.

Figure 2(a) shows the values of I_1 at Hartmann number $0 \cdot 5$ for values of N from 1 to 5 and for r, s up to $N+5$. The data indicate a flattening out in value of the integral from $N = 3$ onwards, and a final choice of $N = 3$ was made to economize on the amount of computation necessary as the array sizes increased for increasing values of N. Inspection of the values at $N = 3$ in Figure 2(a) indicates a suitable choice for r and s equal to $N+1$, as the local minimum here increases for larger values of r and s. Confirmation of this tendency is given in Figure 2(b), where values of the integral for $N = 3$ are shown for selected values of r and s.

The final choice of parameters for the minimization of I_1 was then $N = 3$ and $r, s = 4$. The resulting calculated values of ψ_1 were substituted in the variational principle for the evaluation of v_{1y} and B_{1y} given in Paper I. The results obtained using the new variational form for ψ_1 are similar to those obtained earlier, indicating that the original expression of ψ_1 was an approximate trial form, although derived from incorrect boundary conditions. As in the previous calculation, we obtain a cross flow identical with that found by Tani (1962), together with the net axial current $j_y = \nabla^2\psi_1$ represented in Figures 3(*a*)–3(*c*), where the chosen physical parameters are representative of a laboratory flow.

To first-order in k, there is no change in the net axial flow since v_{1y} is asymmetric. The axial velocity profile for v_{1y} at $z = 0$ is shown in Figure 4; we note that the asymmetry is most pronounced near $M = 3$ and decreases as M becomes large. The cross electric current profiles produced by the first-order contribution are asymmetric and, when combined with the zeroth-order cross currents (Shercliff 1953), the net result is an asymmetry in these profiles.

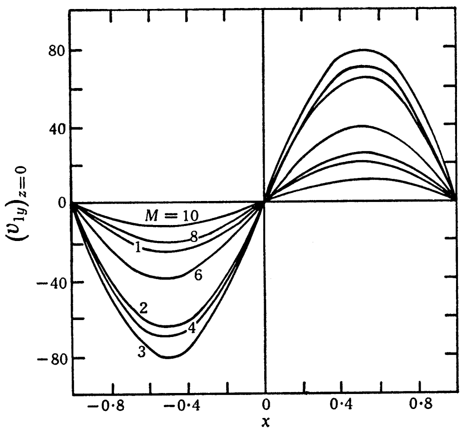


Fig. 4.—Velocity profile of v_{1y} for the first-order contribution to the flow at $z = 0$ for $R = 10^4$, $G = 10^{-2}$, $R_m = 10^{-2}$, and selected values of the Hartmann number M between 1 and 10.

All first-order terms having now been calculated, we proceed to the second-order or k^2 calculations. In this analysis all terms involving R_m in the second-order equations are ignored, although those from the first-order calculation are still retained. Thus the terms that arise as factors of k^2 are: (i) those involving $\{RG\}$ (Tani 1962), (ii) those involving $\{R^2(RG)^3\}$ and, (iii) those involving $\{RR_m(RG)^3\}$; whereas terms of the form $\{R_m^2(RG)^3\}$ are small in comparison with (i), (ii), or (iii) and are neglected.

Since R_m terms are ignored to the second order, we need only consider the following trial forms. Let

$$\chi_2 = xz(1-x^2)^2(1-z^2)^2(g_1+g_2z^2+g_3x^2), \tag{55}$$

satisfying the boundary condition (28), for substitution in equation (34), where g_1, g_2, and g_3 are chosen so that $I_2(\chi_2)$ is stationary, and

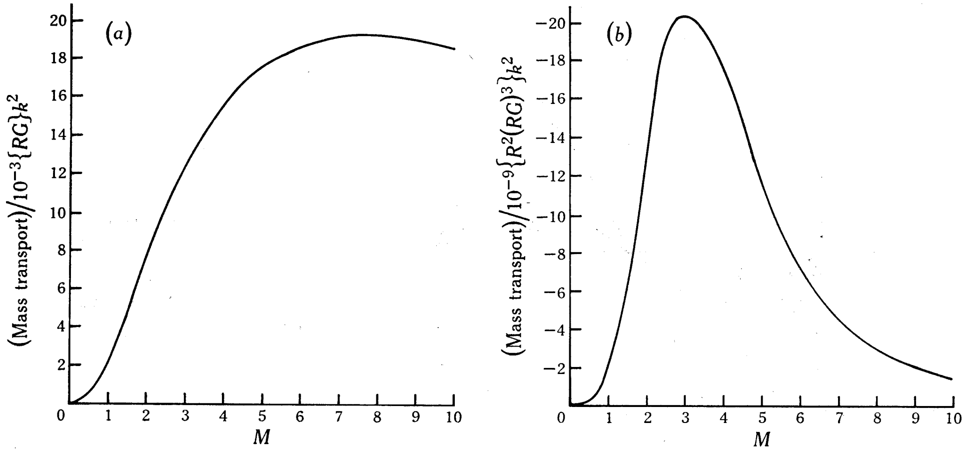
$$v_{2y} = (1-x^2)(1-z^2)(k_1+k_2z^2+k_3x^2) \tag{56}$$

and

$$B_{2y} = z(1-x^2)(1-z^2)(i_1+i_2z^2+i_3x^2), \tag{57}$$

for substitution in equation (36), where k_1, k_2, k_3, i_1, i_2, and i_3 are chosen so that $I_4(v_{2y}, B_{2y})$ is stationary.

The influence of the $\{RG\}$ terms on the mass transport is shown in Figure 5(a). The value at $M = 5$ is in agreement with Tani (1962), namely an *increase* in the flow of $0 \cdot 0175\, k^2 RG$, but additional terms involving R and R_{m}, which were not included by Tani, are also significant. The effects of these terms on the mass transport for values of the Hartmann number up to 10 are shown in Figures 5(b) and 5(c). The contribution by the Reynolds number term $\{R^2(RG)^3\}$ *decreases* the flow, this influence being most marked near $M = 3$ and decreasing as M becomes large. The influence of the magnetic Reynolds number term $\{RR_{\mathrm{m}}(RG)^3\}$ is variable, *increasing* the flow up to about $M = 5 \cdot 5$, with the maximum effect near $M = 3$, and *decreasing* the flow for higher values of M.



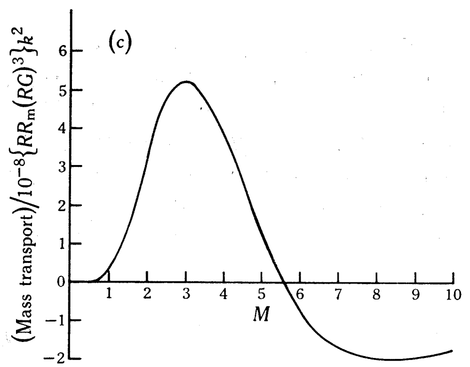


Fig. 5.—Contributions to the mass transport at values of M up to 10 from:
(a) $\{RG\}$ terms, corresponding to the work of Tani (1962),
(b) Reynolds number term $\{R^2(RG)^3\}$, and
(c) magnetic Reynolds number term $\{RR_{\mathrm{m}}(RG)^3\}$.

The relative significance of these terms compared with Tani's (1962) contribution may be seen by considering their ratios at $M = 3$, namely

$$\{R^2(RG)^3\}/\{RG\} \sim -10^{-6}(RG)^2R^2, \qquad \{RR_{\mathrm{m}}(RG)^3\}/\{RG\} \sim +10^{-6}(RG)^2RR_{\mathrm{m}}.$$

For values of $R = 10^4$, $G = 10^{-2}$, and $R_{\mathrm{m}} = 10^{-2}$, which are typical of a laboratory flow, we find $\{R^2(RG)^3\}/\{RG\} \sim 10^6$ and $\{RR_{\mathrm{m}}(RG)^3\}/\{RG\} \sim 1$, that is, the Reynolds number term is of much more significance than Tani's term for the given values of the flow parameters, whilst the magnetic Reynolds number term is of equal significance with Tani's term.

VII. Conclusions

The analysis of the Hall effect in a magnetohydrodynamic flow for a channel with nonconducting walls using perturbation methods has given the following results.

To first order, the characteristic fluid cross flow pattern has been established, although there is no change in the net flow along the channel. However, a net axial electric current flow is established at this order, along with an asymmetric cross current flow, disturbing the generally symmetric cross-current pattern established at zeroth order.

To second order, there are two important influences on the net fluid flow along the channel due to the inclusion of Reynolds number and magnetic Reynolds number terms. For typical laboratory values of the parameters involved, these influences have been found to be of a much greater significance than Tani's (1962) result for the Reynolds number term and of equal significance for the magnetic Reynolds number term. The inclusion of Reynolds number terms in studies of flows of this type is thus seen to be of the utmost importance.

VIII. Acknowledgments

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