ON GRAVITATIONAL ABERRATIONS IN STELLAR IMAGES

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Abstract

On the basis of a cosmological model which is fundamentally of the Friedmann expanding type with a spherically symmetric inhomogeneity superimposed, a study is made of three gravitational aberrations of purely relativistic origin observed in the images of stellar objects: (1) the "gravitational lens" effect, (2) a dispersion effect whereby a point source would produce a diffuse image, and (3) an apparent systematic motion of all light sources towards (or away from) the inhomogeneity. Admissable inhomogeneities in the model must satisfy $PU \lesssim 2\times 10^4$ Mpc, where P is the ratio of the average density of matter within the inhomogeneity to the average density of the universe and U is its diameter in megaparsecs. The assumption is also made that the paths of light rays are described by the null-geodesic equations of the space–time under consideration.

I. Introduction

Although the behaviour of light rays in homogeneous models of the universe has been extensively studied, such work is not expected to give indications of the properties of light propagation when non-uniformities are present. This is not to say, however, that the problem of finding such properties has not already been dealt with. In a recent paper Kristian and Sachs (1966; see also references therein for a history of the problem) have considered the behaviour of light within a density fluctuation in a Friedmann dust universe, where the scale of the fluctuation is to be several orders of magnitude greater than the scale of the visible region (with smaller irregularities smoothed out).

The problem considered in this paper is one where the inhomogeneity may be considerably smaller than the visible region (the scale of which is about 300 Mpc) and neither the source nor observer is necessarily located within the fluctuation. The physical situation thus represented is one where light from a distant (but visible) source passes by a galaxy, cluster, or larger object. The fluctuation is taken to be embedded in a Friedmann model, since this provides for a more realistic treatment of the problem than could be given by using the Schwarzschild metric, and indeed predicts some effects which have no analogue in such static fields.

It is important to note that the fluctuation to be considered does not need to have sharply defined boundaries and may in fact include the entire universe, where in this case the density of matter would have to be spherically symmetric about some point. Such a distribution of matter would occur in the hierarchical models of the universe recently proposed by de Vaucouleurs (1970). It must be stated at once, however, that the restriction to spherical symmetry is a gross oversimplification of

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a general hierarchical model, and the results of the present paper are to be regarded only as indications of what one would expect to observe in a genuinely hierarchical universe. The model outlined above is certainly not isotropic from the observer's viewpoint.

The line element appropriate to the problem has been developed by Cook (1972). In angular coordinates it is

$$\mathrm{d}s^2 = (1-2\eta/S)\mathrm{d}t^2 - S^2c^{-2}(1+2\eta/S)\{\mathrm{d}\psi^2 + k^{-1}\sin^2(k^{\frac{1}{2}}\psi)(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2)\}\,. \tag{1}$$

where k is 1 for a closed model and -1 for an open model, $S = S(t)$ corresponds to the scale factor for a Friedmann model, and $\eta = \eta(\psi)$ is such that η/S is dimensionless and of order $\eta_\psi/S \ll 1$. Equation (1) is approximate to order $(\eta/S)^2$, and is suitable to this order provided the condition $PU \leqslant 2\times 10^4$ Mpc is met, where P is the ratio of the average density of matter within the fluctuation to the average density of the universe and U is the diameter of the fluctuation. In what is to follow, the calculations are for a closed model, although the results may be extended for the case $k = -1$, and by "first order" we shall mean of order η/S.

Since we shall make the fundamental assumption that light rays are described by null geodesics, our program will be to use the null-geodesic equations for the line element (1) to investigate three effects which are due to an inhomogeneous gravitational field:

(*a*) the "gravitational lens" effect whereby the image of an extended object is distorted,

(*b*) a "dispersion" effect whereby point sources of light may produce diffuse images, and

(*c*) an apparent systematic motion of light sources towards (or away from) the inhomogeneity.

II. Solution of Null-geodesic Equations

For the line element (1) the null-geodesic equations for $\theta(\sigma)$ and $\phi(\sigma)$ are

$$S^2(1+2\eta/S)\sin^2\psi\sin^2\theta\,\mathrm{d}\phi/\mathrm{d}\sigma = -\beta c^2 \tag{2}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(-S^2(1+2\eta/S)\sin^2\psi\frac{\mathrm{d}\theta}{\mathrm{d}\sigma}\right)+S^2(1+2\eta/S)\sin^2\psi\sin\theta\cos\theta\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2 = 0\,, \tag{3}$$

where β is a constant of integration. A solution $\theta = \frac{1}{2}\pi$ is of sufficient generality for equation (3). The equation for $\psi(\sigma)$ then becomes

$$\frac{\mathrm{d}}{\mathrm{d}\sigma}\left(-S^2c^{-2}(1+2\eta/S)\frac{\mathrm{d}\psi}{\mathrm{d}\sigma}\right)+\frac{\eta_\psi}{S}\left[\left(\frac{\mathrm{d}t}{\mathrm{d}\sigma}\right)^2+S^2c^{-2}\left\{\left(\frac{\mathrm{d}\psi}{\mathrm{d}\sigma}\right)^2+\sin^2\psi\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2\right\}\right]$$
$$+S^2c^{-2}(1+2\eta/S)\sin\psi\cos\psi\left(\frac{\mathrm{d}\phi}{\mathrm{d}\sigma}\right)^2 = 0\,. \tag{4}$$

We can now eliminate σ between equations (2) and (4) to replace both of these equations by

$$\frac{\mathrm{d}^2\psi}{\mathrm{d}\phi^2}-2\cot\psi\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right)^2-\sin\psi\cos\psi-\frac{c^2\eta_\psi}{S^3}\left[\left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)^2+S^2c^{-2}\left\{\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right)^2+\sin^2\psi\right\}\right]=0. \quad (5)$$

By dividing equation (1) throughout by $\mathrm{d}\sigma^2$ and putting $\mathrm{d}s^2=0$, we have an integral of the system, which after elimination of σ becomes

$$\left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)^2-S^2c^{-2}\left\{\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right)^2+\sin^2\psi\right\}-\frac{2\eta}{S}\left[\left(\frac{\mathrm{d}t}{\mathrm{d}\phi}\right)^2+S^2c^{-2}\left\{\left(\frac{\mathrm{d}\psi}{\mathrm{d}\phi}\right)^2+\sin^2\psi\right\}\right]=0. \quad (6)$$

Since equations (5) and (6) are valid only to first order, we approximate their solution to the same order. To do this we put $\psi=\psi^{(0)}+\psi^{(1)}$ and $t=t^{(0)}+t^{(1)}$, where $t^{(1)}$ and $\psi^{(1)}$ are $O(\eta/S)$, and thus obtain four equations for $\psi^{(0)}$, $\psi^{(1)}$, $t^{(0)}$, and $t^{(1)}$. Of these, the only equations of interest at the moment are those for $\psi^{(0)}$ and $\psi^{(1)}$,

$$\frac{\mathrm{d}^2\psi^{(0)}}{\mathrm{d}\phi^2}-2\cot\psi^{(0)}\left(\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}\phi}\right)^2-\sin\psi^{(0)}\cos\psi^{(0)}=0 \quad (7)$$

and

$$\frac{\mathrm{d}^2\psi^{(1)}}{\mathrm{d}\phi^2}-4\cot\psi^{(0)}\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}\phi}\frac{\mathrm{d}\psi^{(1)}}{\mathrm{d}\phi}+2\operatorname{cosec}^2\psi^{(0)}\left(\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}\phi}\right)^2\psi^{(1)}-(\cos^2\psi^{(0)}-\sin^2\psi^{(0)})\psi^{(1)}$$

$$=\frac{2\eta'(\psi^{(0)})}{S(t^{(0)})}\left\{\left(\frac{\mathrm{d}\psi^{(0)}}{\mathrm{d}\phi}\right)^2+\sin^2\psi^{(0)}\right\}. \quad (8)$$

The general solution of equation (7) is

$$x=\cot\psi^{(0)}=a\cos(\phi-\alpha), \quad (9)$$

where a and α are arbitrary constants. If we put $\psi^{(1)}=(\sin^2\psi^{(0)})y$ equation (8) becomes

$$(a^2-x^2)\frac{\mathrm{d}^2y}{\mathrm{d}x^2}-x\frac{\mathrm{d}y}{\mathrm{d}x}+y=-\frac{2(1+a^2)}{S}\frac{\mathrm{d}\eta}{\mathrm{d}x}. \quad (10)$$

In terms of ϕ this equation is

$$y_{\phi\phi}+y=2\operatorname{cosec}(\phi-\alpha)\,(a+a^{-1})\eta_\phi/S,$$

which has the solution

$$y=2(a+a^{-1})\sin(\phi-\alpha)\int_{\phi_0}^{\phi}\operatorname{cosec}^2(\phi'-\alpha)\,\mathrm{d}\phi'\int_{\phi_0}^{\phi'}\eta_{\phi''}/S\,\mathrm{d}\phi''+\sin(\phi-\phi_0)\left(\frac{\mathrm{d}y}{\mathrm{d}\phi}\right)_{\phi=\phi_0}. \quad (11)$$

III. Uniqueness of Light Path between Source and Observer

Before proceeding to draw conclusions from equation (11) we must consider whether it is possible for two space points P_0 and P_1 to be connected by more than one distinct light path, that is to say, whether we are able to choose $\psi_A^{(0)}(\phi)+\psi_A^{(1)}(\phi)$ and $\psi_B^{(0)}(\phi)+\psi_B^{(1)}(\phi)$ both connecting P_0 and P_1 so that, for some ϕ within the range $\phi_0 < \phi < \phi_1$, the separation between these curves is of at least first order (which is what we mean by distinct in this context). Non-uniqueness in this situation gives rise to the effect (*b*) defined in Section I, and this point is considered further in Section IV(*b*).

We may, without loss of generality, suppose that $\psi_A^{(0)}$ connects $P_0(\psi_0, \phi_0)$ to $P_1(\psi_1, \phi_1)$. Then $\psi_A^{(0)}$ is determined by equation (9), and $\psi_A^{(1)}$ by equation (11) and the end conditions $\psi_A^{(1)}(\phi_0) = \psi_A^{(1)}(\phi_1) = 0$. We now suppose that $\psi_B^{(0)}$ joins P_0 to $P_2(\psi_1+\psi_2, \phi_1)$ and that $\psi_B^{(1)}(\phi_0) = 0$ and $\psi_B^{(1)}(\phi_1) = -\psi_2$. If we now put

$$x_\gamma = \cot\psi_\gamma^{(0)}, \qquad y_\gamma = (1+x_\gamma^2)\psi_\gamma^{(1)}, \qquad y_2 = -\{1+x_A^2(\phi_1)\}\psi_2,$$

where $\gamma = \mathrm{A}, \mathrm{B}$, then to first order we have

$$x_A(\phi_1) + y_2 = x_B(\phi_1).$$

Now define

$$\Theta = y_2 \sin(\phi-\phi_0)/\sin(\phi_1-\phi_0),$$

which gives to first order

$$x_A(\phi) + \Theta(\phi) = x_B(\phi),$$

and impose the restriction $\phi_1 - \phi_0 \leqslant \frac{1}{2}\pi$. If we write $x_A = a\cos(\phi-\alpha)$ then

$$x_A + \Theta = (a+\epsilon)\cos(\phi-\alpha-\delta),$$

where

$$\epsilon = y_2 \cos(\phi_0-\alpha)/\sin(\phi_1-\phi_0)$$

and ϵ/a and δ are first-order quantities. When the left-hand side of equation (10) is made independent of the parameter a by putting $w = x/a$, the functions y_A and y_B are given by the equations

$$(1-w^2)\frac{\mathrm{d}^2 y_A}{\mathrm{d}w^2} - w\frac{\mathrm{d}y_A}{\mathrm{d}w} + y_A = -\frac{2(1+a^2)}{S}\frac{\mathrm{d}\eta}{\mathrm{d}x} \tag{12}$$

and

$$(1-w^2)\frac{\mathrm{d}^2 y_B}{\mathrm{d}w^2} - w\frac{\mathrm{d}y_B}{\mathrm{d}w} + y_B = -\frac{2(1+a^2)}{S}\frac{\mathrm{d}\eta}{\mathrm{d}x} + \epsilon\frac{\partial}{\partial a}\left(-\frac{2(1+a^2)}{S}\frac{\mathrm{d}\eta}{\mathrm{d}x}\right)$$

$$= -\frac{2(1+a^2)}{S}\frac{\mathrm{d}\eta}{\mathrm{d}x} - \frac{2\epsilon(a+a^{-1})x}{S}\frac{\mathrm{d}^2\eta}{\mathrm{d}x^2} + O(\epsilon/a)^2. \tag{13}$$

Then provided $x\,\mathrm{d}^2\eta/\mathrm{d}x^2$ is of first order (and we interpret this condition in Section IV(*b*)), the expressions (12) and (13) are the same differential equation. In this case we let $\psi_{\mathrm{A}}^{(0)}+\psi_{\mathrm{A}}^{(1)} = \psi_{\mathrm{B}}^{(0)}+\psi_{\mathrm{B}}^{(1)}-\xi$, where ξ is a function of at most first order. We then have

$$x_{\mathrm{A}}-y_{\mathrm{A}} = x_{\mathrm{B}}-y_{\mathrm{B}}+(1+x_{\mathrm{B}}^2)\xi$$

but, since $(x_{\mathrm{A}}+\Theta-x_{\mathrm{B}})$ is a second-order quantity,

$$y_{\mathrm{A}}+\Theta = y_{\mathrm{B}}-(1+x_{\mathrm{B}}^2)\xi\,.$$

However, $y_{\mathrm{A}}+\Theta$ is a solution to equation (12) which satisfies the same end conditions as y_{B}, and hence ξ is at most a second-order quantity. We can thus draw the conclusion that, within the accuracy of the model, the light paths between sources and observer are unique, except in the case where η_{xx} is not of first order.

IV. Gravitational Aberrations

(a) *Gravitational Lens Effect*

We now seek to derive an expression for the distortion produced in the image of an extended source. To do this we shall follow Kristian and Sachs (1966) and calculate the eccentricity of the elliptical image that is to be expected from an intrinsically spherical source.

We shall denote the centre of the inhomogeneity (at which is located the origin of coordinates) by C; the observer, with coordinates (ψ_1, ϕ_1), by O; and the centre of the source, with coordinates (ψ_0, ϕ_0), by S. The coordinate system is chosen so that O and S lie in the surface $\theta = \frac{1}{2}\pi$. We now set up at O an orthonormal basis in terms of which observations are to be made. We assume that O may determine the direction of C which is specified by a unit vector $\boldsymbol{i}$. The unit vector $\boldsymbol{j}$ is then normal to the surface OCS at O (determined by $\boldsymbol{i}$ and the direction of light from S) and $\boldsymbol{k}$ is orthogonal to both $\boldsymbol{i}$ and $\boldsymbol{j}$.

Since the space–time under consideration is rotationally symmetric about CO, we deduce that there will be no distortion of the image in the direction $\boldsymbol{j}$. If the angle at O between the direction of light from S and $\boldsymbol{i}$ is ω then we have from the metric (1) that

$$\cot\omega = \operatorname{cosec}\psi_1\,(\mathrm{d}\psi/\mathrm{d}\phi)_{\phi_1},$$

where the light path is described by $\psi = \psi(\phi)$, $\theta = \frac{1}{2}\pi$. If the light paths to O from the points on the edges of the source which lie in the surface OCS are designated ψ_{A} and ψ_{B}, an approximate angular separation between these paths at O is given by

$$\delta\omega = \frac{\sin^2\omega}{\sin\psi_1}\left(\frac{\mathrm{d}\psi_{\mathrm{A}}}{\mathrm{d}\phi}-\frac{\mathrm{d}\psi_{\mathrm{B}}}{\mathrm{d}\phi}\right)_{\phi_1}.$$

Since

$$\delta\omega_0 = \sin^2\omega\operatorname{cosec}\psi_1\,(\psi_{\mathrm{A}\phi}^{(0)}-\psi_{\mathrm{B}\phi}^{(0)})_{\phi_1}$$

is the angular separation which would be measured in a homogeneous universe (as $\psi^{(0)}$ are the solutions to the null-geodesic equations for a homogeneous model), we take $\delta\omega_0$ thus defined as the angular separation in the $\boldsymbol{j}$ direction between rays proceeding from the edges of the source. Hence we have that the eccentricity of the image (i.e. its ratio of major to minor semi-axes) is

$$e = \delta\omega/\delta\omega_0 = (\psi_{A\phi} - \psi_{B\phi})_{\phi_1} / (\psi^{(0)}_{A\phi} - \psi^{(0)}_{B\phi})_{\phi_1} .$$

In terms of x and y as defined in Section III, we have after some manipulation

$$e - 1 = -(y_{A\phi} - y_{B\phi})_{\phi_1} / (x_{A\phi} - x_{B\phi})_{\phi_1} . \tag{14}$$

In equation (9) we choose the zero of ϕ so that $\alpha = 0$ and then suppose

$$x_B = (a + \delta a)\cos(\phi - \delta\alpha) ,$$

where $\delta a/a$ and $\delta\alpha$ are very much less than unity. Thus, with the condition $x_A(\phi_1) = x_B(\phi_1)$ we have

$$(x_A - x_B) \approx (\operatorname{cosec}\phi_1)\delta a$$

and hence from equation (14)

$$e - 1 = -\sin\phi_1 \, (x_{A\phi} - x_{B\phi})/\delta a . \tag{15}$$

In order to proceed with the calculation of e beyond this point, without making further general assumptions regarding the physical situation, we would need to choose a particular distribution of matter and an equation of state in order to determine η and S explicitly as functions of ψ and t respectively. It is possible, however, to introduce considerable simplification into such a calculation if we admit the assumptions:

(i) that a is sufficiently large for only its highest powers to be retained (current depths of observation give $a > 50$; Cook 1971, Section 2.4), and

(ii) that the product of the Hubble parameter with the separation between the rays ψ_A and ψ_B (in light seconds) is small.

With the above assumptions, and using equations (11) and (15), we have after some calculation

$$e - 1 = 2a \operatorname{cosec}(\phi_1 - \phi_0)\{\sin(\phi_1 - \phi_0)M - \cos(\phi_1 - \phi_0)\Lambda + \cos\phi_1 \cos\phi_0 \, M$$
$$+ \cos\phi_1 \sin\phi_0 \, \Lambda + a\cos\phi_0 \, N_1 - a\sin\phi_0 \, N_2\} ,$$

where

$$M = \int_{\phi_0}^{\phi_1} \sin\phi \, \eta x/S \, d\phi , \qquad N_1 = \int_{\phi_0}^{\phi_1} \sin\phi \sin(\phi_1 - \phi) \, \eta x x/S \, d\phi ,$$

$$\Lambda = \int_{\phi_0}^{\phi_1} \cos\phi \, \eta x/S \, d\phi , \qquad N_2 = \int_{\phi_0}^{\phi_1} \cos\phi \sin(\phi_1 - \phi) \, \eta x x/S \, d\phi .$$

We note that the choice of the zero of ϕ determines ϕ_1 and a via the conditions

$$x_1 = \cot\psi_1 = a\cos\phi_1 , \qquad a^2 \approx x_1^2 + (1 + x_1^2)\cot^2\omega .$$

(b) *Dispersion Effect*

In Section III we saw that provided the expression $x\eta_{xx}$ is small the light path connecting a given source and observer is unique to first order. We shall now consider the case where this requirement is not met. The analysis must be of a largely speculative nature since our approximations are only sufficiently accurate to give an indication of the presence of such an effect.

If, in the notation of Section III, we define D to be the separation between any two rays from source to observer, that is,

$$D = (x_{\mathrm{B}}+y_{\mathrm{B}})-(x_{\mathrm{A}}+y_{\mathrm{A}}),$$

then from equations (12) and (13) we have

$$(1-w^2)D_{ww}-wD_w+D = -2\epsilon(a+a^{-1})x\eta_{xx}/S$$

and

$$D(\phi_0) = D(\phi_1) = 0,$$

since $D = y_{\mathrm{B}}-y_{\mathrm{A}}+\Theta$. It follows after some routine calculation that

$$\begin{aligned} D_\phi(\phi_1) = {} & -2\epsilon(1+a^2)\operatorname{cosec}(\phi_1-\phi_0) \\ & \times\left(\sin\phi_0\int_{\phi_0}^{\phi_1}\cos^2\phi\;\eta_{xx}/S\;\mathrm{d}\phi-\cos\phi_0\int_{\phi_0}^{\phi_1}\sin\phi\cos\phi\;\eta_{xx}/S\;\mathrm{d}\phi\right). \qquad (16)\end{aligned}$$

Bearing in mind the considerations of subsection (*a*), we define the angular dispersion at the observer's point to be

$$d = \sin^2\omega\operatorname{cosec}\psi_1\,D_\phi(\phi_1)\,.$$

We note that this dispersion is in the $\boldsymbol{i}$ direction only. Thus if the image of an extended source were to be considered, we would expect its edges in the $\boldsymbol{i}$ direction to be diffuse and the edges in the $\boldsymbol{j}$ direction to be sharp. We should, however, avoid attaching too much quantitative significance to equation (16). Parameters such as ϕ_1 and a may be estimated as in (*a*) above but the quantity ϵ remains virtually undefined, except that ϵ/a is to be much less than unity. We are in any case unable to give a satisfactory treatment of the intensity variation (with distance from a central point) within an image.

Doubt may arise as to whether $x_{\mathrm{A}}+y_{\mathrm{A}}$ and $x_{\mathrm{B}}+y_{\mathrm{B}}$ are not just both inaccurate descriptions of the *one* light path, which would mean of course that the dispersion d was merely a result of the approximation procedure used and not a real effect at all. A discussion of this point is given in the Appendix.

What may we say about the physical conditions which would produce dispersion? It has been shown by Cook (1972) that if $\rho_1(\psi)$ is the density of a fluctuation then

$$\kappa\rho_1 \approx -4c^2\eta_{\psi\psi}/S^3\,,$$

and, since η_{xx} is related to $\eta_{\psi\psi}$ by the relation

$$\eta_{xx} = (1+x^2)^{-2}(\eta_{\psi\psi}+2x\eta_\psi)$$

which for $x \gg 1$ (i.e. a small value of the boundary coordinate of the fluctuation) is approximately

$$\eta_{xx} \approx 3(1+x^2)^{-2}\eta_{\psi\psi},$$

we associate large values of $|\eta_{xx}|$ with a concentration (or rarefaction) of matter. In conclusion, therefore, it seems apparent that dispersion arises from the varying degrees of deflection produced in light paths by the presence of a density fluctuation.

(c) *Apparent Systematic Motion of Light Sources*

The analysis in Sections III and IV(b) has shown that between any source and observer there is, to first order at least, a unique light path at any particular time. Implicit in equation (11), however, is the dependence of y upon the time of emission of a photon. This gives rise to some apparent motions of all light sources, and we shall now consider this effect.

If $t^{(0)}$ is a solution of

$$c^2(t_\phi^{(0)})^2 - S^2(t^{(0)})\{(\psi_\phi^{(0)})^2+\sin^2\psi^{(0)}\} = 0,$$

which is the equation obtained by taking only the terms of highest order in (6), then the function S occurring in equation (11) is of the form $S(t^{(0)}(\phi, t_1))$, where t_1 is the time at which a photon is received by an observer at (ψ_1, ϕ_1). We now wish to replace the term involving $y_\phi(\phi_0)$ in equation (11) with one containing $y_\phi(\phi_1)$. After some calculation, again with $\alpha = 0$, we obtain

$$y(\phi_1, \phi_0, t_1)\cos(\phi_1-\phi_0)$$

$$= -2(a+a^{-1})\left(\cos\phi_0 \int_{\phi_0}^{\phi_1} \eta_\phi/S \,\mathrm{d}\phi + \sin\phi_0 \int_{\phi_0}^{\phi_1} \cot\phi\, \eta_\phi/S\,\mathrm{d}\phi\right) + y_\phi(\phi_1)\sin(\phi_1-\phi_0).$$

For fixed x, y, ϕ_0, and ϕ_1, differentiation of this equation with respect to t_1 gives

$$\sin(\phi_1-\phi_0)\frac{\partial y_\phi(\phi_1)}{\partial t_1} = -\frac{2(a+a^{-1})}{S(t_1)}\left(\cos\phi_0 \int_{\phi_0}^{\phi_1} \frac{\eta_\phi \dot{S}}{S}\,\mathrm{d}\phi + \sin\phi_0 \int_{\phi_0}^{\phi_1} \frac{\cot\phi\,\eta_\phi \dot{S}}{S}\,\mathrm{d}\phi\right),$$

where we have used $\partial t^{(0)}/\partial t_1 = S(t^{(0)})/S(t_1)$. If ω is defined as in Section IV(a) then

$$\operatorname{cosec}^2\omega\, \partial\omega/\partial t_1 = -\sin\psi_1\, \partial y_\phi(\phi_1)/\partial t_1$$

and hence

$$\frac{\partial\omega}{\partial t_1} = \frac{2(a+a^{-1})\sin\psi_1\sin^2\omega}{S(t_1)\sin(\phi_1-\phi_0)}\left\{\cos\phi_0 \int_{\phi_0}^{\phi_1} \frac{\eta_\phi \dot{S}}{S}\,\mathrm{d}\phi + \sin\phi_0 \int_{\phi_0}^{\phi_1} \frac{\cot\phi\,\eta_\phi \dot{S}}{S}\,\mathrm{d}\phi\right\}. \quad (17)$$

Throughout this discussion we have tacitly used the convention that ϕ increases along each light ray (that is, $\phi_1 > \phi_0$). In order to interpret equation (17) in physical terms, we now restrict consideration to rays which do not cross the curves $\phi = 0$ or π. Each ray is then classified according as to whether it travels (i) towards the curve $\phi = 0$, or (ii) away from the curve $\phi = 0$. As a result of these restrictions we have $\sin\phi_0 < 0$ for case (i) and $\sin\phi_0 > 0$ for case (ii). Now, obviously, whatever the sign of η_ϕ for case (i), it will be of opposite sign for case (ii) provided that η_ϕ does not change sign over the path of the photon. We may thus conclude that $\partial\omega/\partial t_1$ has the same sign for both cases (i) and (ii).

For purposes of illustration, we shall suppose $\eta(\psi)$ to be monotonically decreasing (i.e. a concentration of matter at the origin; Cook 1971, Section 2.2). For case (i) above we have $\eta_\phi \geqslant 0$ which implies $\partial\omega/\partial t_1 < 0$. This may be easily seen by writing the expression in braces on the right-hand side of equation (17) as

$$\sin\phi_0 \int_{\phi_0}^{\phi_1} \operatorname{cosec}^2\phi' \, \mathrm{d}\phi' \int_{\phi'}^{\phi_1} \eta_{\phi''} \, \dot{S}/S \, \mathrm{d}\phi'' \, .$$

Similarly, for case (ii) we have $\eta_\phi \leqslant 0$ and hence $\partial\omega/\partial t_1 < 0$. By analogy, were we to suppose $\eta(\psi)$ to be monotonically increasing (i.e. a relative rarefaction of matter at the origin) then we would obtain $\partial\omega/\partial t_1 > 0$ in both cases. We therefore conclude, taking into account the symmetry in θ, that an observer would see all light sources move towards, or away from, the radial geodesic section joining his position to the centre of a concentration, or rarefaction, of matter respectively.

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Appendix

In order to deal with the point raised in Section IV(b) concerning the adequacy of $\psi^{(0)}+\psi^{(1)}$ as an approximation to the solution of equation (5), we need to show that the difference between ψ and $\psi^{(0)}+\psi^{(1)}$ is a quantity of at least second order. If this is so then the analysis in Section IV(b) is acceptable since $\psi_{\mathrm{A}}^{(0)}+\psi_{\mathrm{A}}^{(1)}$ and $\psi_{\mathrm{B}}^{(0)}+\psi_{\mathrm{B}}^{(1)}$ could not then both be valid first-order approximations to the one curve and yet have first-order separation (by "valid" in this context we mean to within the next order of approximation).

If we put $X = \cot\psi$, we may integrate equation (5) to obtain

$$X_\phi^2+X^2+1 = (1+A^2)\left(1+4\int_{\phi_0}^{\phi} \eta_{\phi'}/S \, \mathrm{d}\phi'\right),$$

and we shall investigate the accuracy of successive approximations to the solution of this equation. In what is to follow, "of order n" will be taken to mean of order $(\eta/S)^n A$. If we now put $X = x+Y$, where without loss of generality we define x as a solution of

$$x_\phi^2+x^2+1 = 1+A^2 .$$

we then have

$$2(x_\phi Y_\phi + xY) + Y_\phi^2 + Y^2 = 4(1+A^2)\int_{\phi_0}^{\phi} \eta_{\phi'}/S \, d\phi'. \tag{A1}$$

For $Y = y+Z$, where y is a solution of

$$2(x_\phi y_\phi + xy) = 4(1+A^2)\int_{\phi_0}^{\phi} \eta_{\phi'}/S \, d\phi'$$

Z then satisfies the equation

$$2\{(x_\phi + y_\phi)Z_\phi + (x+y)Z\} + Z_\phi^2 + Z^2 = -(y_\phi^2 + y^2). \tag{A2}$$

Now y, which is given by equation (11) with $a = A$ and $y_\phi(\phi_0) = 0$, is of first order and hence the right-hand side of equation (A2) is of second order (times A). If we assume Z to be of at least first order then (A2) implies that Z is of at least second order (by taking only first-order terms in (A2)).

The remaining possibility is that Z is of zeroth order, in which case Y is also of zeroth order. If we differentiate equation (A1) with respect to ϕ, we obtain

$$Y_{\phi\phi} = -Y + \{2(1+A^2)\eta_\phi/S\}/(x_\phi + Y_\phi). \tag{A3}$$

Without loss of generality we may again assume (as in Section IV(c)) that $\eta(\psi)$ is monotonic increasing and then, provided that initially $Y = 0$ and Y_ϕ is of first order, from equations (A1) and (A3) we have that Y can be of no less than first order, and hence Z must be of at least second order.