APPLICATION OF THE HYDROMAGNETIC ENERGY PRINCIPLE TO A PLASMA BETWEEN ELECTRODES

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Abstract

The approach of Van Kampen and Felderhof (1967) is extended to derive the variation of potential energy δW for a finite magnetized plasma contained by a vacuum magnetic field and in contact with conducting electrodes supported by insulators. The method gives a result in agreement with that of Bernstein *et al.* (1958) except for modification of the vacuum contribution. It leads to proof that the usual interpretation of the surface contribution to δW is incorrect because of modification of the vacuum magnetic field energy associated with second-order distortion of the plasma boundary surface.

I. INTRODUCTION

In the present treatment a fluid theory is adopted in which the influence of collisions is such that the kinetic pressure remains scalar but the electrical conductivity may be assumed infinite. An expression for $\delta W(\xi, \xi)$, the change in system potential energy produced by the Lagrangian displacement ξ, may be obtained by determining the second-order variation of the potential energy function with respect to ξ. However, the literature does not appear to contain a complete derivation of δW by this method although an expression for δW has been obtained by Bernstein *et al.* (1958), effectively by integrating the second-order expression for the rate at which work is done in the system. Van Kampen and Felderhof (1967) used the variational method to derive $\delta \overline{W}$ for an infinite plasma. The present extension of their work to the case of a finite system provides a convenient framework for an explicit treatment of external conductors and insulators, which permits application of the results to the constricted plasma between electrodes (James and Seymour 1971). The analysis principally involves the calculation of the change of magnetic energy external to the plasma, which is first shown to equal the work done in deforming the plasma surface against the pressure of the vacuum magnetic field. A basis is therefore provided for a discussion of the surface term $\delta W_S(\xi, \xi)$ in the original expression for δW (Bernstein *et al.*) in terms of its frequently quoted interpretation as work done against the plasma surface current.

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II. Basic Equations and Boundary Conditions

Using a circumflex to denote vacuum quantities, the vacuum magnetic field $\hat{\boldsymbol{B}}$ at all times satisfies Maxwell's equations without displacement or conduction currents

$$\nabla.\hat{\boldsymbol{B}} = 0 \qquad \text{and} \qquad \nabla\times\hat{\boldsymbol{B}} = 0, \tag{1}$$

and when the system is perturbed

$$\nabla.\hat{\boldsymbol{E}} = 0 \qquad \text{and} \qquad \nabla\times\hat{\boldsymbol{E}} = -\partial\hat{\boldsymbol{B}}/\partial t. \tag{2}$$

Assuming the adiabatic equation of state $\mathrm{d}(p\rho^{-\gamma})/\mathrm{d}t = 0$ to apply to the plasma region, other basic equations required here are:

$$\rho\,\mathrm{d}\boldsymbol{v}/\mathrm{d}t = -\nabla p + \boldsymbol{j}\times\boldsymbol{B}, \tag{3}$$

with permissible neglect of charge accumulation and gravitational force; the infinite electrical conductivity approximation

$$\boldsymbol{E} + \boldsymbol{v}\times\boldsymbol{B} = 0, \tag{4}$$

a justified theoretical constraint since, for example, resistive instabilities may be important in practice even if the conductivity is extremely large; and Maxwell's equations without displacement current

$$\nabla.\boldsymbol{B} = 0, \qquad \nabla\times\boldsymbol{B} = \mu_0\boldsymbol{j}, \qquad \nabla\times\boldsymbol{E} = -\partial\boldsymbol{B}/\partial t. \tag{5a, b, c}$$

Well-known boundary conditions apply at the plasma–vacuum interface (Kruskal and Schwarzschild 1954):

$$\langle p + \tfrac{1}{2}\mu_0^{-1}B^2\rangle = 0, \tag{6}$$

where $\langle X\rangle$ denotes the jump in the quantity X across the interface in the direction of the unit normal $\boldsymbol{n}$ directed out of the plasma;

$$\boldsymbol{n}\times\langle\boldsymbol{E}\rangle = (\boldsymbol{n}.\boldsymbol{v})\langle\boldsymbol{B}\rangle, \tag{7}$$

where $\boldsymbol{v}$ is the velocity of points on the interface (Jeffrey 1966); and

$$\boldsymbol{n}.\langle\boldsymbol{B}\rangle = 0. \tag{8}$$

In the special case of zero internal magnetic field, conditions applicable at the interface at all times are

$$\hat{B}^2 = 2\mu_0 p \qquad \text{and} \qquad \boldsymbol{n}.\hat{\boldsymbol{B}} = 0 \tag{9a, b}$$

from equations (6) and (8) respectively.

Using the subscript zero to denote equilibrium quantities, the absence of an internal magnetic field leads via equation (3) to the condition that p_0 is constant throughout the plasma. Equation (9a) then shows that $\hat{B}_0^2$ is constant at all points on the equilibrium interface between plasma and vacuum.

III. Alternative Derivation of δW

In practice the plasma is a finite body, either a torus closed upon itself or a column terminated by electrodes. For the plasma column it is assumed here that the electrodes are hot enough for the contacting plasma to satisfy the infinite electrical conductivity assumption leading to equation (4). It is also assumed that, to avoid quenching, the plasma contacts only vacuum and very hot rigid electrodes, at which

$$\boldsymbol{n}.\xi = 0. \tag{10}$$

The condition that the region of interest can often be considered to be surrounded by a rigid perfectly-conducting wall (Bernstein *et al.* 1958, p. 19) is not applicable to a plasma between electrodes. If, as in many linear pinch experiments, use is made of a stabilizing conducting shell that is coaxial with the column and continuous along its length, insulating supports must be present to prevent short-circuiting of the discharge and the shell cannot be closed across its ends. Similarly the electrodes must be supported by insulators.

From the treatment of Van Kampen and Felderhof (1967, p. 75) it is possible to obtain for the finite magnetized plasma an expression for $\delta \overline{W}$ representing the second-order variation of potential energy within its volume τ_p. The complete expression for δW can then be obtained by adding to $\delta \overline{W}$ the expression for the second-order variation in W_{BE}, the external magnetic field energy. This derivation, outlined below, leads to a clearer understanding of the surface contribution to δW.

For the finite plasma system the surface terms not taken into consideration by Van Kampen and Felderhof (1967) must be included. Assuming zero gravitational field the required second-order expression for $\delta \overline{W}$ is obtainable from their book (p. 75, equation (21) *et seq.*) as†

$$\begin{aligned}\delta \overline{W} = \tfrac{1}{2}\int_{\tau_p(0)} \mathrm{d}\tau_0\, \{\mu_0^{-1}|\boldsymbol{Q}|^2 - \boldsymbol{j}_0.(\boldsymbol{Q}\times\xi) + \gamma p_0(\nabla.\xi)^2 + (\nabla.\xi)\xi.\nabla p_0\} \\ + \tfrac{1}{2}\mu_0^{-1}\int_{S_p(0)} [(\mu_0 p_0 + \tfrac{1}{2}B_0^2)(\xi.\nabla\xi - \xi\nabla.\xi) \\ + \{\xi.(\xi.\nabla\boldsymbol{B}_0)\}\boldsymbol{B}_0 - \{\xi.(\boldsymbol{B}_0.\nabla\boldsymbol{B}_0)\}\xi].\mathrm{d}\boldsymbol{S}_0,\end{aligned} \tag{11}$$

where

$$\boldsymbol{Q} = \nabla\times(\xi\times\boldsymbol{B}_0), \tag{12}$$

$\xi(\boldsymbol{r}_0, t)$ is the small displacement of a fluid element from its equilibrium position $\boldsymbol{r}_0$ at $t = 0$, $\tau_p(0)$ is the equilibrium plasma volume at $t = 0$, $S_p(0)$ encloses $\tau_p(0)$, and $\boldsymbol{n}_0\,\mathrm{d}S_0 = \mathrm{d}\boldsymbol{S}_0$ is directed out of the plasma.

An immediate simplification can be made by considering the term

$$T = \tfrac{1}{2}\mu_0^{-1}\int_{S_p(0)} \{\xi.(\xi.\nabla\boldsymbol{B}_0)\}\boldsymbol{B}_0.\mathrm{d}\boldsymbol{S}_0 \tag{13}$$

of equation (11). If at any interface carrying a surface current $\boldsymbol{j}^*$ there is no sheet

† The factor $\frac{1}{2}$ has been omitted from the second integral of the last problem on p. 75 of Van Kampen and Felderhof (1967); this has been corrected in the corresponding term of equation (11) above.

mass, the magnetic field must lie in the interface to prevent the occurrence of infinite accelerations. Thus $\boldsymbol{B}_0 . \mathrm{d}\boldsymbol{S}_0$ vanishes on $S_{\mathrm{pv}}(0)$, the interface between plasma and vacuum, making the integrand of term (13) vanish.

On $S_{\mathrm{pc}}(0)$, the interface between plasma and conductor, it is necessary to consider the case in which the magnetic field does not lie in the interface but enters the electrode (e.g. a linear pinch having internal axial field). At $S_{\mathrm{pc}}(0)$ the condition $\mathrm{d}\boldsymbol{S} \times \boldsymbol{E} = 0$ applies and so through equation (4)

$$(\mathrm{d}\boldsymbol{S} . \boldsymbol{B})\boldsymbol{v} = 0\,, \tag{14}$$

since $\mathrm{d}\boldsymbol{S} . \boldsymbol{v}$ vanishes at the fixed and rigid interface.

Since $\boldsymbol{v} = \partial\xi/\partial t$, integration of equation (14) to first order gives

$$(\mathrm{d}\boldsymbol{S}_0 . \boldsymbol{B}_0)\xi = 0\,. \tag{15}$$

Hence, when $\mathrm{d}\boldsymbol{S}_0 . \boldsymbol{B}_0$ does not vanish, the freezing-in effect of plasma infinite conductivity leads from equation (15) to $\xi = 0$ at the plasma–electrode interface, and so $\xi . (\xi . \nabla \boldsymbol{B}_0) = 0$ there. The term (13) thus vanishes over $S_{\mathrm{p}}(0) = S_{\mathrm{pv}}(0) + S_{\mathrm{pc}}(0)$.

Expression (11) gives $\delta \overline{W} = \delta W_{\mathrm{B}} + \delta W_{\mathrm{p}}$, where W_{B} is the energy of the magnetic field within the plasma and W_{p} is the material energy of the plasma. To obtain the complete expression for the change in system potential energy

$$\delta W = \delta \overline{W} + \delta W_{\mathrm{BE}}\,, \tag{16}$$

the variation δW_{BE} in the energy of the magnetic field occupying the volume external to the plasma must be obtained to *second order* in the perturbation ξ. To this end we first obtain $\mathrm{d}W_{\mathrm{BE}}/\mathrm{d}t$.

(*a*) *Determination of* $\mathrm{d}W_{\mathrm{BE}}/\mathrm{d}t$

The magnetic energy external to the plasma is

$$W_{\mathrm{BE}} = \tfrac{1}{2}\mu_0^{-1} \int_{\tau_{\mathrm{c}} + \hat{\tau}(t) + \tau_{\mathrm{i}}} B^2 \,\mathrm{d}\tau\,, \tag{17}$$

where the subscripts c and i refer respectively to the conductor and insulator regions.

Using spherical coordinates it can be proved generally that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_{\tau(t)} f(\boldsymbol{r}, t)\,\mathrm{d}\tau\right) = \int_{\tau(t)} \frac{\partial f(\boldsymbol{r}, t)}{\partial t}\,\mathrm{d}\tau + \int_{S(t)} f(\boldsymbol{r}, t)\boldsymbol{v} . \mathrm{d}\boldsymbol{S}\,, \tag{18}$$

an intuitively obvious result, in which $S(t)$ is the surface enclosing $\tau(t)$, $\boldsymbol{v}$ is the velocity of a point on $S(t)$, and $\mathrm{d}\boldsymbol{S}$ is directed out of $\tau(t)$. Thus

$$\frac{\mathrm{d}W_{\mathrm{BE}}}{\mathrm{d}t} = \mu_0^{-1} \int_{\tau_{\mathrm{c}} + \hat{\tau}(t) + \tau_{\mathrm{i}}} \frac{\partial(\frac{1}{2}B^2)}{\partial t}\,\mathrm{d}\tau + \tfrac{1}{2}\mu_0^{-1} \int_{S(t)} B^2 \boldsymbol{v} . \mathrm{d}\boldsymbol{S}\,. \tag{19}$$

In the rigid material occupying τ_{c} the electrical conductivity is extremely high, and so $\partial \boldsymbol{B}/\partial t$ is sensibly zero there, at least on the rapid time scale of unstable motions under consideration. Hence equation (19) becomes

$$\frac{\mathrm{d}W_{\mathrm{BE}}}{\mathrm{d}t} = \mu_0^{-1}\int_{\hat{\tau}(t)+\tau_{\mathrm{i}}}\frac{\partial(\frac{1}{2}B^2)}{\partial t}\,\mathrm{d}\tau + \tfrac{1}{2}\mu_0^{-1}\int_{S(t)} B^2\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{S}', \tag{20}$$

where

$$S(t) = S_{\mathrm{pv}} + S_{\mathrm{cv}} + S_{\mathrm{ci}} + S_{\infty}. \tag{21}$$

In equation (21) S_{ab} is the interface between media a and b, S_{∞} is the surface at infinity, and $S_{\mathrm{pi}} = 0$ by assumption. In equation (20) the prime on $\mathrm{d}\boldsymbol{S}$ is introduced so that on S_{pv} there will be no confusion between $\mathrm{d}\boldsymbol{S}'$, directed out of the vacuum, and $\mathrm{d}\boldsymbol{S}$, directed out of the plasma, in conformity with its equilibrium form $\mathrm{d}\boldsymbol{S}_0$ of equation (11).

Since no conduction currents flow in $\hat{\tau}(t)$ and τ_{i}, $\nabla \times \boldsymbol{B}$ vanishes in these regions. With the help of equation (5c), a standard vector identity, and Gauss's theorem the volume integral in equation (20) then transforms to the surface integral

$$I_1 = \int_{S(t)} \boldsymbol{B}\,.\,\boldsymbol{E} \times \mathrm{d}\boldsymbol{S}'. \tag{22}$$

Considering the terms of equation (21), $\boldsymbol{E} \times \mathrm{d}\boldsymbol{S}'$ vanishes on the conducting surfaces S_{cv} and S_{ci}, and furthermore it is assumed that the field quantities fall off rapidly enough for the contribution over S_{∞} in (22) to be vanishingly small. I_1 can thus be written

$$I_1 = \int_{S_{\mathrm{pv}}(t)} \hat{\boldsymbol{B}}\,.\,\hat{\boldsymbol{E}} \times \mathrm{d}\boldsymbol{S}'. \tag{23}$$

From equations (4) and (7) and the fact that $\boldsymbol{n}\,.\,\boldsymbol{B} = 0$ on $S_{\mathrm{pv}}(t)$ for the plasma–magnetic field model chosen, it is found that

$$\hat{\boldsymbol{E}} \times \mathrm{d}\boldsymbol{S}' = -\hat{\boldsymbol{B}}(\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{S}') \tag{24}$$

on $S_{\mathrm{pv}}(t)$. Equation (23) now becomes

$$I_1 = -\int_{S_{\mathrm{pv}}(t)} \hat{B}^2\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{S}'. \tag{25}$$

Putting

$$I_2 = \int_{S(t)} B^2\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{S}' \tag{26}$$

for the surface integral in equation (20), rigidity of S_{cv} and S_{ci} implies $\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{S}' = 0$, and so again taking $S_{\mathrm{pi}} = 0$ and the contribution to the integral over S_{∞} vanishingly small,

$$I_2 = \int_{S_{\mathrm{pv}}(t)} \hat{B}^2\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{S}'. \tag{27}$$

From the results (25) and (27), equation (20) now becomes

$$\frac{\mathrm{d}W_{\mathrm{BE}}}{\mathrm{d}t} = \mu_0^{-1}I_1 + \tfrac{1}{2}\mu_0^{-1}I_2 = -\tfrac{1}{2}\mu_0^{-1}\int_{S_{\mathrm{pv}}(t)} \hat{B}^2\boldsymbol{v}\,.\,\mathrm{d}\boldsymbol{S}'. \tag{28}$$

Since $\frac{1}{2}\mu_0^{-1}\hat{B}^2\,\mathrm{d}S'$ is the force directed into the plasma by the pressure of the vacuum magnetic field at the plasma–vacuum interface, integration of equation (28) with respect to time confirms mathematically the physical concept that the change in external magnetic energy is just the work done against the pressure of the vacuum magnetic field in deforming the surface $S_{\mathrm{pv}}(t)$.

(*b*) *Evaluation of* δW_{BE}

To obtain δW_{BE} to second order in the perturbation, equation (28) must be written to second order and integrated. To achieve this, one recalls that from the Lagrangian viewpoint physical properties of a given fluid element at $(\boldsymbol{r}, t)$ are functions of the initial position $\boldsymbol{r}_0$ of that element and of the time t. Thus to first order in ξ, the vacuum magnetic field at the perturbed plasma boundary at time t (see e.g. Schmidt 1966) can be written as

$$\hat{\boldsymbol{B}}(\boldsymbol{r},t) = \hat{\boldsymbol{B}}(\boldsymbol{r}_0,0) + \xi\,.\,\nabla\hat{\boldsymbol{B}}(\boldsymbol{r}_0,0) + \nabla\times\delta\hat{\boldsymbol{A}}\,, \tag{29}$$

where $\delta\hat{\boldsymbol{A}}$ is the first-order perturbation in the vacuum magnetic vector potential $\hat{\boldsymbol{A}}$.

To derive the required expression for $\mathrm{d}\boldsymbol{S}(\boldsymbol{r},t)$, the usual expression for $\mathrm{d}(\mathrm{d}\boldsymbol{S})/\mathrm{d}t$ on a deforming surface,

$$\mathrm{d}(\mathrm{d}\boldsymbol{S})/\mathrm{d}t = (\nabla\,.\,\boldsymbol{v})\,\mathrm{d}\boldsymbol{S} - (\nabla\boldsymbol{v})\,.\,\mathrm{d}\boldsymbol{S} \tag{30}$$

with

$$\boldsymbol{v}(\boldsymbol{r},t) = \partial\xi(\boldsymbol{r}_0,t)/\partial t\,, \tag{31}$$

is here integrated to first order in ξ to obtain

$$\mathrm{d}\boldsymbol{S}(\boldsymbol{r},t) = \mathrm{d}\boldsymbol{S}(\boldsymbol{r}_0,0) + (\nabla\,.\,\xi)\,\mathrm{d}\boldsymbol{S}(\boldsymbol{r}_0,0) - (\nabla\xi)\,.\,\mathrm{d}\boldsymbol{S}(\boldsymbol{r}_0,0)\,. \tag{32}$$

Insertion of equations (29), (31), and (32) into equation (28), together with $2\hat{\boldsymbol{B}}_0\,.\,(\xi\,.\,\nabla)\hat{\boldsymbol{B}}_0 = \xi\,.\,\nabla\hat{B}_0^2$, then yields

$$\mu_0\,\delta W_{\mathrm{BE}} = -\int_0^t \mathrm{d}t' \int_{S_{\mathrm{pv}}(0)} \tfrac{1}{2}\Big\{\hat{B}_0^2 + \xi\,.\,\nabla\hat{B}_0^2 + 2\hat{\boldsymbol{B}}_0\,.\,\nabla\times\delta\hat{\boldsymbol{A}}\Big\}\frac{\partial\xi}{\partial t'}\,.\Big\{\mathrm{d}\boldsymbol{S}_0' + (\nabla\,.\,\xi)\,\mathrm{d}\boldsymbol{S}_0' - (\nabla\xi)\,.\,\mathrm{d}\boldsymbol{S}_0'\Big\}, \tag{33}$$

where $\hat{\boldsymbol{B}}_0 = \hat{\boldsymbol{B}}(\boldsymbol{r}_0,0)$, and $\mathrm{d}\boldsymbol{S}_0' = \mathrm{d}\boldsymbol{S}_{\mathrm{pv}}'(\boldsymbol{r}_0,0)$ is directed into the plasma.

All quantities in equation (33) are functions of $(\boldsymbol{r}_0,t)$ and, since $\boldsymbol{r}_0$ is time independent, the integrations can be commuted. Retaining integrand terms to second order in ξ, the equation becomes

$$\mu_0\,\delta W_{\mathrm{BE}} = L + M + N\,, \tag{34}$$

where

$$L = -\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\,.\int_0^t \mathrm{d}t'\,(\tfrac{1}{2}\hat{B}_0^2)\partial\xi/\partial t' = -\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\,.\,\xi(\tfrac{1}{2}\hat{B}_0^2)\,, \tag{35}$$

$$M = -\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0' \,.\int_0^t \mathrm{d}t' \left\{\tfrac{1}{2}\hat{B}_0^2\left((\nabla\,.\,\xi)\frac{\partial\xi}{\partial t'} - \frac{\partial\xi}{\partial t'}\,.\,\nabla\xi\right) + \frac{\partial\xi}{\partial t'}\,\xi\,.\,\nabla(\tfrac{1}{2}\hat{B}_0^2)\right\}, \tag{36}$$

and

$$N = -\int_{S_{\mathrm{pv}}(0)} \int_0^t \mathrm{d}t' \,(\mathrm{d}\boldsymbol{S}_0'\,.\,\partial\xi/\partial t')(\hat{\boldsymbol{B}}_0\,.\,\nabla \times \delta\hat{\boldsymbol{A}}). \tag{37}$$

The simplification of N is readily achieved. We note first that equations (29), (31), (32), and $\hat{\boldsymbol{E}} = \partial(\delta\hat{\boldsymbol{A}})/\partial t$ enable equation (24) to assume the first-order form

$$\mathrm{d}\boldsymbol{S}_0' \times \partial(\delta\hat{\boldsymbol{A}})/\partial t = -(\mathrm{d}\boldsymbol{S}_0'\,.\,\partial\xi/\partial t)\hat{\boldsymbol{B}}_0, \tag{38}$$

so that equation (37) becomes

$$N = \int_{S_{\mathrm{pv}}(0)} \int_0^t \mathrm{d}t' \,\{\mathrm{d}\boldsymbol{S}_0' \times \partial(\delta\hat{\boldsymbol{A}})/\partial t'\}\,.\,(\nabla \times \delta\hat{\boldsymbol{A}}). \tag{39}$$

Since $\mathrm{d}\boldsymbol{S}_0' \times \partial(\delta\boldsymbol{A})/\partial t$ vanishes on S_{cv} and S_{ci} and the field quantities are assumed negligible at S_∞, the integral (39) can be taken over the surface

$$S(0) = S_{\mathrm{pv}}(0) + S_{\mathrm{cv}} + S_{\mathrm{ci}} + S_\infty, \tag{40}$$

the equilibrium form of equation (21) for the surface bounding the combined vacuum and insulator volume regions. With $\boldsymbol{A}$ the corresponding magnetic vector potential, the integral (39) becomes

$$N = \int_0^t \mathrm{d}t' \int_{S(0)} \mathrm{d}\boldsymbol{S}_0'\,.\left\{\frac{\partial(\delta\boldsymbol{A})}{\partial t'} \times (\nabla \times \delta\boldsymbol{A})\right\}, \tag{41}$$

which, by use of Gauss's theorem and the vanishing electric current condition $\nabla \times (\nabla \times \delta\boldsymbol{A}) = 0$, becomes

$$N = \tfrac{1}{2}\int_{\hat{\tau}(0)+\tau_{\mathrm{i}}} (\nabla \times \delta\boldsymbol{A})^2 \,\mathrm{d}\tau_0, \tag{42}$$

after time integration.

The reduction of M in equation (36) to simpler form presents difficulties. However, the procedure gives useful insight into the underlying physics, and is outlined as follows.

Using a standard vector identity, the expansion of the vector triple product, and the equilibrium form of the pressure balance equation (6), equation (36) becomes

$$M = P + Q, \tag{43}$$

where

$$P = -\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\,.\int_0^t \mathrm{d}t' \left\{\tfrac{1}{2}\hat{B}_0^2\left(\xi\nabla\,.\,\frac{\partial\xi}{\partial t'} - \xi\,.\,\nabla\frac{\partial\xi}{\partial t'}\right) + \xi\frac{\partial\xi}{\partial t'}\,.\,\nabla(\tfrac{1}{2}\hat{B}_0^2)\right\} \tag{44}$$

and

$$Q = -\int_0^t \mathrm{d}t' \int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\,.\left\{(\mu_0 p_0 + \tfrac{1}{2}B_0^2)\nabla \times \left(\frac{\partial\xi}{\partial t'} \times \xi\right) + \nabla(\tfrac{1}{2}\hat{B}_0^2) \times \left(\frac{\partial\xi}{\partial t'} \times \xi\right)\right\}. \tag{45}$$

Equation (44) can immediately be integrated by parts to give, with the help of equation (36),

$$P = -\int_{S_{\rm pv}(0)} \mathrm{d}\boldsymbol{S}_0' \,.\, \{\tfrac{1}{2}\hat{B}_0^2(\xi\nabla\,.\,\xi - \xi\,.\,\nabla\xi) + \xi\xi\,.\,\nabla(\tfrac{1}{2}\hat{B}_0^2)\} - M\,. \tag{46}$$

Integration of equation (45) presents some difficulties. Writing

$$Q = R + S, \tag{47}$$

where

$$R = -\int_0^t \mathrm{d}t' \int_{S_{\rm pv}(0)} \mathrm{d}\boldsymbol{S}_0' \,.\left\{(\mu_0 p_0 + \tfrac{1}{2}B_0^2)\nabla \times \left(\frac{\partial\xi}{\partial t'} \times \xi\right)\right\} \tag{48}$$

and

$$S = -\int_0^t \mathrm{d}t' \int_{S_{\rm pv}(0)} \left(\mathrm{d}\boldsymbol{S}_0' \times \nabla(\tfrac{1}{2}\hat{B}_0^2)\right).\left(\frac{\partial\xi}{\partial t'} \times \xi\right), \tag{49}$$

S can be simplified by introduction of the jump condition (see e.g. Rose and Clark 1961)

$$\langle \mathrm{d}\boldsymbol{S}_0' \times \nabla(\mu_0 p_0 + \tfrac{1}{2}B_0^2)\rangle = 0\,, \tag{50}$$

which permits equation (49) to be written as

$$S = -\int_0^t \mathrm{d}t' \int_{S_{\rm pv}(0)} \mathrm{d}\boldsymbol{S}_0' \,.\left\{\nabla\left(\mu_0 p_0 + \tfrac{1}{2}B_0^2\right) \times \left(\frac{\partial\xi}{\partial t'} \times \xi\right)\right\}. \tag{51}$$

Combination of equations (48) and (51) in equation (47) now yields

$$Q = -\int_0^t \mathrm{d}t' \int_{S_{\rm pv}(0)} \mathrm{d}\boldsymbol{S}_0' \,.\, \nabla \times \left\{\left(\mu_0 p_0 + \tfrac{1}{2}B_0^2\right)\left(\frac{\partial\xi}{\partial t'} \times \xi\right)\right\}. \tag{52}$$

If $S_{\rm pv}(0)$ is a *closed surface* (i.e. if the plasma is in contact with vacuum only), Gauss's theorem applied to equation (52) shows that Q vanishes. If, however, the plasma is in contact with electrodes, $S_{\rm pv}(0)$ is *not* a closed surface. In this case let $C(t)$ be the contour representing the intersection of $S_{\rm pv}(t)$ and $S_{\rm pc}(t)$, and let $\mathrm{d}\boldsymbol{l}$ be an element of path around $C(0)$. Then equation (52) transforms by Stokes's theorem to

$$Q = -\int_0^t \mathrm{d}t' \oint_{C(0)} \mathrm{d}\boldsymbol{l}\,.\left(\mu_0 p_0 + \tfrac{1}{2}B_0^2\right)\left(\frac{\partial\xi}{\partial t'} \times \xi\right). \tag{53}$$

At the beginning of this section (equation (15)) it was shown that, when $\boldsymbol{B}_0\,.\,\mathrm{d}\boldsymbol{S}_{\rm c} \neq 0$, ξ vanishes on $S_{\rm pc}(0)$. Under these conditions Q of equation (53) correspondingly vanishes.

If $\boldsymbol{B}_0\,.\,\mathrm{d}\boldsymbol{S}_{\rm c} = 0$ the displacement of fluid elements on $S_{\rm pc}(0)$ must always be parallel to that surface,

$$\boldsymbol{v}(\boldsymbol{r}, t)\,.\,\mathrm{d}\boldsymbol{S}_{\rm c}(\boldsymbol{r}) = 0\,. \tag{54}$$

Recalling equation (31), and noting the spatial relationship

$$\mathrm{d}\boldsymbol{S}_{\mathrm{c}}(\boldsymbol{r}) = \mathrm{d}\boldsymbol{S}_{\mathrm{c}}(\boldsymbol{r}_0) + \boldsymbol{\xi}\,.\,\nabla\{\mathrm{d}\boldsymbol{S}_{\mathrm{c}}(\boldsymbol{r}_0)\}$$

to first order, equation (54) gives the first-order result

$$(\partial\boldsymbol{\xi}/\partial t)\,.\,\mathrm{d}\boldsymbol{S}_{\mathrm{c}}(\boldsymbol{r}_0) = 0\,, \tag{55}$$

which when integrated to first order yields

$$\boldsymbol{\xi}\,.\,\mathrm{d}\boldsymbol{S}_{\mathrm{c}}(\boldsymbol{r}_0) = 0\,, \tag{56}$$

since $\boldsymbol{\xi} = 0$ at $t = 0$. From equations (55) and (56)

$$\mathrm{d}\boldsymbol{S}_{\mathrm{c}} \times \{(\partial\boldsymbol{\xi}/\partial t) \times \boldsymbol{\xi}\} = 0\,,$$

and so, with C lying in S_{pc}, $\mathrm{d}\boldsymbol{l}\,.\,\mathrm{d}\boldsymbol{S}_{\mathrm{c}} = 0$ and

$$\mathrm{d}\boldsymbol{l}\,.\{(\partial\boldsymbol{\xi}/\partial t) \times \boldsymbol{\xi}\} = 0\,. \tag{57}$$

Applying the result (57) to equation (53), it is seen that in this case Q again vanishes. Equation (43) now reduces to $M = P$, and equation (46) therefore yields

$$M = -\tfrac{1}{2}\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\,.\{\tfrac{1}{2}\hat{B}_0^2(\boldsymbol{\xi}\nabla\,.\,\boldsymbol{\xi} - \boldsymbol{\xi}\,.\,\nabla\boldsymbol{\xi}) + \boldsymbol{\xi}\boldsymbol{\xi}\,.\,\nabla(\tfrac{1}{2}\hat{B}_0^2)\}\,. \tag{58}$$

From equations (35), (42), and (58), equation (34) can now be expressed explicitly as

$$\begin{aligned}\mu_0\,\delta W_{\mathrm{BE}} = &-\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\,.\,\boldsymbol{\xi}(\tfrac{1}{2}\hat{B}_0^2) + \tfrac{1}{2}\int_{\hat{\tau}(0)+\tau_{\mathrm{i}}} (\nabla \times \delta\boldsymbol{A})^2\,\mathrm{d}\tau_0 \\ &-\tfrac{1}{2}\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\,.\{\tfrac{1}{2}\hat{B}_0^2(\boldsymbol{\xi}\nabla\,.\,\boldsymbol{\xi} - \boldsymbol{\xi}\,.\,\nabla\boldsymbol{\xi}) + \boldsymbol{\xi}\boldsymbol{\xi}\,.\,\nabla(\tfrac{1}{2}\hat{B}_0^2)\}\,.\end{aligned} \tag{59}$$

The first-order term in equation (59) represents the contribution from external regions to the first-order variation in system potential energy. Since this first-order variation vanishes for a system originally in equilibrium, it is the second-order terms that are used in the following analysis.

(c) *Determination of* $\delta W = \delta W_{\mathrm{F}} + \delta W_{\mathrm{S}} + \delta W_{\mathrm{E}}$

It was shown above that the term T (equation (13)), which appears in equation (11) vanishes over $S_{\mathrm{p}}(0) = S_{\mathrm{pv}}(0) + S_{\mathrm{pc}}(0)$. Recalling that $\boldsymbol{\xi} = 0$ for the required condition $\boldsymbol{B}_0\,.\,\mathrm{d}\boldsymbol{S}_{\mathrm{c}} \neq 0$, the remaining terms in the surface integral of equation (11) may be taken over $S_{\mathrm{p}}(0) = S_{\mathrm{pv}}(0)$ alone. With the help of equations (3) and (6) in equilibrium form, the former being transformed to give $\boldsymbol{B}_0\,.\,\nabla\boldsymbol{B}_0 = \nabla(\mu_0 p_0 + \tfrac{1}{2}B_0^2)$, substitution of equation (11) and the second-order terms of equation (59) into equation (16) leads, with $\mathrm{d}\boldsymbol{S}_0 = -\mathrm{d}\boldsymbol{S}_0'$, to

$$\delta W = \delta W_{\mathrm{F}} + \delta W_{\mathrm{S}} + \delta W_{\mathrm{E}} \tag{60}$$

for the change in system potential energy. The three terms in equation (60) are:

$$\delta W_{\mathrm{F}} = \tfrac{1}{2}\int_{\tau_{\mathrm{p}}(0)} \mathrm{d}\tau_0 \,\{\mu_0^{-1}\,|\boldsymbol{Q}|^2 - \boldsymbol{j}_0\cdot(\boldsymbol{Q}\times\boldsymbol{\xi}) + \gamma p_0(\nabla\cdot\boldsymbol{\xi})^2 + (\nabla\cdot\boldsymbol{\xi})\boldsymbol{\xi}\cdot\nabla p_0\} \tag{61}$$

the fluid contribution;

$$\delta W_{\mathrm{S}} = \tfrac{1}{2}\int_{S_{\mathrm{pv}}(0)} (\mathrm{d}\boldsymbol{S}_0\cdot\boldsymbol{\xi})\boldsymbol{\xi}\cdot\{\nabla(\tfrac{1}{2}\mu_0^{-1}\hat{B}_0^2) - \nabla(p_0 + \tfrac{1}{2}\mu_0^{-1}B_0^2)\} \tag{62}$$

the surface contribution; and

$$\delta W_{\mathrm{E}} = \tfrac{1}{2}\mu_0^{-1}\int_{\hat{\tau}(0)+\tau_{\mathrm{i}}} (\nabla\times\delta\boldsymbol{A})^2\,\mathrm{d}\tau_0 \tag{63}$$

the contribution from the vacuum and insulator regions external to the plasma.

Expressing $\boldsymbol{\xi} = (\boldsymbol{n}_0\times\boldsymbol{\xi})\times\boldsymbol{n}_0 + \boldsymbol{n}_0(\boldsymbol{n}_0\cdot\boldsymbol{\xi})$ and $\mathrm{d}\boldsymbol{S}_0 = \boldsymbol{n}_0\,\mathrm{d}S_0$, (62) assumes the familiar form

$$\delta W_{\mathrm{S}} = \tfrac{1}{2}\int_{S_{\mathrm{pv}}(0)} \mathrm{d}S_0\,(\boldsymbol{n}_0\cdot\boldsymbol{\xi})^2\langle\boldsymbol{n}_0\cdot\nabla(p_0 + \tfrac{1}{2}\mu_0^{-1}B_0^2)\rangle, \tag{64}$$

in view of the jump condition (50).

The result (63) for δW_{E} can also be expressed as a surface integral over $S_{\mathrm{pv}}(0)$. Noting that $\mathrm{d}\boldsymbol{S}_0'\times\delta\boldsymbol{A}$ also vanishes on S_{cv} and S_{ci}, the argument leading from equation (39) to (42) shows that

$$\int_{S_{\mathrm{pv}}(0)}\int_0^t \mathrm{d}t'\left(\mathrm{d}\boldsymbol{S}_0'\times\delta\hat{\boldsymbol{A}}\right)\cdot\frac{\partial(\nabla\times\delta\hat{\boldsymbol{A}})}{\partial t'} = \tfrac{1}{2}\int_{\hat{\tau}(0)+\tau_{\mathrm{i}}}(\nabla\times\delta\boldsymbol{A})^2\,\mathrm{d}\tau_0 = N, \tag{65}$$

as in (42). Since, from equations (42) and (63), $\delta W_{\mathrm{E}} = \mu_0^{-1}N$, combination of equations (39) and (65) followed by time integration and use of

$$\mathrm{d}\boldsymbol{S}_0'\times\delta\hat{\boldsymbol{A}} = -(\mathrm{d}\boldsymbol{S}_0'\cdot\boldsymbol{\xi})\hat{\boldsymbol{B}}_0, \tag{66}$$

the boundary condition on $S_{\mathrm{pv}}(0)$ obtained by integration of equation (38), leads to

$$\delta W_{\mathrm{E}} = \int_{S_{\mathrm{pv}}(0)} (\mathrm{d}\boldsymbol{S}_0\cdot\boldsymbol{\xi})(\tfrac{1}{2}\mu_0^{-1}\hat{\boldsymbol{B}}_0\cdot\nabla\times\delta\hat{\boldsymbol{A}}), \tag{67}$$

a form not unexpected on physical grounds.

In the approach of Bernstein *et al.* (1958) an expression was obtained for the second-order variation of system potential energy for a region which can often be considered to be surrounded by a rigid perfectly-conducting wall, without specific inclusion of external insulators.

Here the approach of Van Kampen and Felderhof (1967) has been extended to a system comprising a finite plasma-body with external non-shortcircuiting rigid conductors, insulators, and an external magnetic field in a vacuum region extending to infinity, showing that it is permissible to apply the energy principle to the plasma between electrodes. In particular the approach adopted here enables a better understanding to be obtained of the surface term δW_{S} given by equation (64).

IV. Examination of Surface Contribution δW_S

At first sight it seems plausible to identify the surface term (64) appearing in equation (60) as the second-order part of the work performed against the surface current in displacing the boundary by ξ, as has been done, for example, by Schmidt* (1966, p. 125). The following examination of δW_S amplifies its nature and suggests that such a simple one-to-one correspondence between its mathematical form and a specific physical effect is not possible.

If the plausible identification mentioned above is to be generally correct, it must be correct when the internal magnetic field $\boldsymbol{B}_0$ is zero. The "work done against the surface current" is then just the work done against the vacuum magnetic field, and for any ξ it follows that for second-order surface terms

$$\delta W_S = \delta W_{BE} \tag{68}$$

should be true. Noting that the first-order plasma volume change produced by ξ is

$$\delta\tau_1 = -\int_{S_{pv}(0)} \mathrm{d}\boldsymbol{S}'_0 \,.\, \xi\,, \tag{69}$$

we choose for convenience ξ tangential to $S_{pv}(0)$ at all points,

$$\mathrm{d}\boldsymbol{S}'_0 \,.\, \xi = 0\,, \tag{70}$$

and find from equations (59), (63), and (67) that

$$\delta W_{BE} = \tfrac{1}{4}\mu_0^{-1}\hat{B}_0^2 \int_{S_{pv}(0)} \mathrm{d}\boldsymbol{S}'_0 \,.\, (\xi \,.\, \nabla)\xi\,, \tag{71}$$

in view of the last paragraph of Section II.

Thus equation (71) does not necessarily vanish for the ξ field chosen to satisfy equation (70), whereas δW_S given by equation (64) does. The equality (68) is therefore not satisfied, and the plausible identification mentioned at the beginning of this section does not hold in this case. Hence it cannot be true in general.

The non-vanishing of expression (71) for δW_{BE} reflects the fact that equation (70) does not specify zero deformation of the plasma surface to second order in ξ. In fact there exists such a deformation, which requires a second-order amount of work given by expression (71). To understand this more clearly, consider the exact plasma volume change

$$\delta\tau_2 = -\int_0^t \mathrm{d}t' \int_{S_{pv}(0)} \mathrm{d}\boldsymbol{S}' \,.\, \boldsymbol{v}\,. \tag{72}$$

For no deformation of the plasma surface $\delta\tau_2 = 0$, and this is ensured by the condition

$$\mathrm{d}\boldsymbol{S}' \,.\, \boldsymbol{v} = 0\,. \tag{73}$$

* One of us (P.W.S.) wrote on this point to Professor Schmidt, who advised that he did not regard his proof of this identification of δW_S as satisfactory.

With the help of equations (31), (32), and (69), time integration of equation (72) leads to

$$\delta\tau_2 - \delta\tau_1 = -\tfrac{1}{2}\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0' \cdot (\boldsymbol{\xi}\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi}), \tag{74}$$

the second-order change of plasma volume. Using this result and the relation (67), for zero internal magnetic field, equation (59) becomes

$$\delta W_{\mathrm{BE}} = \tfrac{1}{2}\mu_0^{-1}\hat{B}_0^2\,\delta\tau_2 - \tfrac{1}{2}\mu_0^{-1}\int_{S_{\mathrm{pv}}(0)} (\mathrm{d}\boldsymbol{S}_0'\cdot\boldsymbol{\xi})\{\hat{\boldsymbol{B}}_0\cdot\nabla\times\delta\hat{\boldsymbol{A}} + \tfrac{1}{2}\boldsymbol{\xi}\cdot\nabla\hat{B}_0^2\}. \tag{75}$$

For the condition (73), which also implies the weaker condition (70), δW_{BE} of equation (75) vanishes. For condition (70) alone, which ensures the vanishing of $\delta\tau_1$ of equation (69), but which does not make $\delta\tau_2$ of equation (72) zero, δW_{BE} of equation (75) correctly reduces to the second-order energy form (71).

From this treatment it is seen that interpretation of δW_{S} given by equation (64) as the work done against the surface current when the plasma boundary is displaced by $\boldsymbol{\xi}$ is not correct because of modification of the vacuum field magnetic energy arising from second-order distortion of the boundary surface. Indeed, it seems very difficult to give δW_{S} a simple direct physical interpretation, probably because it is a composite term arising from combination of equations (11) and (59) in equation (16) as described. Mathematically the situation is easily understood. In obtaining δW of equation (60) from equations (11) and (59), the term

$$-\tfrac{1}{2}\int_{S_{\mathrm{pv}}(0)} \mathrm{d}\boldsymbol{S}_0'\cdot(\boldsymbol{\xi}\nabla\cdot\boldsymbol{\xi} - \boldsymbol{\xi}\cdot\nabla\boldsymbol{\xi})(\tfrac{1}{2}\hat{B}_0^2)$$

appearing in equation (59) is cancelled, and so it cannot influence δW_{S} appearing in equation (60). On the other hand, it is just this term in equation (59) that gives rise to the non-zero residue (71) for the condition (70), leading in turn to violation of the proposed equality (68) when $\mathrm{d}\boldsymbol{S}_0'\cdot\boldsymbol{\xi} = 0$.

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