POSSIBLE CONSEQUENCES OF THE QUARK MODEL FOR FRAGMENTATION PROCESSES

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Abstract

The quark model is used to predict various sum rules for limiting fragmentation. In particular, the ratio of the cross section of $p \rightarrow p$ to that of $\pi \rightarrow \pi$ is found to be 9 : 4. An initial combination of the quark model with the generalized optical theorem is found to produce results which violate the quark-model selection rules and differ from experimental values by a factor of 10^3–10^6. A possible modification is proposed and further tests are suggested.

I. INTRODUCTION

Inclusive reactions have recently received considerable theoretical (Feynman 1969; Chou and Yang 1970; Mueller 1970; Benecke 1971; Chou 1971; De Tar *et al.* 1971; Peccei and Pignotti 1971) and experimental (Allaby *et al.* 1968; Binon *et al.* 1969; Bushnin *et al.* 1969; Anthony *et al.* 1971; Ko and Lauder 1971*a*, 1971*b*; Ratner *et al.* 1971; Smith 1971) attention. This is undoubtedly because they provide one of the simplest and neatest ways of studying multiparticle production at high energies. Many ideas have been applied to the study of inclusive reactions. Some have been taken from the study of two-body scattering processes while others are newly developed, such as: fragmentation (Chou and Yang 1970), Regge pole phenomenology, duality (Veneziano 1971), scaling, and the generalized optical theorem. These have been applied to the study of inclusive reactions with varying degrees of success. In particular, the quark model, when combined with the idea of fragmentation, has been successfully applied to produce the selection rules for inclusive reactions (Lo and Phua 1972). In the present paper we pursue further the consequences of the quark model and seek more refined results such as sum rules, rather than just qualitative statements of the selection rules. We focus our attention particularly on those aspects of the quark model which cannot easily be reproduced with other models.

Through lack of surer guidance, we shall try to base our treatment on the application of the quark model to two-body reactions (Levin and Frankfurt 1965; Kokedee and Van Hove 1966; Lipkin and Scheck 1966; Lipkin 1969). The fundamental assumption of the quark model for scattering processes is that of linearity: the two-body hadron scattering amplitude is a linear superposition of the two-body quark–quark scattering amplitudes. We assume that this also applies to the amplitude for fragmentation. One aspect of the quark model's particular success in two-body scattering is its provision of sum rules for elastic scattering (Lipkin and Scheck 1966). On the other hand, its success in relation to inelastic scattering can often be achieved

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with other models. Hence our main task is to determine what the quark model can tell us about the favoured fragmentation $a \to a$, the closest analogue in the domain of inclusive reactions to two-body elastic scattering. Answers to this question are provided in Section II. In Section III the compatibility of the quark model and the generalized optical theorem as formulated by Mueller (1970), Chan *et al.* (1971), and De Tar *et al.* (1971) is examined. A discussion of the conclusions reached is given in Section IV.

II. Sum Rules for Favoured Fragmentation

Let us first review what is meant by a favoured fragmentation (Chou and Yang 1970; Lo and Phua 1972). To do this we consider the inclusive reaction

$$a+b \to c+[\text{u.f.}], \tag{1}$$

where [u.f.] stands for any other *unspecified fragment(s)*. In this reaction, the beam a fragments into c and [u.f.]

$$a \xrightarrow{b} c+[\text{u.f.}], \tag{2a}$$

or simply

$$a \to c+[\text{u.f.}], \tag{2b}$$

and the target b fragments into [u.f.]. In general, pionization products, i.e. the slow particles in the centre-of-mass system at infinite energy, are not considered here because they are neither exclusively the fragmentation of the beam nor that of the target.

The quark model suggests that reaction (1) may be regarded as

$$q_a+q_b \to q_c+[\text{u.f.}], \tag{3}$$

where $a = q_a$, $b = q_b$, and $c = q_c$. The fragmentation of a can be regarded as a linear combination of quark transitions:

$$q_a \to q_c \qquad \text{and} \qquad q_a \to q_c+[\text{u.f.}]. \tag{4a,b}$$

In a previous work (Lo and Phua 1972), it was suggested that the quark transition $q_a \to q_c$ is allowed if $q_a = q_c$ and suppressed if $q_a \neq q_c$. Hence fragmentations which contain only allowed quark transitions are *favoured*; those which contain at least one allowed quark transition are *allowed*; and those which contain only suppressed quark transitions are *suppressed*. For example,

$$\mathrm{p} \to \mathrm{p}, \qquad \pi^{\pm} \to \pi^{\pm}, \qquad \text{and} \qquad \mathrm{p} \to \mathrm{p}+[\text{u.f.}] \tag{5a}$$

are favoured fragmentations;

$$\mathrm{p} \to \pi^{\pm} \qquad \text{and} \qquad \mathrm{p} \to \mathrm{K}^{+} \tag{5b}$$

are allowed fragmentations; and

$$\mathrm{p} \to \mathrm{K}^{-} \qquad \text{and} \qquad \pi^{+} \to \pi^{-}+[\text{u.f.}] \tag{5c}$$

are suppressed fragmentations.

The corresponding cases to favoured, allowed, and suppressed fragmentations in two-body scattering are elastic, inelastic, and exotic exchange processes respectively. It is because, in the theory of two-body scattering, the quark model surpasses other high energy models in its ability to correlate elastic scatterings, especially meson–baryon and baryon–baryon scattering, that we concentrate here on the consequences of the quark model in the analogous situation, favoured fragmentations.

We now turn to the sum rules for favoured fragmentation and assume initially that, in the inclusive reaction (1), the beam fragments into itself

$$a \to a \tag{6}$$

and the target fragments in an unspecified manner

$$b \to [\text{u.f.}]. \tag{7}$$

On the basis of the following quark structures

$$(2\pi^+)(\pi^-) = \mathrm{p}\bar{\mathrm{n}} = ppn\bar{p}\bar{n}\bar{n} \qquad \text{and} \qquad (2\pi^-)(\pi^+) = \mathrm{n}\bar{\mathrm{p}} = pnn\bar{p}\bar{p}\bar{n}, \tag{8a, b}$$

we have the following relations between pion, proton, and neutron fragmentations

$$2(\pi^+ \xrightarrow{b} \pi^+) + (\pi^- \xrightarrow{b} \pi^-) = (\mathrm{p} \xrightarrow{b} \mathrm{p}) + (\bar{\mathrm{n}} \xrightarrow{b} \bar{\mathrm{n}}) \tag{9a}$$

and

$$2(\pi^- \xrightarrow{b} \pi^-) + (\pi^+ \xrightarrow{b} \pi^+) = (\mathrm{n} \xrightarrow{b} \mathrm{n}) + (\bar{\mathrm{p}} \xrightarrow{b} \bar{\mathrm{p}}), \tag{9b}$$

where b stands for any hadron target. In the asymptotic energy region, where it is expected that exact SU(3) symmetry holds and quark–antiquark fragmentation is equal, equations (9a) and (9b) become

$$(\mathrm{p} \xrightarrow{\mathrm{p}} \mathrm{p}) \approx \tfrac{3}{2}(\pi \xrightarrow{\mathrm{p}} \pi). \tag{10}$$

The above sum rules (9a), (9b), and (10), whose proof may be found in the Appendix, deal with scattering amplitudes and they imply an inequality in cross section. If we let

$$2w(\mathrm{d}^3\sigma/\mathrm{d}p^3)(a \xrightarrow{b} a) = f(a \xrightarrow{b} a), \tag{11}$$

where p and w are the momentum and energy of the final particle a, then equation (9a) becomes

$$f^{\frac{1}{2}}(\mathrm{p} \to \mathrm{p}) \leqslant f^{\frac{1}{2}}(\bar{\mathrm{n}} \to \bar{\mathrm{n}}) + 2f^{\frac{1}{2}}(\pi^+ \to \pi^+) + f^{\frac{1}{2}}(\pi^- \to \pi^-) \tag{12}$$

and equation (10) becomes

$$f(\mathrm{p} \xrightarrow{\mathrm{p}} \mathrm{p}) = \tfrac{9}{4} f(\pi \xrightarrow{\mathrm{p}} \pi). \tag{13}$$

It is interesting to note that, in the scattering of a and b, there are four closely related processes:

(i) $\qquad a \to a \quad \text{and} \quad b \to b,$

this being the familiar two-body elastic scattering;

(ii) $a \to a$ and $b \to [\mathrm{u.f.}]$;

(iii) $a \to a + [\mathrm{u.f.}]$ and $b \to [\mathrm{u.f.}]$,

this being the usual single-particle distribution in inclusive reactions; and

(iv) $a \to a + [\mathrm{u.f.}]$ and $b \to b + [\mathrm{u.f.}]$.

The linearity assumption of the quark model of the type (8a, b) has been applied to elastic scattering (process (i) above) with success (Lipkin 1969). There appears to be no theoretical reason why linearity should not also hold for processes (ii), (iii), and (iv) as well. There is already some indication from the work of Satz (1967) that multipion production obeys the linearity assumption of the quark model. Hence it would seem vital to test the validity of the relationships (9a) and (9b) for processes (ii), (iii), and (iv) as well. From the experimental point of view, the easiest to check is process (iii). This is especially so, since equation (13) for this process becomes

$$f(\mathrm{p} \to \mathrm{p} + [\mathrm{u.f.}]) = \tfrac{9}{4} f(\pi \to \pi + [\mathrm{u.f.}]), \tag{14}$$

which is the most important form and easiest to test. It is interesting to note that if the factorization of fragmentations a and b is assumed, the ratio 9 : 4 for proton and pion fragmentations would be true for processes (ii) and (iv) if it is true for processes (i) and (iii) respectively.

III. Generalized Optical Theorem and Quark Model

In this section the quark model is combined with the idea of analytical continuation and the generalized optical theorem advocated by Mueller (1970) and Bialas and Czyzewski (1971). Mueller (1970) suggests that the cross section for the inclusive reaction (1) is related to the discontinuity of the off-mass shell elastic scattering of three particles

$$(a + \bar{c}) + b \to (a + \bar{c}) + b. \tag{15}$$

Now the quark model suggests that the scattering amplitude for the process (15) is a linear combination of three quark scattering amplitudes

$$\langle ab\bar{c} | a\bar{c}b \rangle = \Sigma \langle q_a, \bar{q}_c, q_b | q_a, \bar{q}_c, q_b \rangle. \tag{16}$$

Hence the reactions

$$\left.\begin{array}{cc} \mathrm{p} + \mathrm{p} + b \to \mathrm{p} + \mathrm{p} + b, & \mathrm{p} + \pi + b \to \mathrm{p} + \pi + b, \\ \mathrm{p} + \mathrm{K}^+ + b \to \mathrm{p} + \mathrm{K}^+ + b, & \mathrm{p} + \mathrm{K}^- + b \to \mathrm{p} + \mathrm{K}^- + b, \\ \multicolumn{2}{c}{\mathrm{p} + \bar{\mathrm{p}} + b \to \mathrm{p} + \bar{\mathrm{p}} + b,} \end{array}\right\} \tag{17}$$

where b stands for any hadron target, should all have scattering amplitudes of the same order of magnitude.

The fragmentation cross sections for the inclusive reaction (1) according to the generalized optical theorem are

$$\begin{aligned} f(\mathrm{p} \xrightarrow{b} \mathrm{p}+[\mathrm{u.f.}]) &\approx f(\bar{\mathrm{p}} \xrightarrow{b} \bar{\mathrm{p}}+[\mathrm{u.f.}]) \\ &\approx \tfrac{3}{2} f(\mathrm{p} \xrightarrow{b} \pi^{\pm}+[\mathrm{u.f.}]) \\ &\approx \tfrac{3}{2} f(\mathrm{p} \xrightarrow{b} \mathrm{K}^{\pm}+[\mathrm{u.f.}]) . \end{aligned} \tag{18}$$

These not only violate the selection rules of the quark model (Lo and Phua 1972) but also disagree violently with the experimental fact (Allaby *et al.* 1968) that

$$f(\mathrm{p} \xrightarrow{\mathrm{p}} \bar{\mathrm{p}}+[\mathrm{u.f.}])/f(\mathrm{p} \xrightarrow{\mathrm{p}} \mathrm{p}+[\mathrm{u.f.}]) \sim 10^{-3}\text{–}10^{-6} \tag{19}$$

for 19 GeV/c incident protons. Thus, on this initial comparison, the combination of the quark model and the generalized optical theorem produces results that are incompatible with experiment.

We do not know which of the two theoretical ideas, Mueller's generalized optical theorem* or the quark model, is at fault. If the ratio of proton fragmentation $\mathrm{p} \to \mathrm{p}$ to pion fragmentation $\pi \to \pi$ (equations (13) and (14)) turns out not to be 9/4, the linearity of the quark model is clearly not applicable to fragmentation processes. On the other hand, if the ratio indeed turns out to be 9/4, it is obvious that one has to modify the applicability of Mueller's generalized optical theorem at least in its present form.

One possible minimum, although arbitrary, modification of our assumptions is to follow the selection rules and divide the fragmentations into the three classes: favoured, allowed, and suppressed. We then postulate that the combination of the linearity assumption of the quark model, as expressed by equation (16), and the generalized optical theorem can only predict relations among fragmentations of the *same* class but not among fragmentations of different classes. Thus it cannot be used to relate favoured and allowed fragmentations, for instance. The only possible virtue of such a modification is that it still enables us to obtain a large number of sum rules involving cross sections, but not amplitudes, whose validity can be tested experimentally. It may be appropriate to add that, when applied to two-body scattering processes, the quark model cannot be used to relate elastic scattering and inelastic scattering or exotic exchange reactions. Hence our proposed modification is not without precedent.

The sum rules, obtained from equation (16) on our modified assumption, are presented below. For brevity the symbol h is used to represent the unspecified fragment(s). Exact SU(3) symmetry is assumed for three-quark scattering.

The sum rules relating favoured fragmentations are:

$$\begin{aligned} f(\pi^{+} \to \pi^{+}+h) &= f(\pi^{-} \to \pi^{-}+h) \\ &= f(\mathrm{K}^{+} \to \mathrm{K}^{+}+h) \\ &= f(\mathrm{K}^{-} \to \mathrm{K}^{-}+h) . \end{aligned} \tag{20}$$

* Relations obtained for the off-mass shell amplitude may not be applicable to the physical processes.

The sum rules relating allowed fragmentations are:

$$f(\mathrm{p}\to\pi^-+h)-f(\mathrm{p}\to\mathrm{K}^++h)$$
$$=f(\mathrm{p}\to\pi^-+h)-f(\mathrm{p}\to\pi^++h)+f(\mathrm{p}\to\mathrm{K}^++h)-f(\mathrm{n}\to\mathrm{K}^++h),\quad(21\mathrm{a})$$

$$f(\pi^+\to\bar{\mathrm{p}}+h)-f(\pi^+\to\bar{\mathrm{n}}+h)$$
$$=f(\pi^-\to\bar{\mathrm{n}}+h)-f(\pi^-\to\bar{\mathrm{p}}+h)$$
$$=f(\mathrm{p}\to\pi^-+h)-f(\mathrm{p}\to\pi^++h)+f(\pi^+\to\mathrm{K}^++h)-f(\pi^-\to\mathrm{K}^-+h),\quad(21\mathrm{b})$$

$$f(\pi^+\to\mathrm{p}+h)-f(\pi^+\to\mathrm{n}+h)=f(\pi^-\to\mathrm{n}+h)-f(\pi^-\to\mathrm{p}+h),\quad(21\mathrm{c})$$

$$f(\mathrm{K}^+\to\mathrm{p}+h)-f(\mathrm{K}^+\to\mathrm{n}+h)$$
$$=f(\mathrm{p}\to\mathrm{K}^++h)-f(\mathrm{n}\to\mathrm{K}^++h)$$
$$=2f(\pi^+\to\mathrm{p}+h)-f(\pi^+\to\mathrm{n}+h)-f(\mathrm{K}^+\to\mathrm{p}+h),\quad(21\mathrm{d})$$

$$f(\pi^+\to\mathrm{K}^++h)=f(\mathrm{K}^+\to\pi^++h),\quad(21\mathrm{e})$$

$$f(\pi^-\to\mathrm{K}^-+h)=f(\mathrm{K}^-\to\pi^-+h),\quad(21\mathrm{f})$$

$$f(\pi^+\to\mathrm{K}^++h)-f(\pi^-\to\mathrm{K}^-+h)$$
$$=f(\mathrm{K}^+\to\mathrm{p}+h)-f(\mathrm{K}^+\to\mathrm{n}+h)+f(\mathrm{K}^-\to\bar{\mathrm{n}}+h)-f(\mathrm{K}^-\to\bar{\mathrm{p}}+h),\quad(21\mathrm{g})$$

$$f(\bar{\mathrm{p}}\to\pi^-+h)-f(\bar{\mathrm{p}}\to\mathrm{K}^-+h)+f(\mathrm{K}^-\to\bar{\mathrm{n}}+h)-f(\mathrm{K}^-\to\mathrm{p}+h)$$
$$=f(\bar{\mathrm{p}}\to\pi^++h)-f(\bar{\mathrm{p}}\to\pi^-+h),\quad(21\mathrm{h})$$

$$f(\pi^+\to\mathrm{p}+h)-f(\mathrm{K}^+\to\mathrm{p}+h)$$
$$=f(\bar{\mathrm{p}}\to\pi^-+h)-f(\bar{\mathrm{p}}\to\mathrm{K}^-+h)$$
$$=\tfrac{1}{2}\{f(\pi^+\to\mathrm{n}+h)-f(\mathrm{K}^+\to\mathrm{n}+h)\}$$
$$=f(\bar{\mathrm{p}}\to\pi^++h)-f(\bar{\mathrm{p}}\to\pi^-+h)+f(\mathrm{K}^-\to\bar{\mathrm{p}}+h)-f(\mathrm{K}^-\to\bar{\mathrm{n}}+h),\quad(21\mathrm{i})$$

$$f(\mathrm{p}\to\pi^++h)+2f(\mathrm{p}\to\pi^-+h)+f(\bar{\mathrm{p}}\to\bar{\mathrm{n}}+h)$$
$$=f(\pi^+\to\bar{\mathrm{n}}+h)+2f(\pi^+\to\bar{\mathrm{p}}+h)+f(\mathrm{p}\to\mathrm{n}+h).\quad(21\mathrm{j})$$

The sum rules relating suppressed fragmentations are:

$$f(\pi^+\to\mathrm{K}^-+h)=f(\mathrm{K}^+\to\pi^-+h),\quad(22\mathrm{a})$$

$$f(\pi^+\to\pi^-+h)=f(\pi^-\to\pi^++h)$$
$$=f(\mathrm{K}^+\to\mathrm{K}^-+h)$$
$$=f(\mathrm{K}^-\to\mathrm{K}^++h),\quad(22\mathrm{b})$$

$$f(\pi^- \to \mathrm{K}^+ + h) = f(\mathrm{K}^- \to \pi^+ + h), \tag{22c}$$

$$\begin{aligned} f(\pi^+ \to \pi^- + h) - f(\pi^- \to \mathrm{K}^+ + h) &= f(\mathrm{p} \to \mathrm{K}^- + h) - f(\mathrm{n} \to \mathrm{K}^- + h) \\ &= f(\mathrm{p} \to \bar{\mathrm{p}} + h) - f(\mathrm{n} \to \bar{\mathrm{p}} + h), \end{aligned} \tag{22d}$$

$$\begin{aligned} f(\mathrm{K}^- \to \mathrm{p} + h) - f(\mathrm{K}^- \to \mathrm{n} + h) &= f(\pi^+ \to \pi^- + h) - f(\mathrm{K}^+ \to \pi^- + h) \\ &= f(\bar{\mathrm{p}} \to \mathrm{p} + h) - f(\bar{\mathrm{p}} \to \mathrm{n} + h), \end{aligned} \tag{22e}$$

$$\begin{aligned} & f(\mathrm{K}^+ \to \bar{\mathrm{p}} + h) + f(\mathrm{K}^- \to \mathrm{p} + h) \\ & \quad = f(\pi^+ \to \pi^- + h) + f(\pi^- \to \mathrm{K}^+ + h) + f(\mathrm{K}^+ \to \pi^- + h). \end{aligned} \tag{22f}$$

The above sum rules are obtained on the assumption of factorization between fragments of a and b, and are taken at the infinite incident energy limit of a, such that only vacuum exchange occurs between (ac) and b. Since c is analytically continued to $\bar{c}$, we have treated $\bar{c}$ as special in the three-body elastic scattering of $ab\bar{c}$, and have not taken advantage of the formal equality of the elastic scattering of $(a\bar{c})b$ and $(\bar{c}a)b$.

At the moment, there is no experimental evidence to support or reject these sum rules. Anticipating the future results, it is possible to remark:

(1) If the relations (20), (21), and (22) are correct, all the assumptions of the quark model, linearity, and the generalized optical theorem are relevant for the description of fragmentation processes. The crucial question then becomes why these relations should hold only for favoured with favoured fragmentations or allowed with allowed fragmentations but not between different fragmentations.

(2) If the ratio $f(\mathrm{p} \to \mathrm{p}+h)/f(\pi \to \pi+h)$ is 9/4 but the relations (20), (21), and (22) fail, this implies that the quark model is relevant but that the generalized optical theorem as applied by Mueller (1970), Chan *et al.* (1971), and De Tar *et al.* (1971) is doubtful or at least needs an even more drastic modification than the present one, when used with the quark model.

IV. Conclusions

It has been demonstrated that the quark model is capable of providing quantitative predictions for the fragmentation processes of inclusive reactions. The most significant prediction is that the ratio $f(\mathrm{p} \to \mathrm{p})/f(\pi \to \pi) = 9/4$. The experimental confirmation or refutation of this result should be forthcoming soon.

We have not resolved any of the traditional difficulties associated with the quark model for two-body scattering. Thus the problems of what energy and momenta must be used when comparing two different inclusive reaction processes and what is the mysterious origin of the linearity still remain.

The relations derived in Sections II and III for favoured fragmentation are not as numerous as those that have been obtained for elastic scattering. This is due to the fact that exact SU(3) symmetry can be demanded for quark–quark elastic scattering but is of no help in relating the quark + quark → quark + [u.f.] processes to one another.

If the results in Sections II and III are verified, all the finer developments of the quark model (Lipkin 1969) for two-body scattering should then be extrapolated to fragmentation processes. If indeed the ratio 9/4 of equations (13) and (14) is verified, the next step is to try and apply the more refined properties of the quark model, such as spin, spin–orbit coupling, etc. that have been successful for two-body processes to the inclusive reactions.

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Appendix

Let us consider the following exclusive process

$$a+b \rightarrow c+a'+b',$$

where

$$a' = \sum_{i=1}^{n} a_i \qquad \text{and} \qquad b' = \sum_{j=1}^{m} b_j. \tag{A1}$$

We shall assume:

(1) there is no pionization in the sense that any final particle can be traced back to be either the fragmentation of the beam or the target; and

(2) factorization holds, that is,

$$a \rightarrow a'+c \qquad \text{and} \qquad b \rightarrow b'. \tag{A2}$$

The cross section then becomes

$$2w_c\left(\frac{\mathrm{d}\sigma^{nm}}{\mathrm{d}p_c^3}\right) = \sum_{a'b'} |\langle a'c|T|a\rangle\langle b'|T|b\rangle|^2\, \rho_{a'}\rho_{b'}\,\delta^4(p_a+p_b-p_{a'}-p_{b'}-p_c), \tag{A3}$$

where the density-of-states factors $\rho_{a'}$ and $\rho_{b'}$ are defined by

$$\rho_{a'} = \prod_i \mathrm{d}^3 p_{a_i}/2w_{a_i} \qquad \text{and} \qquad \rho_{b'} = \prod_j \mathrm{d}^3 p_{b_j}/2w_{b_j}.$$

Taking out the δ function, we obtain

$$2w_c\left(\frac{\mathrm{d}\sigma^{nm}}{\mathrm{d}p_c^3}\right) = \left(\sum_{a''}^{n} |\langle a''c|T|a\rangle|^2 \rho_{a''}\right)\left(\sum_{b'}^{m} |\langle b'|T|b\rangle|^2 \rho_{b'}\right), \tag{A4}$$

where

$$\rho_{a'} = \rho_{a''}\mathrm{d}^3 p_{a_1}/2w_{a_1}$$

and

$$|\langle a''c|T|a\rangle|^2 = |\langle p_a+p_b-p_{a''}-p_{b'}-p_c, a_2\ldots a_n, c|T|a\rangle|^2\,\delta(p_{a_1}^2+m_{a_1}^2). \tag{A5}$$

The cross section for $a \to c$ only is

$$f_c = \sum_m 2w_c\left(\frac{\mathrm{d}\sigma^{0m}}{\mathrm{d}p_c^3}\right) = |\langle c|T|a\rangle|^2\left(\sum_{b'}^{m} |\langle b'|T|b\rangle|^2 \rho_{b'}\delta^4(p_a+p_b-p_{b'}-p_c)\right). \tag{A6}$$

For comparison between different fragmentations of the kind $a \to c$ with the same target b, we observe that

$$f_c \propto |\langle c|T|a\rangle|^2. \tag{A7}$$

Hence relations (9a), (9b), (10), and (12) are proved.

The cross section for the fragmentation of a into c for the process

$$a \to a' + lc \tag{A8}$$

is

$$2w_c\left(\frac{\mathrm{d}\sigma^{nm}}{\mathrm{d}p_c^3}\right) = \sum_{k=1}^{l}\frac{1}{k!}\left(\sum_{a'}^{n}|\langle a'kc|T|a\rangle|^2\rho_{a'}\rho_c\right)\left(\sum_{b'}^{m}|\langle b'|T|b\rangle|^2\rho_{b'}\right)$$
$$\times\,\delta^4(p_a+p_b-p_{a'}-p_{b'}-p_c), \tag{A9}$$

where

$$p_c = \sum_{k=1}^{l} p_{ck} \qquad \text{and} \qquad \rho_c = \prod_{k=1}^{l}\mathrm{d}^3 p_{ck}/2w_{ck}.$$

For the inclusive spectrum of c, it becomes

$$f_c = \sum_{n,m} 2w_c\left(\frac{\mathrm{d}\sigma^{nm}}{\mathrm{d}p_c^3}\right) = \left(\sum_n A_n\right)\left(\sum_m B_m\right), \tag{A10}$$

where

$$A_n = \sum_{k=1}^{l}\frac{1}{k!}\rho_c|\langle a'kc|T|a\rangle|^2\,\delta^4(p_a+p_b-p_{a'}-p_{b'}-p_c)$$

and

$$B_m = \sum_{j=1}^{m}|\langle b'|T|b\rangle|^2. \tag{A11}$$