THE EFFECT OF ROTATION ON NONLINEAR THERMAL CONVECTION

By R. VAN DER BORGHT* and J. O. MURPHY*

[Manuscript received 17 November 1972]

Abstract

The effect of rotation on nonlinear thermal convection is investigated, in particular at high Rayleigh number. The Boussinesq approximation is adopted in the basic equations and the free boundary conditions are applied. The results derived from asymptotic and perturbation methods are found to be in very good agreement with those obtained by numerical integration.

I. INTRODUCTION

Although the problem of thermal convection in a rotating fluid has not been as thoroughly investigated as the non-rotating case, the corresponding linear stability theory has been considered by a number of authors and an outline of the results is given by Chandrasekhar (1961). Finite amplitude convection in a rotating medium has been the subject of detailed experiments by Rossby (1969), while theoretical investigations have been undertaken by Veronis (1959, 1966, 1968) and Somerville (1971).

An important physical quantity for any comparison between theoretical and experimental results is the Nusselt number, which gives a measure of the total convective and conductive heat flux. In astrophysical applications the Rayleigh number is estimated to be large (Spiegel 1971), and the main aim of the present investigation has been to derive expressions for the Nusselt number as a function of the Rayleigh and Taylor numbers by asymptotic and perturbation methods. The results have then been compared with those derived by numerical integration and very good agreement has been found for high values of the Rayleigh number.

In this paper we consider only rolls and convective cells with square or rectangular planforms. The Boussinesq approximation is adopted and the analysis is carried out for the case of free boundaries, which is the most appropriate condition for astrophysical applications.

II. BASIC EQUATIONS

The basic equations of the problem are derived from a variational principle first introduced by Prigogine and Glansdorff (1964, 1965). This principle states that the actual flow evolves in such a way as to keep the generalized entropy production a minimum with respect to arbitrary variations from it. A modified form, suitable

* Department of Mathematics, Monash University, Clayton, Vic. 3168.

for finding stationary solutions for an incompressible fluid, can be written as (Unno 1969)

$$\delta\phi = \iiint \mathrm{d}x\,\mathrm{d}y\,\mathrm{d}z\,\{\delta\boldsymbol{u}.(\rho\boldsymbol{u}.\nabla\boldsymbol{u}+\nabla p-\rho\boldsymbol{F}-\mu\nabla^2\boldsymbol{u})$$
$$+T^{-1}\delta T(\rho C_{\mathrm{v}}\boldsymbol{u}.\nabla T-K\nabla^2 T)\}\,, \tag{1}$$

where the body force $\boldsymbol{F}$ is given by

$$\boldsymbol{F} = -\nabla\phi-\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r})-2(\boldsymbol{\Omega}\times\boldsymbol{u})-\dot{\boldsymbol{\Omega}}\times\boldsymbol{r} \tag{2}$$

and μ, K, and C_{v} denote respectively the viscosity, conductivity, and specific heat at constant volume of the fluid.

We consider here the case of steady rotation about the z axis, i.e.

$$\boldsymbol{\Omega} = (0,0,\Omega_0)\,,$$

and follow the usual approach of neglecting density variations except in the buoyancy terms. As we have therefore neglected the density variation coupled with the centrifugal acceleration, the following analysis is only valid provided that the Froude number

$$F_{\mathrm{R}} = \Omega_0^2 L/g$$

is small, where L is a characteristic length scale of the fluid (e.g. the thickness of the convective layer; Greenspan 1968) and g is the acceleration due to gravity.

It is convenient to adopt the following expressions for the velocity $\boldsymbol{u}$, temperature T, and density ρ (Chandrasekhar 1961; Roberts 1966)

$$\boldsymbol{u} = \boldsymbol{u}(u,v,w) = \left(\frac{\mathrm{D}W}{k^2}\frac{\partial f}{\partial x}+\frac{Z}{k^2}\frac{\partial f}{\partial y},\ \frac{\mathrm{D}W}{k^2}\frac{\partial f}{\partial y}-\frac{Z}{k^2}\frac{\partial f}{\partial x},\ Wf\right), \tag{3}$$

$$T = T_0+Ff, \tag{4}$$

$$\rho = \rho_0+Pf, \tag{5}$$

where W, Z, T_0, F, and P, functions of the vertical coordinate z, are to be determined. Here D denotes differentiation with respect to z and $f=f(x,y)$ satisfies the differential equation

$$\partial^2 f/\partial x^2+\partial^2 f/\partial y^2 = -k^2 f, \tag{6}$$

k being the horizontal wave number denoting the size of the convective cells. Equation (6) admits a number of solutions corresponding to various planforms of the convective cells (Roberts 1966) and, further,

$$\zeta = \partial v/\partial x-\partial u/\partial y = Zf$$

is the z component of the vorticity.

We use here the notation

$$\langle m\rangle \equiv A\iint (m)\,\mathrm{d}x\,\mathrm{d}y \tag{7}$$

to denote the horizontal average of a quantity m, the constant A being determined by the normalizing condition

$$\langle f^2\rangle \equiv A\iint f^2\,\mathrm{d}x\,\mathrm{d}y = 1. \tag{8}$$

It can be shown that the average $C = \frac{1}{2}\langle f^3\rangle = 0$ for rolls or square and rectangular convective cells, whereas for hexagonal cells $C = 1/\sqrt{6}$. In this paper we shall only consider the case where $C = 0$.

If we intoduce the notation

$$\boldsymbol{M} = \sum_{i=1}^{6} \boldsymbol{M}^i, \tag{9}$$

where

$$\left.\begin{aligned} &\boldsymbol{M}^1 = \rho_0\,\boldsymbol{u}.\nabla\boldsymbol{u}, && \boldsymbol{M}^2 = \nabla p, && \boldsymbol{M}^3 = \rho\,\nabla\phi,\\ &\boldsymbol{M}^4 = \rho_0\,\boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\boldsymbol{r}), && \boldsymbol{M}^5 = 2\rho_0(\boldsymbol{\Omega}\times\boldsymbol{u}), && \boldsymbol{M}^6 = -\mu\,\nabla^2\boldsymbol{u}, \end{aligned}\right\} \tag{10}$$

and use expressions (3) and (4) for the velocity and temperature, it is easily seen that the Euler–Lagrange equations corresponding to the Prigogine and Glansdorff (1964, 1965) variational principle (1) are:

$$\langle \rho_0\,C_\mathrm{v}\,\boldsymbol{u}.\nabla T - K\nabla^2 T\rangle = 0, \tag{11}$$

$$\langle f(\rho_0\,C_\mathrm{v}\,\boldsymbol{u}.\nabla T - K\nabla^2 T)\rangle = 0, \tag{12}$$

$$-k^{-2}\,\mathrm{D}\left\langle\frac{\partial f}{\partial x}M_x + \frac{\partial f}{\partial y}M_y\right\rangle + \langle fM_z\rangle = 0, \tag{13}$$

$$\left\langle\frac{\partial f}{\partial y}M_x - \frac{\partial f}{\partial x}M_y\right\rangle = 0. \tag{14}$$

Making use of the average

$$\langle(\partial f/\partial x)^2 + (\partial f/\partial y)^2\rangle = k^2 \tag{15}$$

and introducing the notation

$$O(\boldsymbol{M}^i) = -k^{-2}\,\mathrm{D}\left\langle\frac{\partial f}{\partial x}M_x^i + \frac{\partial f}{\partial y}M_y^i\right\rangle + \langle fM_z^i\rangle, \tag{16}$$

it can be shown that

$$\left.\begin{aligned} &O(\boldsymbol{M}^1) = 0, && O(\boldsymbol{M}^2) = 0, && O(\boldsymbol{M}^3) = gk^2P,\\ &O(\boldsymbol{M}^4) = 0, && O(\boldsymbol{M}^5) = -2\rho_0\,\Omega_0\,\mathrm{D}Z, && O(\boldsymbol{M}^6) = \mu(\mathrm{D}^2 - k^2)^2 W. \end{aligned}\right\} \tag{17}$$

Substitution of all these expressions in equation (13) then gives

$$\mu(\mathrm{D}^2 - k^2)^2 W = -gk^2 P + 2\rho_0\,\Omega_0\,\mathrm{D}Z. \tag{18}$$

Proceeding in a similar way with equations (11), (12), and (14) we also obtain

$$\rho_0 C_v \mathrm{D}(FW) = K\mathrm{D}^2 T_0, \tag{19}$$

$$K(\mathrm{D}^2 - k^2)F = \rho_0 C_v W \mathrm{D} T_0, \tag{20}$$

$$\mu(D^2 - k^2)Z = -2\rho_0 \Omega_0 \mathrm{D} W. \tag{21}$$

For an incompressible fluid the equation of state is

$$\rho = \rho_0 + \alpha\rho_0(T_0 - T) \tag{22}$$

or, on introducing equations (4) and (5),

$$Pf = -\alpha\rho_0 Ff. \tag{23}$$

After multiplying throughout by f and taking the horizontal average,

$$P = -\alpha\rho_0 F, \tag{24}$$

and substitution of this value of P in equation (18) then gives

$$\mu(\mathrm{D}^2 - k^2)^2 W = \alpha\rho_0 g k^2 F + 2\rho_0 \Omega_0 \mathrm{D} Z. \tag{25}$$

The dimensionless forms of equations (19), (20), (21), and (25) are obtained by making the substitutions

$$\left.\begin{array}{lll} \mathrm{D} \to \mathrm{D}/d, & k^2 \to a^2/d^2, & T_0 \to T_0(\Delta T), \\ W \to (\kappa/d)W, & Z \to (\kappa/d^2)\tau^{\frac{1}{2}} Z, & F \to F(\Delta T), \end{array}\right\} \tag{26}$$

where the thermal diffusivity κ and Taylor number τ are given by

$$\kappa = K/\rho_0 C_v \qquad \text{and} \qquad \tau = 4d^4\Omega_0^2/\nu^2, \tag{27}$$

a is a dimensionless wave number, and ΔT is the temperature difference across the layer of thickness d. On introducing the Rayleigh number

$$R = g\alpha \Delta T d^3/\nu\kappa, \tag{28}$$

where $\nu = \mu/\rho_0$ is the kinematic viscosity, the basic equations of the problem can be written

$$(\mathrm{D}^2 - a^2)^2 W = Ra^2 F + \tau \mathrm{D} Z, \tag{29}$$

$$(\mathrm{D}^2 - a^2)F = W \mathrm{D} T_0, \tag{30}$$

$$\mathrm{D}^2 T_0 = \mathrm{D}(FW), \tag{31}$$

$$(\mathrm{D}^2 - a^2)Z = -\mathrm{D} W. \tag{32}$$

In the case of free boundaries, which we consider here, the basic equations have to be solved subject to the boundary conditions

$$W = \mathrm{D}^2 W = F = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1\,, \tag{33}$$

$$T_0 = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad T_0 = -1 \quad \text{at} \quad z = 1\,. \tag{34}$$

In addition the following boundary conditions on the normal component of the vorticity have to be satisfied (Chandrasekhar 1961)

$$\mathrm{D}Z = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad z = 1\,. \tag{35}$$

Equations (29)–(32) have been solved, subject to the conditions (33)–(35), for a number of values of the Rayleigh and Taylor numbers using asymptotic, perturbation, and numerical methods. An outline of these methods together with the results obtained are given in the following sections.

III. Asymptotic Method

Following the method outlined by Van der Borght *et al.* (1972) it can be shown that in the main body of the fluid layer, and for large Rayleigh number with a Taylor number of order unity, the differential equation that should be satisfied is

$$W(\mathrm{D}^2 - a^2)^2 W = Ra^2 N + \tau W \mathrm{D}Z \tag{36}$$

and, introducing the variables ψ and χ defined by

$$W = (RaN)^{\frac{1}{2}}\psi\,, \qquad Z = (RaN)^{\frac{1}{2}}\chi\,, \tag{37a, b}$$

this equation can be written as

$$\psi(\mathrm{D}^2 - a^2)^2\psi = 1 + \tau\psi\,\mathrm{D}\chi\,. \tag{38}$$

In addition, using the same variables, equation (32) takes the form

$$(\mathrm{D}^2 - a^2)\chi = -\mathrm{D}\psi\,. \tag{39}$$

Adopting

$$\psi = \sum_{k=1}^{\infty} A_k \sin(k\pi z)\,, \qquad \chi = \sum_{k=1}^{\infty} B_k \cos(k\pi z)\,, \tag{40a, b}$$

and noting that these expressions satisfy the free boundary conditions

$$\psi = \mathrm{D}^2\psi = 0\,, \quad \mathrm{D}\chi = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad 1\,, \tag{41}$$

we obtain on substituting the forms (40) into equation (39)

$$B_k = \{k\pi/(k^2\pi^2 + a^2)\}A_k\,, \qquad k = 1, 2, 3 \dots. \tag{42}$$

Equation (38) can also be written as

$$(\mathrm{D}^2-a^2)^2\psi = \psi^{-1} + \tau \mathrm{D}\chi, \tag{43}$$

and multiplying this equation by $2\sin(k\pi z)$, integrating over z from 0 to 1, making use of equations (42), and retaining only the first two terms in the expansion for ψ we finally get

$$k = 1, \qquad \left((\pi^2+a^2)^2 + \frac{\pi^2\tau}{\pi^2+a^2}\right)A_1 = \frac{2}{A_1}\left(1-\frac{A_3}{A_1}\right); \tag{44a}$$

$$k = 3, \qquad \left((9\pi^2+a^2)^2 + \frac{9\pi^2\tau}{9\pi^2+a^2}\right)A_3 = \frac{2}{A_1}\left(1-\frac{3A_3}{A_1}\right). \tag{44b}$$

These equations can now be solved explicitly for A_1 and A_3.

If in the boundary layer

$$\psi = k_0 z \tag{45}$$

then it follows from equation (40a) that

$$k_0 = \pi A_1(1+3A_3/A_1), \tag{46}$$

and substitution of the values of A_1 and A_3 obtained from equations (44) gives

$$k_0 \approx \frac{\pi\sqrt{2}}{\{(\pi^2+a^2)^2 + \pi^2\tau/(\pi^2+a^2)\}^{\frac{1}{2}}}\left(1 + \frac{5}{2}\,\frac{(\pi^2+a^2)^2 + \pi^2\tau/(\pi^2+a^2)}{(9\pi^2+a^2)^2 + 9\pi^2\tau/(9\pi^2+a^2)}\right). \tag{47}$$

As shown by Howard (1965), the Nusselt number is given by

$$N = (2K)^{-4/3}(k_0 a)^{2/3} R^{1/3}, \tag{48}$$

where

$$K = \Gamma(\tfrac{3}{4})^2/\sqrt{2} = 1{\cdot}062. \tag{49}$$

Therefore, in this case

$$N \approx \left(\frac{1}{2{\cdot}124}\right)^{4/3}\left(\frac{2\pi^2 a^2}{(\pi^2+a^2)^2 + \pi^2\tau/(\pi^2+a^2)}\right)^{1/3} \times \left(1 + \frac{5}{3}\,\frac{(\pi^2+a^2)^2 + \pi^2\tau/(\pi^2+a^2)}{(9\pi^2+a^2)^2 + 9\pi^2\tau/(9\pi^2+a^2)}\right)R^{1/3}. \tag{50}$$

A comparison has been made between the values of the Nusselt number obtained from the asymptotic formula (50) and those obtained by direct numerical integration. It can be seen from the results presented in Figure 2 below that the agreement is excellent for large values of the Rayleigh number, as long as the Taylor number is not too large.

IV. Perturbation Method

The asymptotic values of N in Section III were obtained under the assumption that the Taylor number was of order unity. We now consider the nonlinear case where both R and τ are large.

The linear theory predicts that the value of the critical Rayleigh number for the fundamental mode solution will be given by

$$R = \{(\pi^2+a^2)^3+\pi^2\tau\}/a^2\,, \tag{51}$$

and for large τ this reduces to

$$R \approx (\pi^2/a^2)\tau\,. \tag{52}$$

Elimination of T_0 and F from the basic equations (29)–(32) leads to the following differential equation in W and Z

$$(\mathrm{D}^2-a^2)^3 W = W^2\{(\mathrm{D}^2-a^2)^2 W-\tau\,\mathrm{D}Z\}-Ra^2NW-\tau\,\mathrm{D}^2 W. \tag{53}$$

Now let

$$R = \lambda\tau \tag{54}$$

and assume that τ is large, in which case equation (53) reduces to

$$\mathrm{D}^2 W+W^2\,\mathrm{D}Z+\gamma W = 0\,, \tag{55}$$

where $\gamma = \lambda a^2 N$.

If we introduce the expansions

$$Z = \sum_{k=1,3} \zeta_k\cos(k\pi z)\,, \qquad W = \sum_{k=1,3} w_k\sin(k\pi z) \tag{56a, b}$$

into equation (32), we find

$$\zeta_k = \{k\pi/(k^2\pi^2+a^2)\}w_k\,, \qquad k = 1,3\,. \tag{57}$$

After multiplication of equation (55) by $\sin(k\pi z)$ and subsequent integration with respect to z from 0 to 1, making use of the result (57), it can be shown that w_3/w_1 satisfies the second-degree equation

$$\left(\gamma-\pi^2\right)\left\{-1+2\left(2+\frac{9(\pi^2+a^2)}{9\pi^2+a^2}\right)\frac{w_3}{w_1}\right\}$$

$$= 3\left(\gamma-9\pi^2\right)\frac{w_3}{w_1}-\left(\gamma-9\pi^2\right)\left(2+\frac{9(\pi^2+a^2)}{9\pi^2+a^2}\right)\left(\frac{w_3}{w_1}\right)^2 \tag{58}$$

and w_1 is then given by

$$\frac{4(\pi^2+a^2)}{\pi^2}\left(\gamma-\pi^2\right) = w_1^2\left\{3-\left(2+\frac{9(\pi^2+a^2)}{9\pi^2+a^2}\right)\frac{w_3}{w_1}\right\}. \tag{59}$$

On introducing the notation

$$\phi = F/N, \tag{60}$$

it follows from equations (30) and (31) that

$$\mathrm{D}^2\phi = a^2\phi + W^2\phi - W, \tag{61}$$

and, adopting an expansion for ϕ such that

$$\phi = \phi_1 \sin(\pi z) + \phi_3 \sin(3\pi z) \tag{62}$$

and then proceeding as before, the following expressions are obtained for ϕ_3/ϕ_1 and ϕ_1 in terms of w_3/w_1 and w_1

$$\frac{\phi_3}{\phi_1} = \frac{w_1^2(1-4w_3/w_1)+(w_3/w_1)\{4(\pi^2+a^2)+w_1^2(3-2w_3/w_1)\}}{4(9\pi^2+a^2)+2w_1^2+w_1^2(w_3/w_1)} \tag{63}$$

and

$$\phi_1 = \frac{w_1}{(\pi^2+a^2)+\frac{1}{4}w_1^2\{3-(\phi_3/\phi_1+2w_3/w_1)\}}. \tag{64}$$

The Nusselt number is then given by

$$N = \{1 - \tfrac{1}{2}\phi_1 w_1(1 + \phi_3 w_3/\phi_1 w_1)\}^{-1}. \tag{65}$$

A comparison of the values of N obtained from the formula (65), when $a = \pi$, with those obtained by numerical methods is included in Figure 2(*a*) and it can again be seen that the agreement is excellent for large values of R and τ. In fact, at large Rayleigh number the analytical results obtained from Sections III and IV, when used in conjunction, give quite good results for the Nusselt number over the whole range of values of the Taylor number that support convection.

V. Numerical Method and Results

Numerical procedures described by Van der Borght *et al.* (1972) have been used here to obtain solutions of the equations which now incorporate the effect of rotation on thermal convection. It is convenient to first eliminate the vertical component of the vorticity $Z(z)$ from equation (29), using (32), and then introduce the following Fourier representations of the variables into the resulting equation and also into equations (30) and (31):

$$W = \sum_{n=1}^{M} W_n \sin\{(2n-1)\pi z\}, \tag{66a}$$

$$F = \sum_{n=1}^{M} f_n \sin\{(2n-1)\pi z\}, \tag{66b}$$

$$T_0 = \sum_{n=1}^{M} t_n \sin(2n\pi z) - z. \tag{66c}$$

Expansions including only sine terms are appropriate when considering the free boundary conditions and the symmetry of the solutions about $z = \frac{1}{2}$ requires only the even or odd modes to be retained. The unknown coefficients in the expansions (66) are determined, using the generalized Newton–Raphson method, from the nonlinear system of equations

$$\left(\{(2n-1)^2+\alpha^2\}+\frac{\tau(2n-1)^2}{\pi^4\{(2n-1)^2+\alpha^2\}^2}\right)w_n = \left(\frac{R_1\alpha^2}{\{(2n-1)^2+\alpha^2\}}\right)f_n, \tag{67}$$

$$\{(2n-1)^2+\alpha^2\}f_n = w_n - \pi \sum_{p=1}^{M} pt_p\{w_{n+p}+Y(2n-1-2p)\,w_{\frac{1}{2}|2n-1-2p|+\frac{1}{2}}\}, \tag{68}$$

$$nt_n = \tfrac{1}{4}\pi \sum_{p=1}^{M} w_p\{f_{n+p}+Y(2p-1-2n)\,f_{\frac{1}{2}|2n-2p+1|+\frac{1}{2}}\}, \tag{69}$$

where

$$R_1 = R/\pi^4, \qquad w_n = W_n/\pi^2, \qquad \alpha = a/\pi,$$

and

$$\begin{aligned} Y(x) &= -1 \quad \text{for} \quad x<0,\\ &= \ \ 0 \qquad\qquad x=0,\\ &= \ \ 1 \qquad\qquad x>0. \end{aligned}$$

Solutions have been obtained for values of R up to 10^7, with M ranging between 30 at $R = 10^4$ and 90 at $R = 10^7$ to ensure the constancy of the Nusselt number N, over a wide range of values of a and τ such that

$$\gamma = \{(\pi^2+a^2)^3+\pi^2\tau\}/a^2R \leqslant 1. \tag{70}$$

Following the substitutions (26), the vertical component ζ of the vorticity is now related to $Z(z)$ by

$$\zeta = \tau^{\frac{1}{2}}Zf, \tag{71}$$

where

$$Z(z) = \sum_{n=1}^{M}\left(\frac{(2n-1)W_n}{\pi\{(2n-1)^2+\alpha^2\}}\right)\cos\{(2n-1)\pi z\} \tag{72}$$

in the case of free boundaries.

Figures 1(a)–1(d), which are of particular physical interest, illustrate the dependence of the Nusselt number on the wave number for increasing values of the Taylor number when the Rayleigh number varies from $R = 10^4$ to 10^7. The inhibiting effect of rotation on the convective processes is clearly illustrated, with the maximum value of N now occurring at a higher wave number or smaller cell size. It may also be concluded that, at high Rayleigh number, values of the Taylor number up to $\tau \approx R^{\frac{1}{2}}$ have little effect on the maximum heat transfer but this maximum is shifted to higher wave number. On comparing Figure 1(d), for example, with the corresponding figure in the magnetic case (Van der Borght *et al.* 1972, Fig. 1(c)), it is seen that qualitatively Q and τ act in the same way.

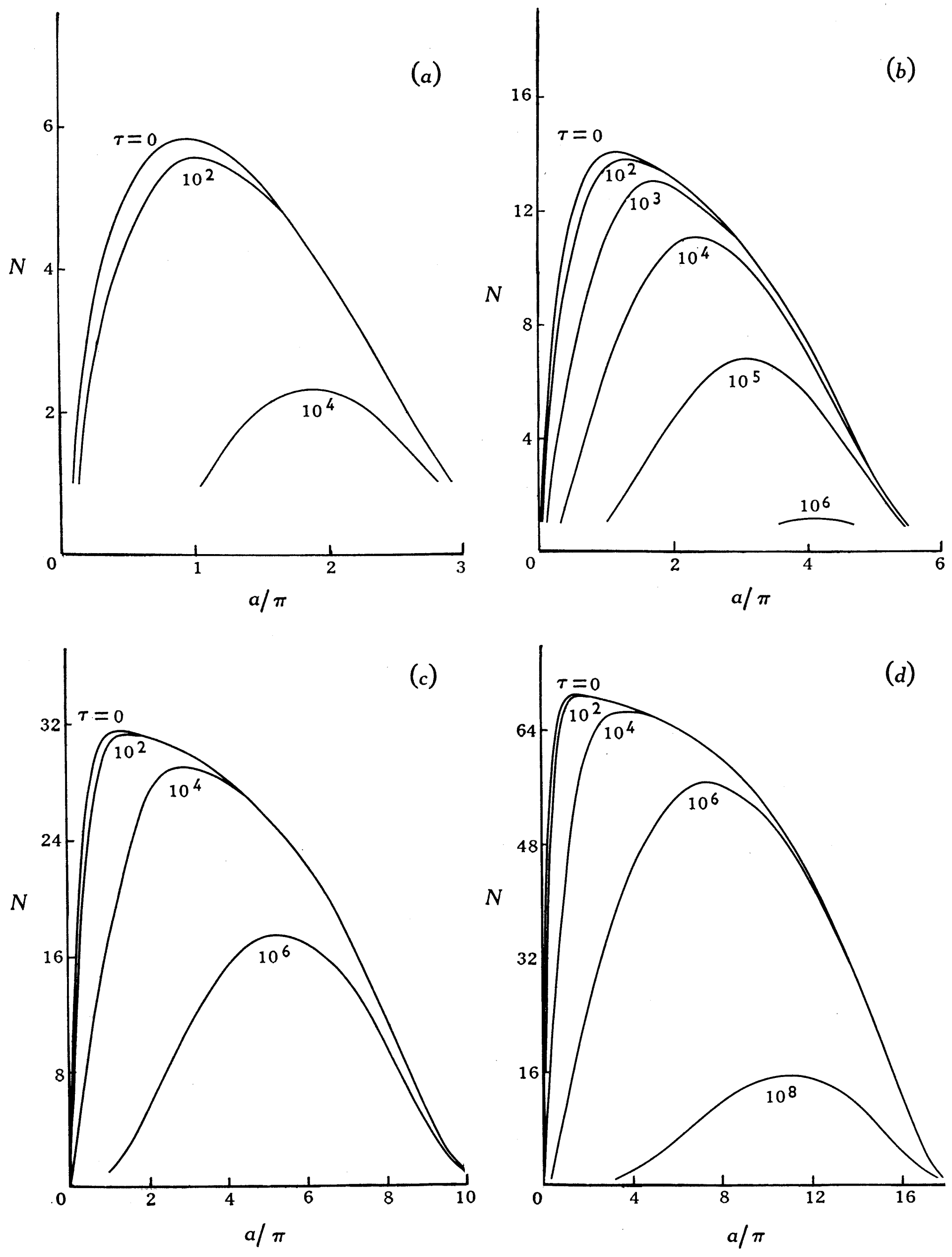

Fig. 1.—Numerical results showing the dependence of the Nusselt number N on a/π for the indicated values of the Taylor number τ when the Rayleigh number R is equal to (*a*) 10^4, (*b*) 10^5, (*c*) 10^6, and (*d*) 10^7.

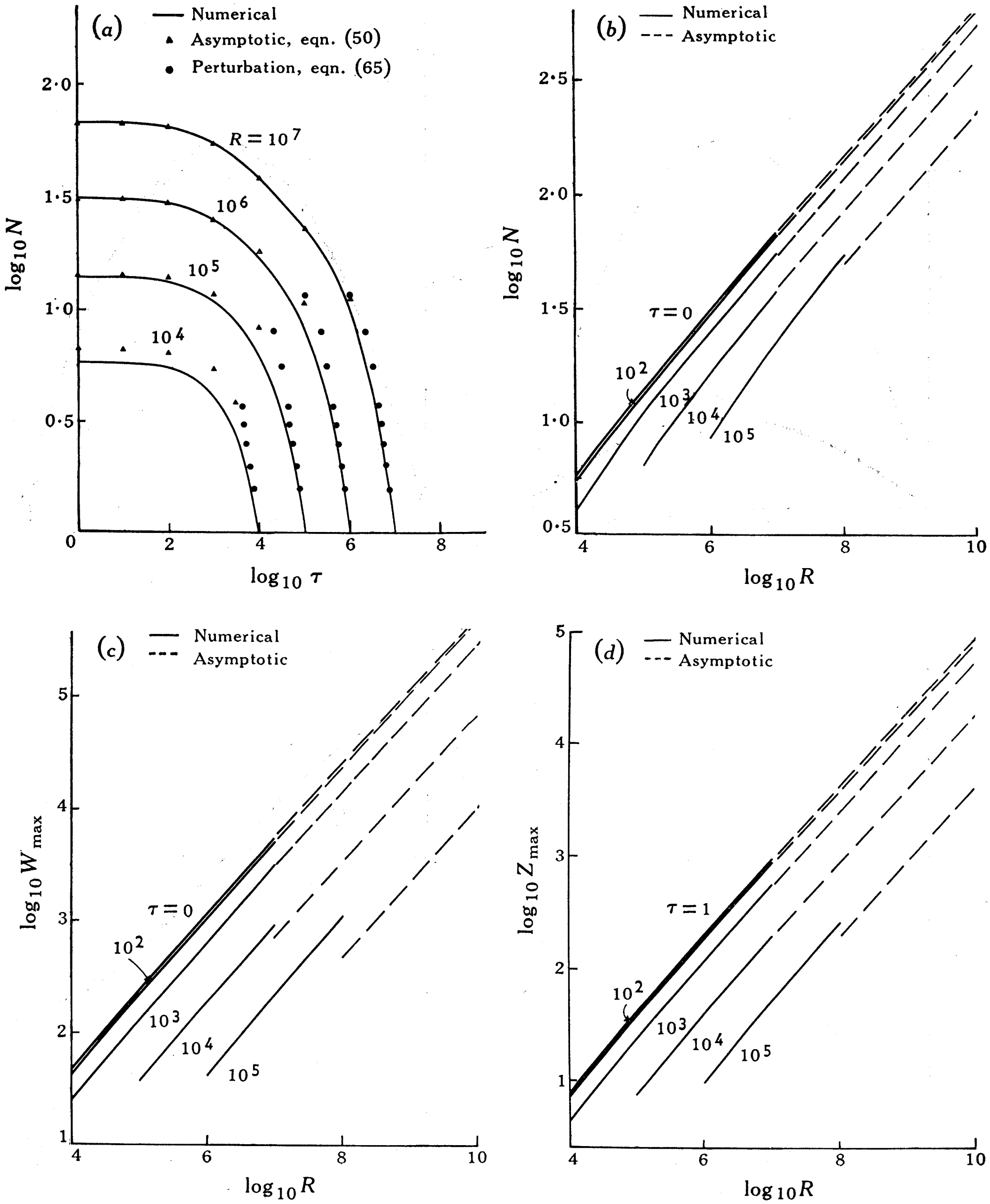

Fig. 2.—Comparison of analytical and numerical results for $a = \pi$: (*a*) N as a function of τ for four values of R, (*b*) N as a function of R for increasing τ, (*c*) the maximum value of the vertical velocity $W(z)$ as a function of R for increasing τ, and (*d*) the maximum value of the vorticity $Z(z)$ as a function of R for increasing τ.

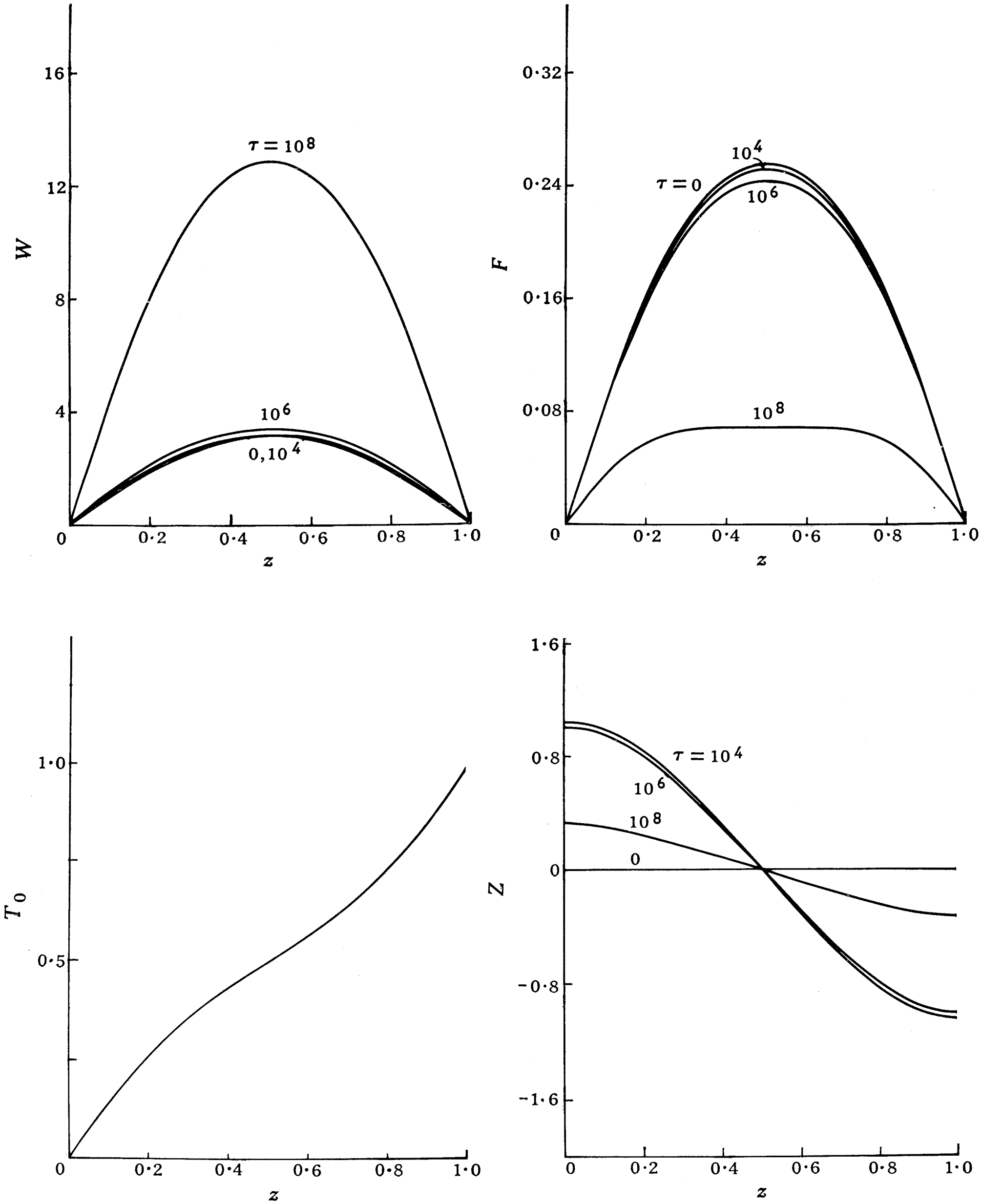

Fig. 3(*a*). $R = 10^7$, $\gamma = 0 \cdot 80$ ($a \ll \pi$).

Figs. 3(*a*)–3(*d*).—Numerical results for W, F, T_0, and Z across the fluid layer, $0 \leqslant z \leqslant 1$, when $R = 10^7$ for the indicated values of τ. The solutions can be broadly grouped into four types: (*a*) γ, as defined by equation (70), $= 0 \cdot 80$ ($a \ll \pi$), (*b*) $a = \pi$, (*c*) $a = 8\pi$, (*d*) $\gamma = 0 \cdot 85$ ($a \gg \pi$).

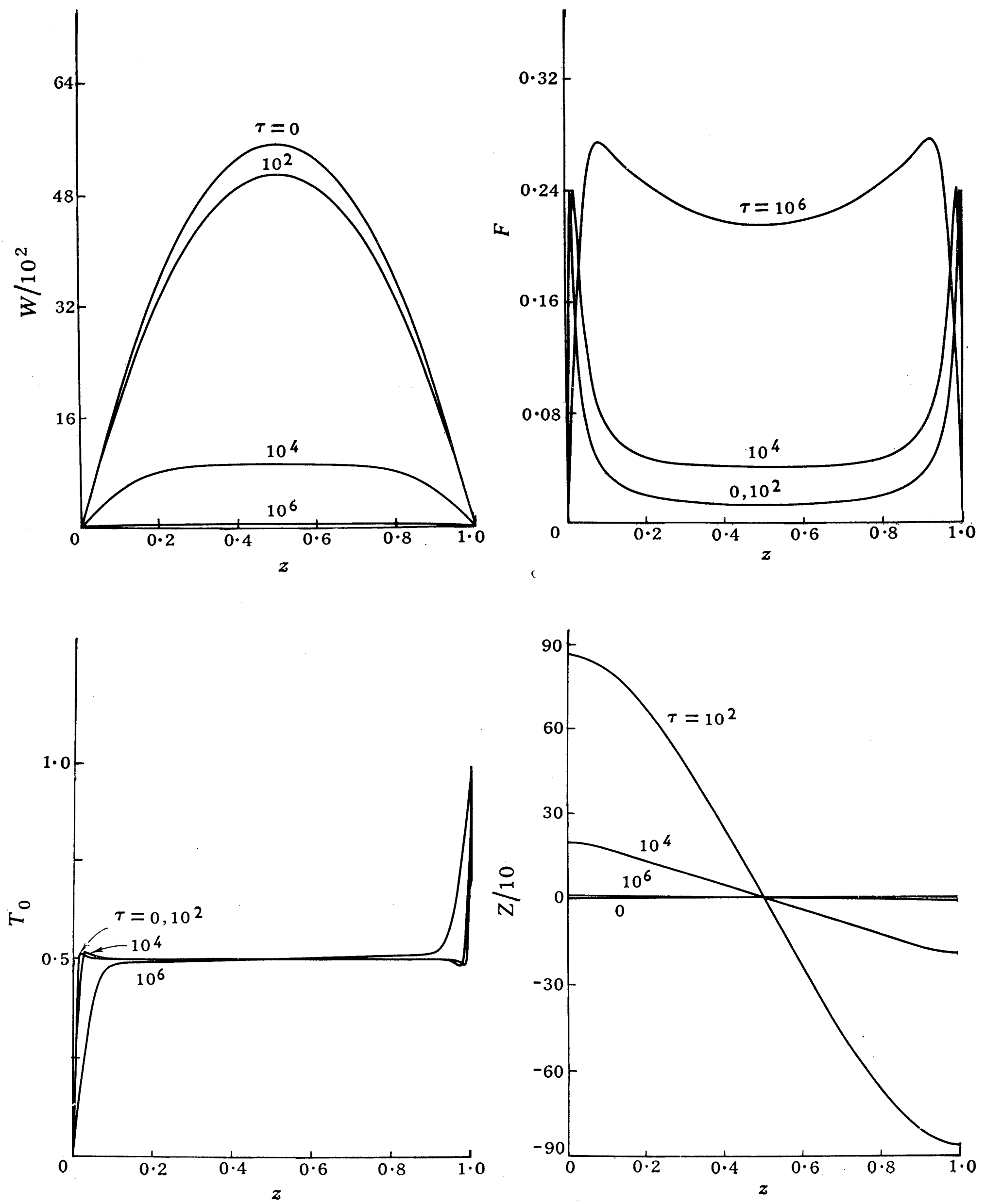

Fig. 3(*b*). $R = 10^7, a = \pi$.

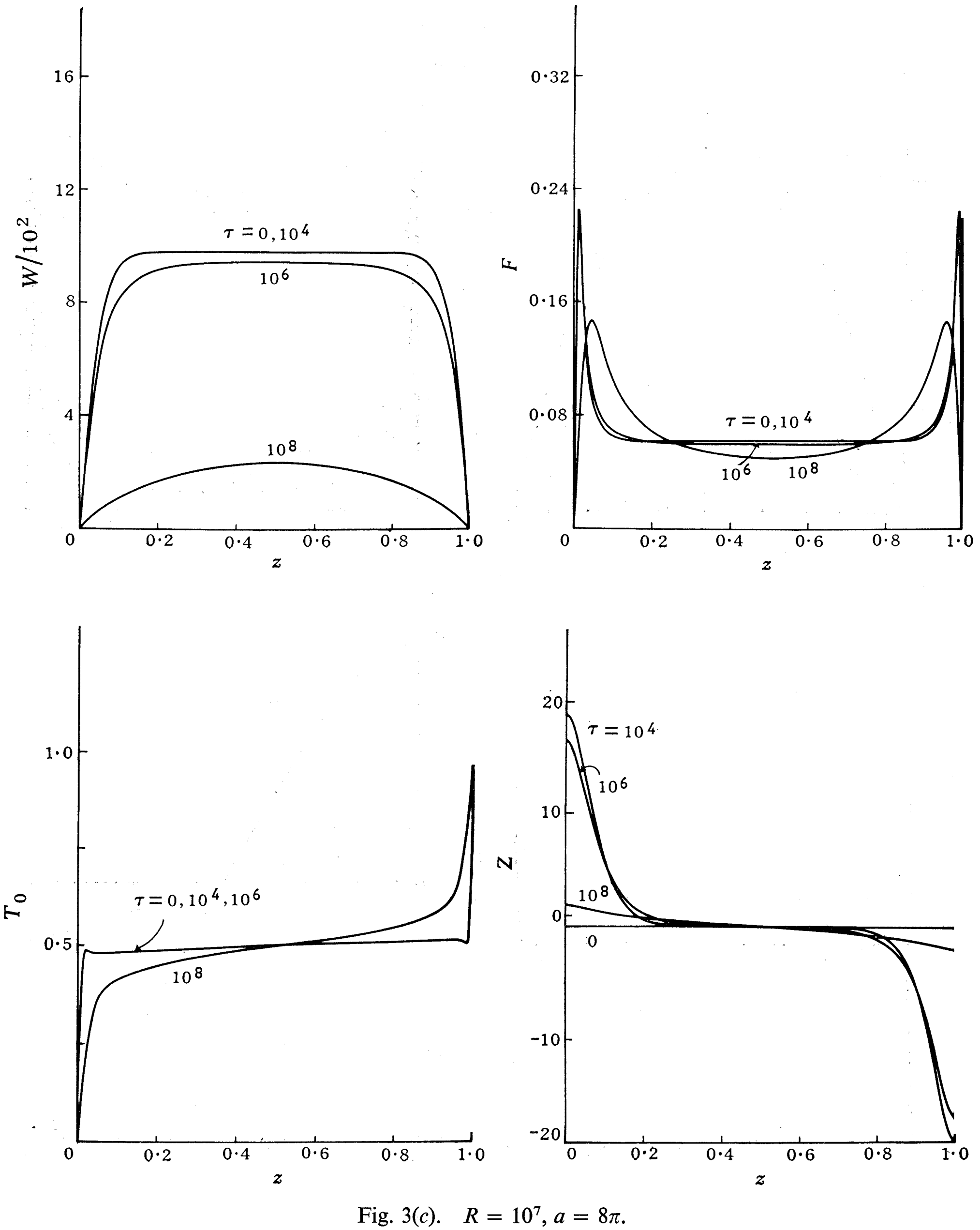

Fig. 3(*c*). $R = 10^7, a = 8\pi$.

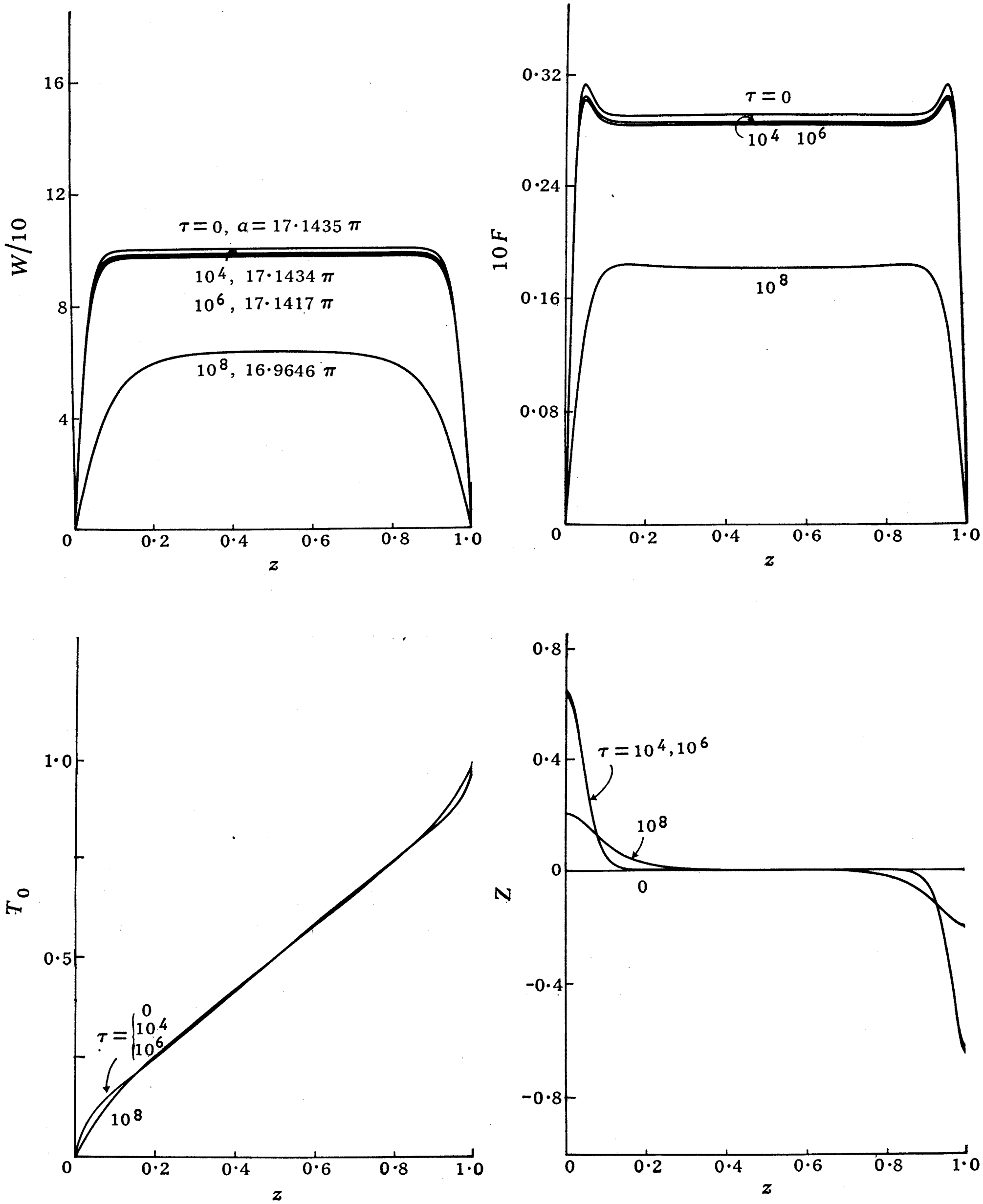

Fig. 3(*d*). $R = 10^7$, $\gamma = 0{\cdot}85$ ($a \gg \pi$).

Further quantitative evidence of the inhibiting effect of rotation, obtained from the numerical solutions, is given in Figure 2(*a*), which shows the $\log N$ versus $\log \tau$ dependence for $a = \pi$ and four values of R. Also included in this diagram are the asymptotic and perturbation results obtained from the solutions in Sections III and IV. At $R = 10^7$ it is seen that the agreement is very good and, in the case of the solutions from equation (50) in Section III, is over a surprisingly large range of τ. Further comparisons between the asymptotic and numerical solutions are given in Figures 2(*b*)–2(*d*), where agreement with varying τ is again very good.

The characteristics of the detailed solutions obtained for W, F, T_0, and Z, with $0 \leqslant z \leqslant 1$, are represented in Figures 3(*a*)–3(*d*) for $R = 10^7$, varying values of τ, and a specified value of a or γ. At high Rayleigh number and on the basis of wave-number dependence, the solutions obtained can be broadly grouped into four types, as illustrated for $R = 10^7$: (i) In Figure 3(*a*), where $\gamma = 0 \cdot 80$ and $a \ll \pi$, a good representation of the solution is given by retaining only two terms in the expansions (66). (ii) The wide isothermal region in Figure 3(*b*), i.e. constant T_0, together with small thermal boundary layers characterize the $a = \pi$ type solutions. (iii) When $a = 8\pi$ the greatest changes in T_0 are still in the boundary layers, but now a small mean temperature gradient exists over the central region and, in addition, the W solutions in Figure 3(*c*) no longer have the sine type profile. (iv) For $\gamma = 0 \cdot 85$ and $a \gg \pi$, the magnitude of the vertical velocity is considerably reduced and the W solution has a flat profile corresponding to a near constant convective vertical velocity over most of the layer (Fig. 3(*d*)), a property observed in the magnetic case; moreover, the T_0 curve closely approximates the conductive profile and Z is zero over most of the layer except for sharp changes near the free boundaries. Another group of solutions, which arises for $\gamma \approx 1$, $a \gg \pi$, and R large is well represented also by only two terms in the expansions (66) and is similar to type (i) above.

VI. Conclusions

It can be seen from Figure 1 that the introduction of rotation has a marked inhibiting effect on thermal convection and that maximum heat transfer occurs at a smaller cell size than before. There is no indication in this work of the non-monotonicity of the Nusselt number as observed by Rossby (1969) at low values of the Rayleigh number. Somerville (1971) also could not detect this effect in his theoretical results unless he used the observed horizontal wave number. This discrepancy between the observed and theoretical values for the critical wave numbers has still to be completely explained.

Veronis (1968) has indicated that the effects of rotation, in fluids with large Prandtl number, can be directly observed by noting the changes in the general features of the temperature field as τ is increased. If we take $f = \sqrt{2}\cos(ax)$, corresponding to rolls in the y direction, then we have a qualitative comparison with the two-dimensional studies of Veronis (1968) for free boundaries and Somerville (1971) for rigid boundaries. Without rotation, the temperature field for rolls at low Rayleigh number exhibits the same anvil-like structure observed by Veronis, although not as pronounced, resulting from warm fluid being convected upwards and spreading horizontally at the top of the layer and cold descending fluid also spreading horizontally at the bottom of the layer. With increasing τ the isotherm diagrams show that this spreading is gradually contained. Figure 4 gives the vertical isotherm diagram for rolls for the stated values of the parameters.

Overall the present work has shown that excellent agreement exists between the values of the Nusselt number as predicted by the asymptotic or perturbation theories and the computed values. As indicated in the Introduction, the Rayleigh number for astrophysical applications will be large and it is unlikely that the Boussinesq approximation will hold in these problems. Alternatively, the full compressibility equations (Van der Borght 1971) would have to be solved and this is likely to be a very difficult task. Asymptotic methods will again prove very useful and the present work shows that the predictions of such a theory are very good indeed and can be accepted with confidence.

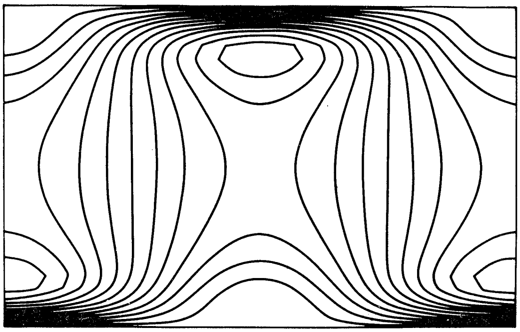

Fig. 4.—Isotherms calculated for rolls in the y direction for the parameter values: $R = 10^4$, $a = \pi$, and $\tau = 10^2$.

VII. References

CHANDRASEKHAR, S. (1961).—"Hydrodynamic and Hydromagnetic Stability." (Clarendon Press: Oxford.)

GREENSPAN, H. P. (1968).—"The Theory of Rotating Fluids." (Cambridge Univ. Press.)

HOWARD, L. N. (1965).—Notes from Summer Study Program in Geophysical Fluid Dynamics, Woods Hole Oceanographic Inst., 65–51, Vol. 1, p. 125.

PRIGOGINE, I., and GLANSDORFF, P. (1964).—*Physica, 's Grav.* **30**, 351.

PRIGOGINE, I., and GLANSDORFF, P. (1965).—*Physica, 's Grav.* **31**, 1242.

ROBERTS, P. H. (1966).—*In* "Non-equilibrium Thermodynamics, Variational Techniques and Stability". (Eds. R. J. Donnelly, R. Herman, and I. Prigogine.) p. 125. (Chicago Univ. Press.)

ROSSBY, H. T. (1969).—*J. Fluid Mech.* **36**, 309.

SOMERVILLE, R. C. J. (1971).—*Geophys. Fluid Dyn.* **2**, 247.

SPIEGEL, E. A. (1971).—*A. Rev. Astr. Astrophys.* **9**, 323.

UNNO, W. (1969).—*Publs astr. Soc. Japan* **21**, 240.

VAN DER BORGHT, R. (1971).—*Publs astr. Soc. Japan* **23**, 539.

VAN DER BORGHT, R., MURPHY, J. O., and SPIEGEL, E. A. (1972).—*Aust. J. Phys.* **25**, 703.

VERONIS, G. (1959).—*J. Fluid Mech.* **5**, 401.

VERONIS, G. (1966).—*J. Fluid Mech.* **24**, 545.

VERONIS, G. (1968).—*J. Fluid Mech.* **31**, 113.