FINITE REST MASSES OF WAVE QUANTA IN INHOMOGENEOUS MATERIAL MEDIA

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Abstract

Snell's law is put in such a form that the bending of wave trajectories in inhomogeneous media can be compared with the bending due to a mechanical force field as deduced in an earlier paper (Cole 1971). Two experiments are proposed to detect this bending effect. Though small in terrestrial conditions, the effect may be significant in the vicinity of some massive celestial objects such as pulsars. It is shown that the Coulomb law in a cold plasma is of Yukawa form in which the scale distance is c/ω_p where ω_p is the plasma frequency. This distance is the Compton wavelength of the particle defined by wave quanta in the plasma.

I. Introduction

In a previous paper (Cole 1971, hereinafter referred to as Paper I) a wave quantum in a material medium was represented as a particle with finite rest mass and this approach led to the calculation of a new form of bending of a photon trajectory in a homogeneous medium due to the presence of a gravitational field. The present work furthers the application of this idea with a theoretical consideration of the observation of the effect in more realistic inhomogeneous media. For this purpose Snell's law is restated in a form suitable for comparison with the new bending of trajectories.

It has been shown in Paper I that a quantum of any form of wave energy specified by

$$E = \hbar\omega \qquad \text{and} \qquad \boldsymbol{p} = \hbar\boldsymbol{k} \tag{1a, b}$$

can be considered for some purposes to be a particle in a vacuum with velocity

$$\boldsymbol{v} = c^2\boldsymbol{k}/\omega, \qquad v = cn, \tag{2a, b}$$

where $n = ck/\omega$ is the refractive index of the medium and c is the invariant speed of light. The particle has a rest mass m_p given by

$$m_p = p(n^{-2}-1)^{\frac{1}{2}}/c = \hbar\omega(1-n^2)^{\frac{1}{2}}/c^2. \tag{3}$$

These definitions of the parameters for the particle are consistent with the special theory of relativity which specifies that

$$E^2 = p^2c^2 + m_p^2c^4. \tag{4}$$

The movement of photons in a uniform plasma in a gravitational field was discussed in Paper I using these concepts in conjunction with the dynamics of the problem.

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We now consider a general dynamical analysis of the movement of wave quanta in a medium of variable refractive index under the influence of a mechanical force field. Consistency with existing theory is found but some new effects are delineated.

Although the basic equations of the theory are identical in form with those of de Broglie (1929), the present application and emphasis are different. de Broglie was concerned with obtaining a wave representation for what had been hitherto known as particles in the form of "matter waves", which have never been observed directly, and he wrote of "the refractive index of the vacuum n for these waves". The view of the present work is that there is a particle representation in material media of well-known and directly observable waves, e.g. elastic and electromagnetic waves, for which one refers to the refractive index *n of the medium*. Within the context of wave–particle duality the present work complements that of de Broglie.

Lucas (1969) has discussed the propagation of electromagnetic waves in media from a photon point of view. Although similar in spirit to the present approach, his analysis followed the work of Greenberg and Greenberg (1968), which was referred to in Paper I and with which the present author disagreed in the definition of velocity of the particle associated with the photon. Later Lucas (1970) took the appropriate velocity to be the group velocity instead of the phase velocity and deduced an energy of interaction for electromagnetic waves with media. There are important differences between the work of Lucas and the present analysis. Firstly, the group velocity and the velocity (2a) are only identical under certain circumstances (see Section II). Secondly, the energy of interaction is expressed in the present work by $m_{\mathrm{p}} c^2$ (see equation (3)), this being the amount of energy of the wave quantum not associated with its momentum, i.e. the energy that is not kinetic. This expression is different from that found by Lucas. Further, the present work relates to wave quanta other than just electromagnetic ones.

The last feature mentioned in the preceding paragraph also distinguishes the present work from that of Treder (1971). There is, however, another important difference. Treder's analysis has its origins in the fact that for a certain approximation the space–time metric can be presented in the form of a law of propagation of light in Euclidean space (see Fock 1964). It is then possible to define an "effective refractive index" of a vacuum in the presence of a gravitational field and to discuss the bending of light trajectories in terms of this effective refractive index. Treder's paper extends this analogy by invoking a nonlinear interaction of light with a radiation field proposed by Freundlich (1954) and then generating a modified metric to obtain a modified expression for the effective refractive index. This is in contrast to the present work which is a linear theory, does not involve modification of the metric, and adopts the approach that light travels on a null geodesic in a vacuum but on an ordinary geodesic in a medium (see Paper I).

II. Snell's Law

Consider initially a medium of variable refractive index that is not in a field of force. It is presupposed that the wavelength of the waves concerned is very much less than the characteristic distance over which the refractive index shows appreciable change, i.e.

$$2\pi/k \ll n/\nabla n\,. \tag{5}$$

The usual expression of Snell's law is

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 , \tag{6}$$

where n_1 and n_2 are the refractive indices on opposite sides of a given plane and θ is the angle between the wave direction and the normal to that plane. Since no change in frequency occurs during refraction, equation (6) can be restated as

$$k_1 \sin\theta_1 = k_2 \sin\theta_2 . \tag{7}$$

We also have

$$k_1 n_2 = k_2 n_1 \tag{8}$$

and, from equations (7) and (8), for the case where $n_2 = n_1 + \mathrm{d}n$ with $\mathrm{d}n$ small it is readily shown that

$$\mathrm{d}\boldsymbol{k} = k(\nabla n/n)\mathrm{d}s , \tag{9}$$

where $\mathrm{d}s$ is an element of length measured in the direction of $\boldsymbol{k}$. The alternative form (9) of Snell's law is now suited to our purpose.

Instead of considering a wave quantum in the medium we replace it by an equivalent particle in a vacuum (as in Paper I) so that in equation (9) $\mathrm{d}s$ is an element of length travelled by the equivalent particle according to equation (2a). From equation (9) we then have

$$\mathrm{d}(\hbar\boldsymbol{k})/\mathrm{d}t = \hbar ck\, \nabla n . \tag{10}$$

The movement of the wave quantum therefore can be considered to be that of the equivalent particle of constant mass m $(= \hbar\omega/c^2)$ in a field of potential N given by

$$N = -\tfrac{1}{2}c^2n^2 + \text{const.}, \tag{11}$$

for in this case we have

$$\mathrm{d}\boldsymbol{p}/\mathrm{d}t = -m\nabla N \tag{12}$$

which is equivalent to equation (10). Since m is constant, equation (12) may be written as

$$\mathrm{d}\boldsymbol{v}/\mathrm{d}t = -\nabla N = cn\nabla n . \tag{13}$$

The constancy of m here is in contrast to equation (14) in Section III below, in which a "mechanical" force F may be derivable from a potential. Equations (11), (12), and (13) then are the forms that Snell's law takes in the present particle formulation of wave properties.

Maupertuis's principle of least action states that the trajectory of a particle between two points P_1 and P_2 in a stationary force field is such that

$$J = \int_{P_1}^{P_2} \boldsymbol{p} \cdot \mathrm{d}\boldsymbol{s}$$

is a minimum, where $\mathrm{d}\boldsymbol{s}$ is an element of length along the trajectory. If we replace $\boldsymbol{p}$ by $(\hbar\omega/c^2)\boldsymbol{v}$ in this integral and use the expression (2a) for the velocity of the particle associated with the quantum, then Fermat's principle follows, for we have

$$J = (\hbar\omega/c)\int_{P_1}^{P_2} n\,\mathrm{d}s = \hbar\omega\int_{P_1}^{P_2}\mathrm{d}t\,.$$

de Broglie (1929) established the equivalence of these principles in his treatment of matter waves, and this result is also seen to be implicit in the context of the present work. In discussions on geometrical optics these two principles are often quoted as being analogous (see e.g. Born and Wolf 1965). As a particular example, when a refractive medium is spherically symmetric $|\boldsymbol{n}\times\boldsymbol{r}|$ is constant along a ray, where $\boldsymbol{r}$ is the radius vector from the centre of symmetry. This result, known as Bouguer's formula, is quoted by Born and Wolf (1965) as "being the analogue of a well-known formula in dynamics, which expresses the conservation of angular momentum of a particle moving under the action of central forces". In terms of the present theory this result follows, not as an analogue, but as a direct consequence of the "particle" nature of the wave propagation.

It is of interest to note that Weinberg (1962) has extended the eikonal method of Hamilton to discuss the propagation of electromagnetic fields in general media and he has suggested that the group path is identical with the path described by Fermat's principle only when the dispersion relation $D(\omega,\boldsymbol{k}) = 0$ is such that $D(\alpha\omega,\alpha\boldsymbol{k}) = \alpha^m D(\omega,\boldsymbol{k})$, that is, when it is homogeneous in ω and $\boldsymbol{k}$. However, it is suggested here that this condition is too restrictive and that a more general condition is that $\partial\omega/\partial\boldsymbol{k}$ be parallel to $\boldsymbol{k}$. This happens, for example, in the case of electromagnetic waves in a cold plasma (see equation (21) in Section III), a case explicitly excluded by Weinberg.

A further point of interest is the relation between the velocity $\boldsymbol{v}$ of the particle specified here and in Paper I and the group velocity $\partial\omega/\partial\boldsymbol{k}$. From equations (2a) and (4) this is found to be

$$\boldsymbol{v} = c^2\boldsymbol{k}/\omega = \partial\omega/\partial\boldsymbol{k} - (m_{\mathrm{p}}\,c^4/\hbar^2\omega)\partial m_{\mathrm{p}}/\partial\boldsymbol{k}\,.$$

Therefore only when $\partial m_{\mathrm{p}}/\partial\boldsymbol{k} \neq 0$, where m_{p} is defined by equation (3), does the trajectory defined by Fermat's principle not correspond to the group path. In all cases, however, the Fermat trajectory corresponds to the Maupertuis trajectory of the particle defined by equations (1)–(4).

III. Movement of Wave Quanta in Mechanical Force Fields

(*a*) *Uniform Media*

Consider now a wave quantum moving through a uniform medium in the presence of a mechanical force field. In terms of its particle representation we may write

$$\mathrm{d}\boldsymbol{p}/\mathrm{d}t = m\boldsymbol{F} = -\nabla U, \tag{14}$$

where U is a scalar potential function and $\boldsymbol{F}$ is the mechanical force per unit mass acting on the particle. In the case where m changes very little over the trajectory, to a good approximation we have

$$\mathrm{d}\boldsymbol{p}/\mathrm{d}t = -m\nabla\phi \qquad \text{or} \qquad \mathrm{d}\boldsymbol{k}/\mathrm{d}t = -(\omega/c^2)\nabla\phi\,. \tag{15a, b}$$

As an example, equations (15) will be very good approximations for movement in a gravitational field defined by the Schwarzschild metric when $m'/a \ll 1$, where m' is half the Schwarzschild radius and a is the actual radius of a celestial object. It should perhaps be pointed out that adopting an equation such as (15a) in the case of a gravitating particle produces an advance in the perihelion of the orbit of only one-sixth that in the full general relativistic treatment. However, we are not concerned here with this second-order effect but are primarily interested in the first-order change in the basic trajectory itself (i.e. in the basic ellipse).

Equations (15) allow a simple explanation of the large bending of trajectories of radio photons near the plasma frequency in a gravitational field (Paper I). For these photons, v as defined by equation (2b) is very small (comparable with or less than the escape velocity in the gravitational field, because $n \to 0$ near the plasma frequency), so that the trajectory may have considerable curvature.

It is to be noted that equation (15b) implies that ω changes as $\boldsymbol{k}$ changes, which is distinct from the situation described by equation (10). However, the changes in ω will generally be extremely small. In the special case when $m_{\mathrm{p}} = 0$ ($v = c$) equation (15b) yields for radial propagation from a star

$$k_r = k_a \exp\{(\phi_a - \phi_r)/c^2\} \qquad \text{or} \qquad \lambda_r = \lambda_a \exp\{(\phi_r - \phi_a)/c^2\}, \tag{16}$$

where the subscripts r and a denote values of the parameters at radial distances r and a in the gravitational field (assumed spherically symmetric).

(*b*) *Variable Media and Weak Fields*

In the situation where the quantum responds not only to variations in the refractive index but also to a weak mechanical force, the two effects in equations (10) and (15b) are superimposed so that the particle moves as if subjected to a potential $\phi - \frac{1}{2}c^2n^2$. Thus

$$\mathrm{d}\boldsymbol{p}/\mathrm{d}t = -m\nabla(\phi - \tfrac{1}{2}c^2n^2). \tag{17}$$

It is assumed that changes in ω and hence in m due to $\nabla\phi$ are relatively small so that any dependence of n on ω can be neglected, as will be the case in weak gravitational fields. From (1b), equation (17) may be written as

$$\mathrm{d}\boldsymbol{k}/\mathrm{d}t = (\omega/c^2)\nabla(\phi - \tfrac{1}{2}c^2n^2) \tag{18}$$

or

$$\ln(k/k_0) = -\int_{s_0}^{s} (k/c^2n^2)\nabla(\phi - \tfrac{1}{2}c^2n^2)\,\mathrm{d}s, \tag{19}$$

where the integration is performed along the path of the particle. Equations (17), (18), and (19) may be considered to be general forms of Snell's law.

The relative importance of refractive effects and mechanical force effects on the trajectory may be gauged by comparing $\nabla\phi$ and $c^2n\nabla n$. The potential ϕ could arise, for example, in a gravitational field or from a centrifugal force (as in a laboratory experiment with a rapidly rotating material medium). In the case of gravitation, if g is the local acceleration due to gravity then the gravitational component in the deformation of the trajectory is significant for

$$\nabla n^2 < g/c^2 . \tag{20}$$

In particular the gravitational deflection is dominant for $\nabla n^2 = 0$. In the case of electromagnetic waves propagating in a plasma without a magnetic field

$$n^2 = 1-\omega_p^2/\omega^2 , \tag{21}$$

where

$$\omega_p^2 = 4\pi n_e e^2/m_e , \tag{22}$$

e and m_e being the electronic charge and mass and n_e the electron density. For this example $\nabla n^2 = 0$ would imply $\nabla n_e = 0$. If there were an altitude region over which $\nabla n_e = 0$, such as a peak or "valley" in the ionosphere, it is conceivable that the particle associated with the photon, if of correct frequency, would move in a circular satellite orbit in this region under the influence of gravity. This would require

$$gr = v^2 = c^2(1-\omega_p^2/\omega^2) ,$$

that is,

$$\omega \approx \omega_p(1+gr/2c^2) . \tag{23}$$

In many cases of interest, such as the Earth's ionosphere and the solar corona, equation (23) virtually implies $\omega = \omega_p$, as it is clear that in these cases gr is very much less than c^2. This further implies $n^2 = gr/c^2 \approx 0$. Now Stix (1962) has shown that if a general magnetic field exists in a plasma then $n^2 = 0$ not only when $\omega = \omega_p$ but also when

$$\omega_p^2/\omega^2 = 1\pm\omega_{ce}/\omega ,$$

where ω_{ce} is the angular gyrofrequency of the electrons, that is, when

$$\omega = \tfrac{1}{2}\omega_{ce}\{1\pm(1+4\omega_p^2/\omega_{ce}^2)^{\frac{1}{2}}\} . \tag{24}$$

Thus the theoretical possibility exists for a gravitational field to have a significant influence, in comparison to refraction effects, on the trajectory of a radio photon through a plasma. For the Earth and the Sun, where gr/c^2 has values of 6×10^{-11} and 3×10^{-7} respectively, the experimental consequences of the effect would not be great, but they could be significant in the gravitational fields of more massive celestial objects such as pulsars and quasars.

IV. Possible Experimental Tests of the Basic Theory

(a) *Laboratory Experiment*

There are virtually only two types of mechanical force fields available for experimentation in the context of this theory, namely gravitation and the pseudo-forces such as centrifugal and coriolis forces, although there is perhaps an exotic third class involving gradients of pressure in a photon gas. Let us consider the propagation of any form of wave through a medium which manifests a resonance for the wave (e.g. electromagnetic waves often in the ultraviolet range in a dense medium; Born and Wolf 1965). If the waves are emitted horizontally into the medium then under the influence of gravity the wave trajectories will bend downwards an amount z given by

$$z = \tfrac{1}{2}g(l/v(\omega))^2, \tag{25}$$

where $v(\omega)$ is the speed of the particle associated with the wave of frequency ω (equation (2a)) and l is the length of the medium traversed. It is supposed initially that the medium is homogeneous, but this condition is reconsidered below. It then follows that

$$z = \frac{g}{2c^2}\left(\frac{l}{n(\omega)}\right)^2 \quad \text{or} \quad n(\omega) = \frac{l}{c}\left(\frac{g}{2z}\right)^{\frac{1}{2}}. \tag{26}$$

For electromagnetic waves in a dense medium with, say, $z = 100$ Å and $l = 120$ cm, equation (26) would require $n(\omega) \sim 10^{-4}$. This means that the frequency ω would be extremely close to the resonance frequency, and could also be in a region of much absorption, although the latter difficulty could be avoided if the experiment were conducted at a temperature near absolute zero. Such an experiment would appear to be difficult to perform. In principle equation (26) also illustrates how $n(\omega)$ may be found experimentally in the region of resonance, i.e. by measurement of z.

There remains an experimental problem regarding the sensitivity of the trajectory to variations in the refractive index (see the inequality (20)). A possible solution could be as follows. With the apparatus first oriented so that the waves travelled vertically between a suitably aligned source and receiver, the bulk of the trajectory variation would be related to ∇n^2 and gravity would have little effect. If the whole apparatus were then rigidly rotated so that the waves passed through the same region of the material but in a horizontal direction, presumably the necessary change in alignment of the source and receiver would be dominantly due to gravity. The difference between deflections in the possible horizontal positions should be $2z$. A systematic gravitational deflection could then be averaged out from the effects of ∇n^2.

(*b*) *Ionospheric Experiment*

An analogous experiment might be performed with radio reflection from the Earth's ionosphere. When the frequency of a transmitter equals the plasma frequency at a region in the ionosphere where $\nabla n^2 = 0$, abnormal transmission over longer than usual distances may be obtained. Although it could be difficult to positively identify this propagation under conditions where anomalous transmission may occur by many other means (e.g. scattering from irregularities), an electron density profile of the ionosphere taken at about midway between transmitter and receiver at these times could assist in the identification. The experiment would be difficult owing to the lack of control over ionospheric conditions but nevertheless could possibly yield results with proper statistical analysis. The laboratory experiment suggested in subsection (*a*) above, however, appears to be more feasible because of the possibility of performing it near the absolute zero of temperature.

V. Coulomb Law in Cold Plasma without Magnetic Field

It is of interest to note that the Coulomb law appropriate for a medium may be deduced from the form that the dispersion of electromagnetic waves takes as the frequency approaches zero. Thus, in the simple case of a cold plasma, replacing ω by

$\mathrm{i}\,\partial/\partial t$ and $\boldsymbol{k}$ by $-\mathrm{i}\,\partial/\partial \boldsymbol{r}$ in equation (21), the field is defined by the Klein–Gordon equation

$$c^2\nabla^2\phi - \partial^2\phi/\partial t^2 = \omega_{\mathrm{p}}^2\phi\,. \tag{27}$$

This equation has the spherically symmetric static solution ($\partial/\partial t = 0$)

$$\phi = (q/r)\exp(-r\omega_{\mathrm{p}}/c)\,, \tag{28}$$

which shows that the appropriate form ϕ of Coulomb's law in a cold plasma is a Yukawa type potential with characteristic distance c/ω_{p}. It is to be noted that this is the Compton wavelength of the particle as defined by equations (22) and (3). The characteristic distance may also be written $\hbar/m_{\mathrm{p}}c$, where m_{p} is the rest mass of the photon in the plasma (see Paper I) given by $m_{\mathrm{p}} = \hbar\omega_{\mathrm{p}}/c^2$.* Expressed in this way the Coulomb law is similar to the Yukawa law for a particle of rest mass m_{p}. Equation (28) also reduces to the correct vacuum form when $\omega_{\mathrm{p}} = 0$.

The expression (28) for ϕ should not be confused with one of similar form which occurs in Debye shielding and has a characteristic distance $L_{\mathrm{D}} = V_{\mathrm{th}}/\omega_{\mathrm{p}}$, where V_{th} is the thermal speed in a plasma. The quantity L_{D} is derived by applying Maxwell's equations and allowing only for thermal motions of the particles. This is in contrast to (28) which is derived from Maxwell's equations while allowing only for characteristic collective oscillations in the cold ($T = 0$) plasma. It is noted that L_{D} is very much less than c/ω_{p} for most terrestrial plasmas except of course those in regions near the absolute zero of temperature. A simple application of Debye theory when $T = 0$ would suggest that electrostatic fields have a zero characteristic length L_{D} and there would be no penetration of fields into a cold plasma.

The shielding implied by equation (28) may be considered to be due to the fact that photons of frequency less than ω_{p} do not propagate in the plasma. It is to be noted that the parameter c/ω_{p} has emerged in earlier work concerning the potential for the magnetic field due to a moving electron in a cold plasma (see Bekefi 1966, his equation (5.5)). The origin of this effect in the analysis of the disturbance magnetic field from an electron in a cold plasma not pervaded by a magnetic field is the same, namely the cutoff in propagation for photons of frequency below ω_{p}. In fact, the result given by Bekefi (1966) is the Lorentz transformation of equation (28) above, as it should be. However, the present author believes that the interpretation in terms of a departure of the Coulomb law from its form in a vacuum, or alternatively in terms of the finite rest mass of a photon in the plasma, is new. This represents a point of consistency between accepted theory and the approach in Paper I and the present work. Of course, it has been known for a long time that if photons had a finite rest mass in a vacuum then this would show up as a Yukawa type modification of the Coulomb law (see Goldhaber and Nieto 1971); but such a possibility is not the concern of the present paper, in whose terms the rest mass of photons in a vacuum is taken to be zero.

It is of interest to note that, when a magnetic field pervades a plasma, hydromagnetic waves extend the possible spectrum of electromagnetic waves to zero. In this case the dispersion equation yields two Klein–Gordon equations (one for each mode) which, under static conditions, can be combined to produce Laplace's equation.

* Equation (19) in Paper I is incorrect in that h should be replaced by $\hbar$.

It follows that the addition of a magnetic field to a plasma restores the Coulomb law to its vacuum form.

A final point of interest is the similarity between this analysis of the Coulomb law in a cold plasma and that of the phenomenon of superconductivity in solids (see e.g. Kittel 1967). The parameter that distinguishes a plasma without a magnetic field in a superconducting state from a plasma with an internal magnetic field (the normal conducting state) is c/ω_{p}, which occurs in the solid state literature and is known as the London penetration depth. Thus, as in the superconducting state, the decrease of the Coulomb interaction from its vacuum value is associated with the absence of a magnetic field. It may be inferred from equation (28) that an external magnetic field penetrates a cold plasma without a magnetic field to a depth of c/ω_{p} in the steady state.

VI. Conclusions

By restating Snell's law in an alternative form it has been possible to compare directly the bending of wave trajectories by mechanical force fields and by inhomogeneities in the medium. Although the bending due to mechanical force fields may be difficult to observe on Earth, two possible experiments have been proposed. This bending could be of interest in some cosmological problems.

The theoretical implications of this work could also be of wide interest. The analyses in Paper I and in the present paper have shown that dynamics can be applied to the particle equivalent of the wave quantum to solve a range of problems in wave propagation. In particular, the velocity of the medium and velocities with respect to it retain physical significance in a way that is consistent with the special theory of relativity (see Paper I). Finally, these methods lead to the implication that the Coulomb law in a cold plasma without a magnetic field is of Yukawa form. The addition of a magnetic field restores the law to its vacuum form.

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