FINITE DEFORMATIONS OF THE NUCLEUS

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Abstract

Deformations of the nucleus are examined with a model based on the irrotational incompressible motion of an inhomogeneous fluid. Restrictions of previous treatments to small deformations are noted and the model is extended to arbitrary deformations. The particular case of ellipsoidal deformation is considered and compared with experiment. It is concluded that the contribution of ellipsoidal deformation to hexadecapole deformation is not negligible.

I. Introduction

Collective states of a nucleus can be described as oscillations of the shape of the nucleus or as rotations of a deformed nucleus. The density of nucleons in a deformed nucleus can be written as

$$D(\boldsymbol{r}) = \sum_{lm} D_{lm}(r)\, Y_{lm}(\theta, \phi). \tag{1}$$

The spherically symmetric part of the density distribution for the ground state is fairly well known, for instance, in the case of the charge density from experiments on the elastic scattering of high energy electrons. Experiments are now yielding information about the remaining $D_{lm}(r)$ terms. It is of interest whether there is any relation between the $D_{lm}(r)$ and, in particular, their relation to $D_0(r)$; in other words, as to how the nucleus deforms. Attention has been drawn to this question by Satchler (1972), who pointed out that the results obtained from experiments on nuclear scattering depend on the assumed mode of deformation of the nucleus.

In principle the question of nuclear deformation can be avoided by calculating $D(\boldsymbol{r})$ directly from a microscopic theory of the nucleus, provided the results of such calculations fit the experimental data. However, until such calculations are available throughout the periodic table, it is useful to have simpler models to analyse experiments. In previous work a model based on irrotational incompressible motion of an inhomogeneous fluid has been used to describe nuclear vibrations (Tassie 1956, 1958) and the rotation of a deformed nucleus (Tassie 1960). These analyses treated the deformed nucleus as a "frozen" vibration of an originally spherically symmetric nucleus. It must be emphasized that such work was valid only for small deformations. The fluid model and modifications of it have been used extensively for the calculation of electron scattering (Onley *et al.* 1963, 1964) and for the analysis of electron scattering experiments (Überall 1971; Buskirk *et al.* 1972; Fukuda and Torizuka 1972; Nakada

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and Torizuka 1972). The model also has been extended by Überall and Ugincius (1969) to vibrations of a deformed nucleus.

Nuclear deformations are not always small, however, and in fact the fluid model described above has been compared with experiments in which either the deformation or amplitude of vibration is not small. It seems worth while to extend the model to arbitrary deformations, and this is done in Section II of the present paper. The particular case of ellipsoidal deformation is considered in Section III, while in Section IV the results of experiments are discussed and compared with the predictions of ellipsoidal deformation.

II. General Theory

The term $D(\boldsymbol{r},t)$ is to be treated as a classical density. For large deformations it must not be assumed that this density has a harmonically vibrating form $\exp(\mathrm{i}\omega t)$, as was assumed by Tassie (1956) and Überall (1971), since it is clear that, for a uniform density distribution vibrating about a spherical equilibrium shape with a finite amplitude of vibration, the time dependence of $D(\boldsymbol{r},t)$ is a series of step functions. Some care is therefore needed in formulation.

If $\boldsymbol{v}$ is the velocity of the nuclear fluid then

$$\mathrm{d}D/\mathrm{d}t = \boldsymbol{v}\,.\,\nabla D + \partial D/\partial t\,. \tag{2}$$

Assuming the nuclear fluid to be incompressible,

$$\mathrm{d}D/\mathrm{d}t = 0 \qquad \text{and} \qquad \partial D/\partial t = -\boldsymbol{v}\,.\,\nabla D\,. \tag{3a, b}$$

The equation of continuity,

$$\nabla\,.\,(D\boldsymbol{v}) + \partial D/\partial t = 0\,, \tag{4}$$

together with equation (3b) yields the condition

$$\nabla\,.\,\boldsymbol{v} = 0\,. \tag{5}$$

For irrotational flow

$$\boldsymbol{v} = \nabla\Phi \tag{6}$$

and so

$$\nabla^2\Phi = 0 \tag{7}$$

with the solutions

$$\Phi(\boldsymbol{r},t) = \sum_{lm} \gamma_{lm}(t)\, r^l\, Y_{lm}(\theta,\phi)\,. \tag{8}$$

The density can then be determined by solving the equation

$$\partial D/\partial t = -\nabla\Phi\,.\,\nabla D\,. \tag{9}$$

Because the motion is incompressible, the density can be written as

$$D(\boldsymbol{r},t) = D^0(\boldsymbol{r}_0)\,, \tag{10}$$

where D^0 is the density in the initial spherical nucleus and the element of nuclear fluid with initial position $\boldsymbol{r}_0$ has position $\boldsymbol{r}$ at time t. For use in equation (10), it is convenient to write

$$r - r_0 = \sum_{lm} a_{lm}(r,t)\, Y_{lm}(\theta,\phi)\,. \tag{11}$$

The motion specified by equation (11) must be consistent with the velocity potential given by (8). It should be noted that the equation used by Rayleigh (1945), Tassie (1956), Überall and Ugincius (1969), and Überall (1971) holds only to first order in the deformation, and in general

$$(\partial r/\partial t)_\rho \neq \partial \Phi/\partial r\,. \tag{12}$$

A correct treatment of this condition has been given by Pal (1972) who shows that, writing the equation of a surface as

$$F(\boldsymbol{r},t) = 0\,, \tag{13}$$

the condition of matching the normal component of the velocity at the surface is

$$\nabla F\,.\,\nabla \Phi = -\partial F/\partial t\,. \tag{14}$$

For simplicity we assume axial symmetry and take only terms with $m = 0$ in equations (8) and (11), i.e. we write $a_{l0} = a_l$ and $\gamma_{l0} = \gamma_l$. Equation (13) for a surface of constant density $D^0(r_0)$ becomes

$$F = r - \sum_l a_l(r,t)\, Y_{l0} - r_0\,. \tag{15}$$

Substituting equations (15) and (8) in equation (14), we find

$$l\gamma_l r^{l-1} Y_{l0} - \sum_{l_1 l_2}\left(\frac{\partial a_{l_1}}{\partial r} l_2 \gamma_{l_2} r^{l_2-1} Y_{l_1 0} Y_{l_2 0} + a_{l_1}\gamma_{l_2} r^{l_2-2}\frac{\mathrm{d}Y_{l_1 0}}{\mathrm{d}\theta}\frac{\mathrm{d}Y_{l_2 0}}{\mathrm{d}\theta}\right) = \dot{a}_l Y_{l0}\,. \tag{16}$$

In general a_l is not proportional to a power of r and equation (16) is not very useful. There are, however, two simple cases: (1) For small deformations, we have

$$\dot{a}_l = l\gamma_l r^{l-1}\,, \tag{17}$$

as used by Tassie (1956). (2) For all $\gamma_l = 0$ except γ_2, we can write

$$a_l = \alpha_l r\,,$$

$$\tfrac{1}{2}\dot{\alpha}_l = \delta_{l2}\gamma_2 - \sum_{l_1}\left(\alpha_{l_1}\gamma_2 \int Y_{l0} Y_{l_1 0} Y_{20}\,\mathrm{d}\Omega + \tfrac{1}{2}\alpha_{l_1}\gamma_2 \int Y_{l0}\frac{\mathrm{d}Y_{l_1 0}}{\mathrm{d}\theta}\frac{\mathrm{d}Y_{20}}{\mathrm{d}\theta}\,\mathrm{d}\Omega\right), \tag{18}$$

so that α_l is independent of r. This case corresponds to ellipsoidal deformation, as is shown in the next section.

III. Ellipsoidal Deformation

Pal (1972) has considered in detail the ellipsoidal deformation of a homogeneous liquid drop. The equation of the ellipsoidal surface is

$$\sum_{i=1}^{3} x_i^2/a_i^2 = 1. \tag{19}$$

With r_0 the radius of a sphere having the same volume as contained in the ellipsoidal surface, that is,

$$a_1 a_2 a_3 = r_0^3, \tag{20}$$

Pal finds for ellipsoidal vibrations that

$$\Phi = +\tfrac{1}{2}\sum_{i=1}^{3} (\dot{a}_i/a_i)x_i^2 \tag{21}$$

(where the velocity potential Φ here is equivalent to $-\phi$ as used by Pal). With restriction to the axial symmetry $a_1 = a_2$ and writing

$$a_2 = a_1 = Ar_0 \qquad \text{and} \qquad a_3 = A^{-2}r_0, \tag{22}$$

the velocity potential (21) becomes

$$\Phi = \tfrac{1}{2}(\dot{A}/A)r^2(1-3\cos^2\theta) = (4\pi/5)^{\frac{1}{2}}(\dot{A}/A)r^2\, Y_{20}(\theta), \tag{23}$$

which is of the form of equation (8) with

$$\gamma_2 = (4\pi/5)^{\frac{1}{2}}\dot{A}/A \tag{24}$$

and all other $\gamma_l = 0$. Substituting equations (22) into (19) yields the result

$$r = Ar_0\{1+(A^6-1)\cos^2\theta\}^{-\frac{1}{2}} = r_0\Big(1+\sum_{l\ \text{even}}\beta_l\, Y_{l0}(\theta)\Big), \tag{25}$$

where

$$\beta_l = \int Y_{l0}(\theta)\,[A\{1+(A^6-1)\cos^2\theta\}^{\frac{1}{2}}-1]\,\mathrm{d}\Omega. \tag{26}$$

Then

$$\beta_0 = (4\pi)^{\frac{1}{2}}\{Ay-1\}, \tag{27a}$$

$$\beta_2 = (4\pi)^{\frac{1}{2}}\sqrt{\tfrac{5}{16}}\,A\{-(2+3d)y+3dA^3\}, \tag{27b}$$

$$\beta_4 = (4\pi)^{\frac{1}{2}}\tfrac{3}{16}A\{3(2+10d+\tfrac{35}{4}d^2)y-\tfrac{5}{2}(5+\tfrac{21}{2}d)A^3\}, \tag{27c}$$

where

$$d = (A^6-1)^{-1} \tag{28}$$

and

$$y = (1-A^6)^{-\frac{1}{2}}\arcsin\{(1-A^6)^{\frac{1}{2}}\} \qquad \text{for} \qquad A < 1, \tag{29a}$$

$$= (A^6-1)^{-\frac{1}{2}}\ln\{A^3+(A^6-1)^{\frac{1}{2}}\} \qquad \text{for} \qquad A > 1. \tag{29b}$$

For small deformations,

$$\beta_0 \approx -(4\pi)^{\frac{1}{2}}\frac{4\varepsilon^2}{5}, \qquad \beta_2 \approx -(4\pi)^{\frac{1}{2}}\frac{2\varepsilon}{\sqrt{5}}, \tag{30a, b}$$

$$\beta_4 \approx (4\pi)^{\frac{1}{2}}\frac{36\varepsilon^2}{35}, \qquad \beta_6 \approx -(4\pi)^{\frac{1}{2}}\frac{5\varepsilon^3}{3\times 7\times 11\times\sqrt{13}}, \tag{30c, d}$$

with

$$\varepsilon = A-1\,. \tag{31}$$

The curve derived from equations (27) for β_4 as a function of β_2 is compared with experimental data in Figure 1. Although the difference between the results of equations (27) and the corresponding ones of equations (30) are not negligible, the

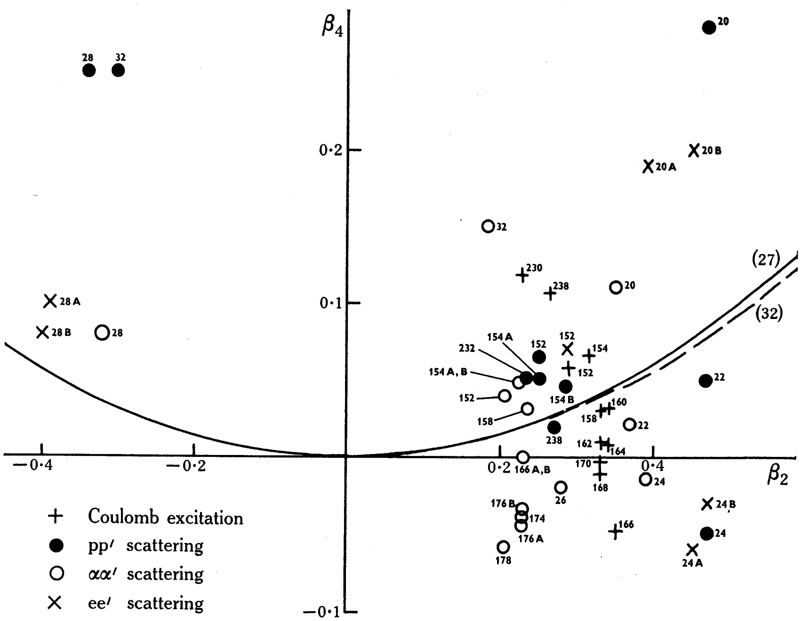

Fig. 1.—Plots of β_4 as a function of β_2 with experimental values from Coulomb excitation and inelastic scattering of protons, α-particles, and electrons. The full curve is derived from equations (27) for ellipsoidal deformation while the dashed curve is the approximation (32). The experimental values are labelled by the mass number A of the nucleus and by A or B where two measurements have been made by the same method for the same nucleus. The results are: pp′ data by de Swiniarski *et al.* (1972) for ^{20}Ne and ^{22}Ne; αα′ data by Rebel *et al.* (1972) for ^{20}Ne, ^{22}Ne, ^{24}Mg, ^{26}Mg, and ^{28}Si; ee′ data by Horikawa *et al.* (1971) for ^{20A}Ne, ^{24A}Mg, and ^{28A}Si; ee′ data by Nakada and Torizuka (1972) for ^{20B}Ne, ^{24B}Mg, and ^{28B}Si; pp′ data by de Swiniarski *et al.* (1969) for ^{24}Mg, ^{28}Si, and ^{32}S; αα′ data by Schweimer *et al.* (1972) for ^{32}S; Coulomb excitation data by Erb *et al.* (1972) for ^{152}Sm, ^{154}Sm, ^{158}Gd, ^{160}Gd, ^{162}Dy, ^{164}Dy, ^{166}Er, ^{168}Er, and ^{170}Er; ee′ data by Bertozzi *et al.* (1972) for ^{152}Sm; αα′ data by Hendrie *et al.* (1968) for ^{152}Sm, ^{154A}Sm, ^{158}Gd, ^{166A}Er, ^{174}Yb, ^{176A}Yb, and ^{178}Hf; pp′ data by Kurepin and Lombard (1971) for ^{152}Sm; pp′ data by Brown and Stoler (1970) for ^{154A}Sm; αα′ data by Aponick *et al.* (1970) for ^{154B}Sm, ^{166B}Er, and ^{176B}Yb; pp′ data by Kurepin *et al.* (1972) for ^{154B}Sm; Coulomb excitation data by McGowan *et al.* (1971) for ^{230}Th and ^{238}U; pp′ data by Moss *et al.* (1971) for ^{232}Th and ^{238}U.

ratio of equations (30c) and (30b), namely

$$\beta_4 \approx \tfrac{9}{7}(4\pi)^{-\frac{1}{2}}\beta_2^2, \tag{32}$$

is a good approximation, as shown by the dashed curve in Figure 1. From equations (30d) and (30b),

$$\beta_6 \approx \tfrac{25}{1848}\sqrt{\tfrac{5}{13}}(4\pi)^{-1}\beta_2^3. \tag{33}$$

IV. Discussion

For small deformations, the angular distribution of scattered particles is the same for both rotational and vibrational excitation of the nucleus (Tassie 1960), but this equality no longer holds when the deformations are finite (Reiner and Tassie 1965). In the treatment of vibrational nuclei, the transition density is obtained from the matrix elements of the D_l. This can be done by expressing the γ_l and β_l in terms of phonon creation and annihilation operators. The resulting transition density depends not only on the model of nuclear deformation but also on the details of the nuclear vibrational wavefunctions, e.g. on the amount of mixing of one-phonon and two-phonon states. In the case of spherical nuclei, for example, the corrections of second order in the deformation vanish for pure one-phonon transitions. The analysis of experimental results foı vibrational nuclei is more sensitive to details of the vibrational wavefunctions than to details of how the nucleus deforms.

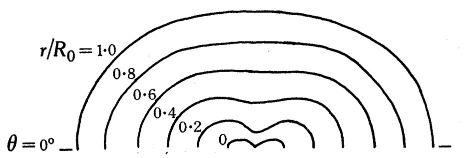

Fig. 2.—Equipotential surfaces of the nucleus for a deformation described by equation (36) for $\beta_2 = 0{\cdot}3$.

The excitation of rotational states is more sensitive to terms of higher order in the deformation. Figure 1 shows many values of β_2 and β_4 obtained from experiments on the excitation of rotational states. The observed values of β_4 are of roughly the same order of magnitude as that required for ellipsoidal deformations, but the figure does not seem to indicate that the nucleus is ellipsoidal. However, the deduction of β_l from experiment depends on the choice of model for the nuclear deformation (Satchler 1972). In particular, the analyses of the experiments on inelastic proton and α-particle scattering generally use a deformed nuclear potential of the form

$$V(r) = v_0[1+\exp\{(r-R(\theta))/a\}]^{-1} \tag{34}$$

with

$$R(\theta) = R_0\left(1+\sum_l \beta_l Y_{l0}\right), \tag{35}$$

so that the deformation of the nuclear fluid is

$$r = r_0 + \sum_l \beta_l Y_{l0}(\theta) R_0. \tag{36}$$

Such a deformation of the nucleus seems rather peculiar, and in fact is singular at the origin. The equipotential surfaces for $\beta_2 = 0 \cdot 3$ are shown in Figure 2. The outer part of the nucleus is less deformed than the inner part, so that such an analysis of processes which are very dependent on the outer part of the nucleus can lead to higher apparent values of β_l than an analysis with a deformation of the type

$$r = r_0\left(1 + \sum_l \beta_l Y_{l0}\right).$$

There are also other uncertainties in the deduction of β_l values from the experimental data (Satchler 1972). The wide scatter of different results for the same nucleus can be seen in Figure 1. A further possible source of error is that some of the lighter nuclei may not be very good examples of rotational nuclei.

On present evidence the only possible conclusion is that the contribution of ellipsoidal deformation to β_4 is not negligible. From equation (33), it is expected that for ellipsoidal deformation with $|\beta_2| < 0 \cdot 6$ we will have $|\beta_6| < 0 \cdot 00015$, which is about two orders of magnitude less than values deduced from experiment by Hendrie *et al.* (1968), Aponick *et al.* (1970), and Moss *et al.* (1971).

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