PERIODIC NONLINEAR DIFFUSION: AN INTEGRAL RELATION AND ITS PHYSICAL CONSEQUENCES

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Abstract

A general integral relation in steady periodic nonlinear diffusion is established through a Kirchhoff transformation. The time average $\bar{\Theta}$ of the transformed concentration Θ satisfies Laplace's equation and may therefore be evaluated readily. Results for $\bar{\Theta}$ not only serve as simple and exact checks on detailed numerical solutions of the nonlinear diffusion equation but also provide, immediately and exactly, the principal information often sought about these solutions. The results are especially simple in many steady periodic nonlinear diffusion problems where the time average of the flux density is everywhere zero and $\bar{\Theta}$ is constant. The method is applied to examples in fluid mechanics and geophysics: spatially periodic laminar boundary layers, tidal influence and seasonal stream effects on groundwater levels, and soil temperature waves.

I. INTRODUCTION

The nonlinear diffusion equation arises in many applications, including physical chemistry, heat and mass transfer, the physics of solids, metallurgy, fluid mechanics, and the earth sciences. Although solutions of the equation satisfying any well-posed set of conditions may always be found, in principle, by the direct use of finite-difference methods on high speed computers, for both intellectual and economic reasons it is desirable to take the study of various problems in nonlinear diffusion as far as possible by the methods of mathematical analysis. Treatments which have proved fruitful include similarity techniques (Boltzmann 1894; Philip 1955), perturbation of similarity solutions (Philip 1966, 1969), and integral methods (Macey 1959; Parlange 1971; Philip and Knight 1973); a short general review of the subject has been given by Philip (1973). The methods developed to date, however, are ill-fitted to consideration of *periodic* nonlinear diffusion, and there seems to have been no progress with analytical approaches to this class of problem. The present paper establishes a general integral relation in steady periodic nonlinear diffusion. As well as affording a precise, simple, and continuous check on the numerical analysis of any such problem, this relation provides, exactly and at once, important information (in fact, what is often the principal information sought) about the solution. The usefulness of the result is illustrated here through several examples drawn from fluid mechanics and geophysics.

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II. General Integral Relation

We are concerned with nonlinear diffusion described by the equation

$$\partial\theta/\partial t = \nabla.(D\nabla\theta), \tag{1}$$

where θ is the concentration, t is the time, and the diffusivity D is a non-negative function of θ. The superficially more general "nonlinear heat conduction equation"

$$C\,\partial\theta/\partial t = \nabla.(D\nabla\theta),$$

with C also a non-negative function of θ, reduces to the form (1) through the transformation

$$W = \int_{\theta_0}^{\theta} C(\theta')\,\mathrm{d}\theta'.$$

The transformation of Kirchhoff (1894)

$$\Theta = \int_{\theta_0}^{\theta} D(\theta')\,\mathrm{d}\theta'$$

enables us to rewrite equation (1) in the form

$$\partial\theta/\partial t = \nabla^2\Theta. \tag{2}$$

In both of the preceding integrals, the lower limit θ_0 is an arbitrary constant.

We consider any stationary periodic phenomenon satisfying equation (1) and such that

$$\theta(\boldsymbol{r}, t) \equiv \theta(\boldsymbol{r}, t+T),$$

where $\boldsymbol{r}$ is the position vector and the period T is a positive constant. Integration of equation (2) over any period $t = t_0$ to t_0+T shows that

$$\nabla^2\bar{\Theta} = 0, \tag{3}$$

where

$$\bar{\Theta}(\boldsymbol{r}) = T^{-1}\int_{t_0}^{t_0+T} \Theta(\boldsymbol{r}, t)\,\mathrm{d}t$$

is the mean value of $\Theta(\boldsymbol{r}, t)$. We thus have the useful general result that $\bar{\Theta}$ satisfies Laplace's equation.

We now examine the boundary conditions on equation (3).

(i) At boundaries where the condition on equation (1) is of the concentration type

$$\theta = \theta_1(t),$$

where θ_1 is a function of t with period T, the corresponding condition on equation (3) is

$$\bar{\Theta} = T^{-1}\int_{t_0}^{t_0+T}\left(\int_{\theta_0}^{\theta_1(t)} D(\theta')\,\mathrm{d}\theta'\right)\mathrm{d}t = \bar{\Theta}_1. \tag{4}$$

$\bar{\Theta}_1$ may vary with position on the boundary if $\theta_1(t)$ does.

(ii) At boundaries where the condition on equation (1) is of the flux type

$$D\,\partial\theta/\partial n = F(t),$$

where F is a function of t with period T, the corresponding condition on equation (3) is

$$\partial\bar{\Theta}/\partial n = T^{-1}\int_{t_0}^{t_0+T} F(t)\,\mathrm{d}t = \bar{F}. \tag{5}$$

$\bar{F}$ may also vary with position on the boundary if $F(t)$ does. Here and in what follows $\partial/\partial n$ signifies differentiation with respect to the normal to the boundary.

We see that for all steady periodic nonlinear diffusion phenomena we may calculate $\bar{\Theta}$ through the, generally rather simple, process of solving Laplace's equation subject to the appropriate conditions. It is therefore a straightforward matter to construct a dictionary of solutions $\bar{\Theta}(\boldsymbol{r})$ to problems of this class involving various geometrical configurations and boundary conditions. These *exact* results $\bar{\Theta}(\boldsymbol{r})$ then serve as a check on detailed numerical solutions in the form of $\Theta(\boldsymbol{r},t)$ or $\theta(\boldsymbol{r},t)$. We will not pursue this in detail here, but will limit consideration to some simple special results which immediately provide important information about steady periodic nonlinear diffusion processes arising, for example, in fluid mechanics and geophysics.

III. Systems with Zero Average Flux Density

Many steady periodic nonlinear diffusion phenomena are such that the flux density averaged over one period is zero at every point of the system. Some phenomena of this type are enumerated below.

(i) *One-dimensional Systems*

(1) Finite system $0 \leqslant x \leqslant L$, say, with the conditions $\theta = \theta_1(t)$ at $x = 0$ and $\partial\theta/\partial x = 0$ at $x = L$.

(2) Finite system $0 \leqslant x \leqslant 2L$ with the condition $\theta = \theta_1(t)$ at $x = 0$ and $2L$.

(3) Semi-infinite system $0 \leqslant x$ with the conditions $\theta = \theta_1(t)$ at $x = 0$ and $\lim_{x\to\infty} \partial\theta/\partial x = 0$.

(ii) *Two- and Three-dimensional Systems*

(1) Finite system bounded by one closed curve (two dimensions) or one closed surface (three dimensions), at which the condition $\theta = \theta_1(t)$ holds, with θ_1 independent of position on the boundary.

(2) Finite system bounded by two closed curves (two dimensions) or two closed surfaces (three dimensions), where the condition $\theta = \theta_1(t)$, with θ_1 independent of position, holds on one boundary and the condition $\partial\theta/\partial n = 0$ holds on the other.

(3) Infinite system bounded internally by one closed curve (two dimensions) or one closed surface (three dimensions), at which the condition $\theta = \theta_1(t)$, with θ_1 independent of position, holds while the second condition is $\lim_{r\to\infty} \partial\theta/\partial r = 0$. (In two dimensions $r = (x^2+y^2)^{\frac{1}{2}}$ and in three dimensions $r = (x^2+y^2+z^2)^{\frac{1}{2}}$, the origins of (x, y) and (x, y, z) being taken inside the closed curve and closed surface respectively.)

For each of these systems and also for certain systems in other configurations (and for more general, but perhaps rather artificial, systems with the foregoing zero flux conditions $F = 0$ replaced by the weaker $\bar{F} = 0$) the solution of equation (3) is simply

$$\bar{\Theta} = \text{const.} = \bar{\Theta}_1. \tag{6}$$

In all the above systems the amplitude of fluctuations in θ decreases with increasing distance from the boundary at which $\theta_1(t)$ is imposed. It follows that, far enough from that boundary, the periodic fluctuation in Θ is also small so that we have to a good approximation there

$$\Theta = \text{const.} = \bar{\Theta}_1. \tag{7}$$

In the same approximation then

$$\theta = \text{const.} = \theta^*,$$

with θ^* such that

$$\int_{\theta_0}^{\theta^*} D(\theta')\,\mathrm{d}\theta' = \bar{\Theta}_1. \tag{8}$$

For this class of problem we may therefore evaluate at once the essentially steady concentration θ^* which is in equilibrium with a fluctuating concentration imposed at a sufficiently distant boundary. Illustrations of this result in various applications are considered in the following section.

IV. Applications in Fluid Mechanics and Geophysics

(*a*) *Spatially Periodic Boundary Layers: Cavity Flows*

Particular forms of the preceding results (6), (7), and (8) hold for spatially periodic boundary layers and are relevant to related flow problems involving closed streamlines. The von Mises (1927) form of the laminar boundary layer equation with zero pressure gradient is

$$\frac{\partial u}{\partial x} = \frac{\partial}{\partial \phi}\left(\nu u \frac{\partial u}{\partial \phi}\right), \tag{9}$$

where u denotes the component of fluid velocity in the x direction (i.e. parallel to the wall), ν is the kinematic viscosity, and the stream function ϕ is defined by

$$u = \partial\phi/\partial y, \qquad v = -\partial\phi/\partial x, \tag{10}$$

v being the component of fluid velocity in the y direction (i.e. normal to the wall). It follows at once that the zero-pressure-gradient spatially periodic steady flow for $y \geqslant 0$ produced by imposition at the wall of the condition

$$y = 0, \qquad u = u_0(x),$$

with u_0 a periodic function of x, has the properties:

(1) along every streamline $\overline{u^2}$ is constant and equal to $\overline{u_0^2}$; and

(2) outside the boundary layer u is constant and equal to $(\overline{u_0^2})^{\frac{1}{2}}$.

The first of these is a specialization of equation (6) and the second a specialization of equations (7) and (8).

Wood (1957) was the first to establish the special results for this problem. The region of constant u outside the spatially periodic boundary layer is closely related to the central core of constant vorticity in cavity flows and other closed-streamline flows (see the review by Burggraf 1966). In the paper which pioneered boundary-layer theory, Prandtl (1905) asserted the existence of the constant-vorticity core in appropriate flows. Batchelor (1956), following work by Pillow (1952) and Carrier (1953), used the Wood (1957) result to evaluate core vorticity, while Feynman and Lagerstrom (1957) gave the same result independently.

(*b*) *Groundwater Levels in Equilibrium with Tides*

The Boussinesq (1904) equation for unsteady groundwater flow in a homogeneous soil mass overlying a horizontal impermeable base, with no gain or loss of water at the soil surface, may be written as

$$\frac{\partial z}{\partial t} = \frac{\partial}{\partial x}\left(D\frac{\partial z}{\partial x}\right) + \frac{\partial}{\partial y}\left(D\frac{\partial z}{\partial y}\right). \tag{11}$$

where t is time, x and y are horizontal rectangular cartesian coordinates, and z is the elevation of the free groundwater surface (groundwater level) above the impermeable base. The diffusivity D is given by

$$D = Kz/s\,, \tag{12}$$

with K the hydraulic conductivity and s the specific yield.

It will be noted that equation (11) applies also to *heterogeneous* soils with K varying with depth. In that case equation (12) is replaced by

$$D = \int_0^z s^{-1}K(z')\,\mathrm{d}z'\,.$$

When both K and s vary with z, the equation replacing (11) again reduces to the nonlinear diffusion form. The discussion which follows in this subsection and the next applies specifically to homogeneous soils, but it will be evident that similar results hold for heterogeneous soils. When K increases with z, deviations from the linear theory exceed those for homogeneous soils, which are evaluated below.

A number of investigators (Jacob 1950; Ferris 1951; Werner and Noren 1951) have used a linearization of equation (11) to study the influence of tides on groundwater levels in coastal regions, and the results are embodied in hydrological texts and handbooks (see e.g. Todd 1959, 1964). One important result of this linear analysis is that, at points far enough inland, the essentially constant groundwater level in equilibrium with the tide is equal to the mean sea level. It will be evident from Section III that the exact nonlinear result is readily available; as shown below, the linear analysis may be considerably in error and the inland groundwater level may lie significantly above the mean sea level.

Let H be the elevation of the mean sea level above the impermeable base and let αH be the tidal semi-amplitude. We assume that the tidal variation of sea level is sinusoidal and that the land–sea interface is effectively vertical. It then follows from equations (7) and (8) that, in the simplest case with $\alpha \leqslant 1$, the inland groundwater surface is $\{(1+\frac{1}{2}\alpha^2)^{\frac{1}{2}}-1\}H$ above the mean sea level. For the case $\alpha = 1$ the inland groundwater level lies above the mean sea level by about 23% of the tidal semi-amplitude.

Elevation of the inland groundwater level due to nonlinear tidal influence is, of course, separate and distinct from groundwater elevation associated with hydrostatic equilibrium between marine salt water and terrestrial fresh water (Badon-Ghyben 1888; Herzberg 1901). Presumably the two effects reinforce each other in some circumstances.

(c) *Groundwater Levels in Equilibrium with Seasonal Streams*

Fluctuating groundwater levels occur also in connection with groundwater bodies contiguous with seasonally variable streams. Although stream levels will not, of course, be truly periodic, a periodic model preserves some important features of stream–groundwater interaction. We take an example which models the behaviour of ephemeral streams in tropical regions, which flow only for brief periods during the wet season and which tend to be either in full flood or dry. As before, we suppose that the stream (bounded by effectively vertical banks) and the groundwater body overly a horizontal impermeable base.

Equations (11) and (12) apply for this example, and we assume that the stream level varies periodically, being at an elevation Z (flood level) above the base for a fraction ε of the time and at an elevation βZ (river bed level) for the remainder of the time. It then follows from equations (7) and (8) that, according to the nonlinear analysis based on equations (11) and (12), the essentially constant groundwater level distant from the stream will be $\{\varepsilon+(1-\varepsilon)\beta^2\}^{\frac{1}{2}}Z$. The corresponding linear analysis, on the other hand, predicts that the level will be $\{\varepsilon+(1-\varepsilon)\beta\}Z$. For the case $\varepsilon = 0 \cdot 04$ and $\beta = 0$, the nonlinear result is $0 \cdot 2Z$ whereas the linear result is $0 \cdot 04Z$. It is evident that the linear theory can be greatly in error.

(d) *Soil Temperature in Depth*

The classical analysis of diurnal and annual soil temperature waves is based on the linear heat conduction equation

$$\partial\theta/\partial t = D\,\partial^2\theta/\partial z^2\,, \tag{13}$$

where θ is the temperature, t is the time, z is the vertical coordinate, and the thermal diffusivity D is treated as a constant. One well-known and important consequence of this linear theory is that the soil temperature in depth asymptotically approaches the time-average of the surface temperature (Keen 1931; van Wijk and de Vries 1963). In reality, however, the soil thermal diffusivity is temperature-dependent, and this dependence is quite strong in dry soils (de Vries 1963). An examination of the magnitude of the errors of the linear theory is therefore desirable.

In the nonlinear analysis, equation (13) is replaced by

$$\frac{\partial\theta}{\partial t} = \frac{\partial}{\partial z}\left(D\frac{\partial\theta}{\partial z}\right), \tag{14}$$

where D is now a function of θ. We consider the case where surface temperature fluctuates sinusoidally between the values θ_0+A and θ_0-A and D varies linearly with θ such that

$$D(\theta_0+A) = \gamma\, D(\theta_0-A). \tag{15}$$

It then follows from equations (7) and (8) that θ^*, the essentially constant value of θ in depth, is given by

$$\theta^* = \theta_0+[\{(\tfrac{3}{2}\gamma^2+\gamma+\tfrac{3}{2})^{\frac{1}{2}}-\gamma-1\}/(\gamma-1)]A.$$

In dry soils in arid regions we may have $A = 20°\mathrm{C}$ and $\gamma = 3$ (de Vries 1963; Geiger 1965), and for these values the linear theory, which gives $\theta^* = \theta_0$, underestimates the soil temperature in depth by 2·4°C, a significant discrepancy in many applications.

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