QUANTUM–MECHANICAL WAVE EQUATION FOR TWO PARTICLES OF ARBITRARY SPIN WITH INSTANTANEOUS INTERACTION

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Abstract

A quantum-mechanical wave equation for two particles of arbitrary spin is derived for any instantaneous interaction. The starting point is an integral equation which is a relativistic generalization of the Schrödinger integral equation and is similar to the Bethe–Salpeter equation. The final two-particle wave equation contains the sum of the Hamiltonians of the two particles and has only one time variable. It reduces to the two-particle Schrödinger equation in the non-relativistic limit.

I. INTRODUCTION

Previous discussions of the quantum-mechanical problem of a two-body system have been limited mostly to spin-0 and spin-$\frac{1}{2}$ particles. Thus the two-particle wave equation set up by Breit (1929) describes a system of two electrons whereas the well-known four-dimensional relativistic Bethe–Salpeter equation (Salpeter and Bethe 1951), which can be derived from theoretical field considerations, has been solved mainly for particles of spin 0 and $\frac{1}{2}$. This limitation is largely due to the fact that the wave equation for a single particle of higher spin is rather complicated and involves matrices which are difficult to handle.

It is the objective of this work to derive a quantum-mechanical wave equation for two particles of any spin with an arbitrary instantaneous interaction. The derivation starts with an integral equation which is a relativistic generalization of the Schrödinger integral equation and is similar to the Bethe–Salpeter integral equation. Use is also made of the fact that the wave equation for a particle of any spin can be written in Hamiltonian form (Weaver *et al.* 1964). The problem is treated purely as a quantum-mechanical one, and there is no claim that the resulting equation will describe any theoretical field processes such as pair creation and annihilation of particles and antiparticles. Since the two-particle wavefunction contains a single time variable, it avoids the difficulty of having to interpret the relative time. The wave equation also does not contain extra solutions, as can be seen from the fact that the solution of the equation for two free particles is just the product of two free particle wavefunctions, one for each particle. Furthermore, the equation can be shown to reduce to the two-particle Schrödinger equation in the non-relativistic limit.

Throughout this paper, quantities are expressed in natural units, in which $\hbar$, Planck's constant divided by 2π, and c, the speed of light, are taken as unity,

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$\hbar = c = 1$. The imaginary fourth component convention is also used in which the space–time four-vectors are $x_\mu = (x, y, z, it)$ and the invariant is

$$x_\mu x_\mu = x^2 + y^2 + z^2 - t^2 .$$

The scalar product of two real four-vectors a and b is denoted by

$$ab = a_\mu b_\mu = a_1 b_1 + a_2 b_2 + a_3 b_3 - a_0 b_0 ,$$

where $a_4 = \mathrm{i} a_0$, $b_4 = i b_0$, and a_0 and b_0 are real. Greek indices μ are used to represent summations over all four components while Latin indices i represent summations over the three space components.

II. Integral Equation

The non-relativistic Schrödinger equation

$$\left(\frac{\;^2}{2m} - V + \mathrm{i}\frac{\partial}{\partial t}\right)\psi = 0 \tag{1}$$

can be cast into an integral form by means of the non-relativistic free particle Green's function G. This function satisfies the equation

$$\left(\frac{\nabla^2}{2m} + \mathrm{i}\frac{\partial}{\partial t}\right) G(\boldsymbol{x}-\boldsymbol{x}', t-t') = \mathrm{i}\,\delta(\boldsymbol{x}-\boldsymbol{x}')\,\delta(t-t'),$$

and can be written in integral form as

$$G(\boldsymbol{x}-\boldsymbol{x}', t-t') = \frac{\mathrm{i}}{(2\pi)^4}\int \frac{\exp\{\mathrm{i}p(x-x')\}}{-\boldsymbol{p}^2/2m + E}\,\mathrm{d}^3 p\,\mathrm{d}E. \tag{2}$$

With this Green's function, equation (1) can be put into the integral form

$$\psi(\boldsymbol{x}, t) = \psi_0(\boldsymbol{x}, t) - \mathrm{i}\int G(\boldsymbol{x}-\boldsymbol{x}', t-t')\,V(\boldsymbol{x}')\,\psi(\boldsymbol{x}', t')\,\mathrm{d}^3 x'\,\mathrm{d}t', \tag{3}$$

where $\psi_0(\boldsymbol{x}, t)$ is the free particle wavefunction.

Equation (3) can be generalized to give the following relativistic integral equation for two particles with an arbitrary interaction V,

$$\psi(x_1, x_2) = \psi_0(x_1, x_2) - \mathrm{i}\int G_1(x_1 - x_1')\,G_2(x_2 - x_2')\,V(x_1', x_2')\,\psi(x_1', x_2')\,\mathrm{d}^4 x_1'\,\mathrm{d}^4 x_2', \tag{4}$$

where $\psi_0(x_1, x_2)$ is the wavefunction for two free particles, $G_1(x_1 - x_2')$ and $G_2(x_2 - x_2')$ are the free particle Green's functions for particles 1 and 2 respectively, and

$$\mathrm{d}^4 x_1' = \mathrm{d}^3 x_1'\,\mathrm{d}t_1' \qquad \text{and} \qquad \mathrm{d}^4 x_2' = \mathrm{d}^3 x_2'\,\mathrm{d}t_2'.$$

Equation (4) is similar to the Bethe–Salpeter equation which was derived from theoretical field considerations. The only difference between the two equations is that the Green's function in (4) is taken to be a retarded function like the one used in equation (3) whereas the Feynman Green's function is used in the Bethe–Salpeter equation. Equation (4) is assumed to hold true for particles of any spin.

For two particles of specified spin, the Green's functions $G_1(x_1-x_1')$ and $G_2(x_2-x_2')$ in equation (4) are to be determined from the corresponding single-particle wave equations. These latter equations can be written in the Hamiltonian form

$$H(\nabla)\psi = \mathrm{i}\,\partial\psi/\partial t, \tag{5}$$

where ψ is the wavefunction and $H(\nabla)$ is the Hamiltonian operator which depends on the spin of the particle:

(i) For a spin-0 particle (see Appendix)

$$H(\nabla) = \nabla^2\sigma/2m - m\rho,$$

where

$$\sigma = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \qquad \rho = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

(ii) For a spin-$\frac{1}{2}$ particle

$$H(\nabla) = -\mathrm{i}\boldsymbol{\alpha}.\nabla + \beta m,$$

where $\boldsymbol{\alpha}$ and β are the usual matrices in Dirac theory.

(iii) For a spin-1 particle (see e.g. Weaver *et al.* 1964)

$$H(\nabla) = \beta E - \frac{\mathrm{i}\,2E^2(\boldsymbol{\alpha}.\nabla)}{2E^2-m^2} + \frac{2E\beta(\boldsymbol{\alpha}.\nabla)^2}{2E^2-m^2},$$

where

$$\alpha_i = \begin{bmatrix} 0 & -\mathrm{i}S_i \\ \mathrm{i}S_i & 0 \end{bmatrix}, \qquad \beta = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad E = (-\nabla^2 + m^2)^{\frac{1}{2}},$$

the S_i being the 3×3 spin-1 matrices.

The Green's function for equation (5) satisfies the equation

$$\left(H(\nabla') - \mathrm{i}\frac{\partial}{\partial t'}\right)G(x'-x) = \mathrm{i}\,\delta^4(x'-x)$$

and can be written as

$$G(x'-x) = \frac{\mathrm{i}}{(2\pi)^4}\int \frac{H(\mathrm{i}\boldsymbol{p}) + E}{\boldsymbol{p}^2 + m^2 - E^2}\exp\{\mathrm{i}p(x'-x)\}\,\mathrm{d}^3p\,\mathrm{d}E, \tag{6}$$

where we have used the fact that $H^2(\mathrm{i}\boldsymbol{p}) = \boldsymbol{p}^2 + m^2$ for any relativistic Hamiltonian

$H(\mathrm{i}\boldsymbol{p})$. Figure 1 shows the contour of integration in the E plane, in which the two poles are displaced an infinitesimal distance below the real axis. This corresponds to choosing $G(x'-x)$ to be retarded so that, for $t' < t$, we have $G(x'-x) = 0$. The Green's function therefore relates the wavefunction at a particular time to its value at an earlier time. After evaluating the E integral, we obtain from equation (6)

$$G(x'-x) = 0\,, \qquad t' < t\,, \tag{7a}$$

$$= -\tfrac{1}{2}\delta^3(\boldsymbol{x}'-\boldsymbol{x})\,, \qquad t' = t\,, \tag{7b}$$

$$= -\frac{1}{2(2\pi)^3}\int \mathrm{d}^3p \exp\{\mathrm{i}\boldsymbol{p}.(\boldsymbol{x}'-\boldsymbol{x})\}\left\{\mathscr{E}+\frac{1}{\mathscr{E}}+\frac{H(\mathrm{i}\boldsymbol{p})}{E}\left(\frac{1}{\mathscr{E}}-\mathscr{E}\right)\right\}, \quad t' > t\,, \tag{7c}$$

with

$$\mathscr{E} = \exp\{\mathrm{i}E(t'-t)\}\,, \qquad E = (p^2+m^2)^{\frac{1}{2}}\,,$$

and

$$\lim_{t'\to t+} G(x'-x) = -\delta^3(\boldsymbol{x}'-\boldsymbol{x})\,.$$

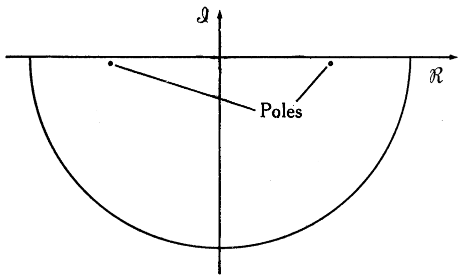

Fig. 1.—Contour of integration in the E plane for evaluation of the Green's function (6). The two poles are displaced an infinitesimal distance below the real axis.

For a general instantaneous potential, equation (4) becomes

$$\psi(x_1,x_2) = \psi_0(x_1,x_2) - \mathrm{i}\int G_1(x_1-x_1')\,G_2(x_2-x_2')\,V(\boldsymbol{x}_1',\boldsymbol{x}_2')\,\delta(t_1'-t_2') \times\psi(x_1',x_2')\,\mathrm{d}^4x_1'\,\mathrm{d}^4x_2'\,, \tag{8}$$

with the Green's functions given by the relations (7). The expression (8) is an integral equation for two particles, each separately obeying an equation of the form (5) in the absence of interactions. Operating on both sides of equation (8) by $H_1(\nabla_1) - \mathrm{i}\,\partial/\partial t_1$, we obtain

$$\begin{aligned}\left(H_1(\nabla_1) - \mathrm{i}\frac{\partial}{\partial t_1}\right)\psi(x_1,x_2) &= \int \delta(x_1-x_1')\,G_2(x_2-x_2')\,V(\boldsymbol{x}_1',\boldsymbol{x}_2')\,\delta(t_1'-t_2') \\ &\qquad \times\psi(x_1',x_2')\,\mathrm{d}^4x_1'\,\mathrm{d}^4x_2' \\ &= \int G_2(x_2-x_2')\,V(\boldsymbol{x}_1,\boldsymbol{x}_2')\,\delta(t_1-t_2')\,\psi(x_1,x_2')\,\mathrm{d}^4x_2' \\ &= \int G_2(\boldsymbol{x}_2-\boldsymbol{x}_2',t_2-t_1)\,V(\boldsymbol{x}_1,\boldsymbol{x}_2')\,\psi(\boldsymbol{x}_1,t_1;\boldsymbol{x}_2',t_1)\,\mathrm{d}^3x_2'\,. \end{aligned} \tag{9}$$

The corresponding equation that is obtained from (8) by operating on both sides by $H_2(\nabla_2) - \mathrm{i}\,\partial/\partial t_2$ is

$$\left(H_2(\nabla_2) - \mathrm{i}\frac{\partial}{\partial t_2}\right)\psi(x_1, x_2) = \int G_1(\boldsymbol{x}_1 - \boldsymbol{x}_1', t_1 - t_2)\, V(\boldsymbol{x}_1', \boldsymbol{x}_2)\, \psi(\boldsymbol{x}_1', t_2; \boldsymbol{x}_2, t_2)\, \mathrm{d}^3 x_1' . \quad (10)$$

Adding equations (9) and (10) and using the relations (7) and the fact that

$$\lim_{t_1, t_2 \to t} \left(\frac{\partial}{\partial t_1} + \frac{\partial}{\partial t_2}\right)\psi(\boldsymbol{x}_1, t_1; \boldsymbol{x}_2, t_2) = \frac{\partial \psi(\boldsymbol{x}_1, \boldsymbol{x}_2, t)}{\partial t},$$

we finally obtain

$$\left(H_1(\nabla_1) + H_2(\nabla_2) - \mathrm{i}\frac{\partial}{\partial t}\right)\psi(x_1, x_2, t) = -V(\boldsymbol{x}_1, \boldsymbol{x}_2)\,\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, t). \quad (11)$$

The result (11) is valid for all possibilities $t_1 > t_2$, $t_1 = t_2$, and $t_1 < t_2$.

For stationary state solutions with

$$\psi(\boldsymbol{x}_1, \boldsymbol{x}_2, t) = \psi(\boldsymbol{x}_1, \boldsymbol{x}_2)\exp(-\mathrm{i}Et),$$

equation (11) becomes

$$\{H_1(\nabla_1) + H_2(\nabla_2) + V(\boldsymbol{x}_1, \boldsymbol{x}_2)\}\,\psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = E\psi(\boldsymbol{x}_1, \boldsymbol{x}_2), \quad (12)$$

where E is the total energy of the system of two particles. This relation contains the sum of two Hamiltonians and has the same form as the Breit equation for spin-$\frac{1}{2}$ particles. In fact, the Breit equation is a special case of (12).

Two Free Particles

For the case of two free particles, with the interaction $V(\boldsymbol{x}_1, \boldsymbol{x}_2)$ set equal to zero, equation (12) becomes

$$\{H_1(\nabla_1) + H_2(\nabla_2)\}\,\psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = E\psi(\boldsymbol{x}_1, \boldsymbol{x}_2).$$

The solutions of this equation are

$$\psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \psi_1(\boldsymbol{x}_1)\,\psi_2(\boldsymbol{x}_2),$$

with

$$H_1(\nabla_1)\,\psi_1(\boldsymbol{x}_1) = E_1\,\psi_1(\boldsymbol{x}_1), \qquad H_2(\nabla_2)\,\psi_2(\boldsymbol{x}_2) = E_2\,\psi_2(\boldsymbol{x}_2),$$

and $E_1 + E_2 = E$.

Non-relativistic Limit

Assume in the non-relativistic limit that

$$E_i - m_i = \varepsilon_i \sim O(v^2), \qquad i = 1, 2,$$

$$V(\boldsymbol{x}_1, \boldsymbol{x}_2) \sim O(v^2),$$

and

$$\nabla V(\boldsymbol{x}_1, \boldsymbol{x}_2) \sim a^{-1} V(\boldsymbol{x}_1, \boldsymbol{x}_2) \sim O(v^3),$$

where we have used the same approximation as Schiff (1968), and a represents the linear dimensions of the system with

$$a^{-1} \sim p \sim mv.$$

Putting $t_2 = t_1$ in equation (9), $t_1 = t_2$ in equation (10), and writing

$$\psi(x_1, x_2) = \psi(\boldsymbol{x}_1, \boldsymbol{x}_2) \exp\{-\mathrm{i}(E_1 t_1 + E_2 t_2)\},$$

we have

$$H_i(\nabla_i) \psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \{E_i - \tfrac{1}{2} V(\boldsymbol{x}_1, \boldsymbol{x}_2)\} \psi(\boldsymbol{x}_1, \boldsymbol{x}_2). \tag{13}$$

Operating on both sides of equation (13) by $H_i(\nabla_i)$ then leads to the expression

$$H_i^2(\nabla_i) \psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \{E_i - \tfrac{1}{2} V(\boldsymbol{x}_1, \boldsymbol{x}_2)\}^2 \psi(\boldsymbol{x}_1, \boldsymbol{x}_2) + \dots, \tag{14}$$

The order of magnitude of the additional terms is $\nabla V(\boldsymbol{x}_1, \boldsymbol{x}_2) \sim O(v^3)$ or higher. Simplifying equation (14) and collecting terms up to order v^2, we obtain

$$-\nabla_i^2 \psi(\boldsymbol{x}_1, \boldsymbol{x}_2)/2m_i = \{\varepsilon_i - \tfrac{1}{2} V(\boldsymbol{x}_1, \boldsymbol{x}_2)\} \psi(\boldsymbol{x}_1, \boldsymbol{x}_2), \tag{15}$$

with $i = 1, 2$. Addition of both equations of (15) then gives the two-particle Schrödinger equation

$$\left(-\frac{\nabla_1^2}{2m_1} - \frac{\nabla_2^2}{2m_2} + V(\boldsymbol{x}_1, \boldsymbol{x}_2)\right) \psi(\boldsymbol{x}_1, \boldsymbol{x}_2) = \varepsilon \psi(\boldsymbol{x}_1, \boldsymbol{x}_2),$$

where $\varepsilon = \varepsilon_1 + \varepsilon_2$.

III. Conclusions

The two-particle wave equation derived here is simple in form and contains the sum of the two Hamiltonians for the particles rather than the product, as is the case in the Bethe–Salpeter differential equation. It is therefore of the same order as the single-particle equation. This new equation could be useful in the investigation of problems such as the quark–antiquark model and high energy scattering. The solution of the equation for some interactions of particles with spin 0, $\frac{1}{2}$, and 1 will be considered in subsequent papers.

IV. Acknowledgment

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V. References

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Appendix

The wave equation for a spin-0 particle is the Klein–Gordon equation

$$\partial^2\psi/\partial x_\mu\,\partial x_\mu = m^2\psi\,. \tag{A1}$$

By defining

$$\partial\psi/\partial x_\mu = m\phi_\mu\,, \tag{A2}$$

equation (A1) can be written as

$$\partial\phi_\mu/\partial x_\mu = m\psi\,, \tag{A3}$$

or in matrix form

$$\begin{bmatrix} 0 & \partial/\partial x_1 & \partial/\partial x_2 & \partial/\partial x_3 & \partial/\partial x_4 \\ \partial/\partial x_1 & 0 & 0 & 0 & 0 \\ \partial/\partial x_2 & 0 & 0 & 0 & 0 \\ \partial/\partial x_3 & 0 & 0 & 0 & 0 \\ \partial/\partial x_4 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix} = m \begin{bmatrix} \psi \\ \phi_1 \\ \phi_2 \\ \phi_3 \\ \phi_4 \end{bmatrix}.$$

Defining the vector $\boldsymbol{\phi}$ as $\phi_1 \boldsymbol{i} + \phi_2 \boldsymbol{j} + \phi_3 \boldsymbol{k}$, equation (A2) becomes

$$\partial\psi/\partial x_4 = m\phi_4\,, \qquad \nabla\psi = m\boldsymbol{\phi}\,, \tag{A4a, b}$$

and, from equation (A4b), equation (A3) can be expressed as

$$\partial\phi_4/\partial x_4 = m\psi - \nabla^2\psi/m\,. \tag{A5}$$

Equations (A4a) and (A5) can then be put in the matrix form

$$\frac{\partial}{\partial x_4}\begin{bmatrix} \psi \\ \phi_4 \end{bmatrix} = \begin{bmatrix} 0 & m \\ m - \nabla^2/m & 0 \end{bmatrix} \begin{bmatrix} \psi \\ \phi_4 \end{bmatrix}$$

or

$$\mathrm{i}\frac{\partial}{\partial t}\begin{bmatrix} \psi \\ \phi_4 \end{bmatrix} = \left(\frac{\nabla^2\sigma}{2m} - m\rho\right)\begin{bmatrix} \psi \\ \phi_4 \end{bmatrix},$$

where

$$\sigma = \begin{bmatrix} 0 & 0 \\ 2 & 0 \end{bmatrix}, \qquad \rho = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$