ELIMINATION OF PRIMITIVE DIVERGENTS FROM FIELD THEORY BY MEANS OF COMPLEX COUPLING CONSTANTS

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[Manuscript received 18 January 1973]

Abstract

LSZ iteration theory is extended to accommodate quantum fields coupled by complex constants, while retaining a positive metric and a Hermitian Hamiltonian. Interpolating and particle ($\sim$in, out) fields are linked by an operator $U(t)$ which is non-unitary, so that Haag's theorem may be avoided. It is shown that $U(t)$ may be rendered sufficiently well-behaved as $t \to \pm\infty$ to allow development of the iteration series for the τ function. For certain combinations of fields the coupling constants and masses can then be chosen so as to eliminate the primitive divergents from the iteration series for any S-matrix element. The theory is illustrated by two models: four spinor plus two scalar fields, and the electromagnetic plus several spinor fields. In the second model not every spinor field corresponds to a stable physical particle, and the *LSZ* formalism is extended to allow for this.

I. INTRODUCTION

An attempt is made here to formulate for coupled fields an *LSZ* iteration theory with a positive metric and a Hermitian Hamiltonian but in which the coupling constants may be complex so that, for certain combinations of fields, they may be chosen in such a way as to eliminate the primitive divergents arising in the iteration series for any S-matrix element. The idea of cancelling divergences by choosing the right set of fields and couplings is a well-known one. The present work attempts to exploit more fully the limits of the relations between the particle ($\sim$in, out) fields and the interpolating fields that exist in the *LSZ* formulation of field theory and its iteration ("perturbation") development, as presented, for example, by Bjorken and Drell (1965) and Roman (1969). By retaining a positive metric and a Hermitian Hamiltonian it is possible to avoid many of the problems that arise in the indefinite-metric electrodynamics of Lee and Wick (1970) or in the indefinite-metric *LSZ* theory of Taylor (1970).

The state space is a Hilbert space, spanned by vectors generated by particle field operators, and is of positive metric because of the forms postulated for the equal-time commutation relations. The Hamiltonian, which is a conventional free-field form in the particle operators and is Hermitian, is written in terms of the interpolating field operators with no restriction on the complexity of the coupling constants. In consequence, the products of interpolating field operators in the Hamiltonian in general lack their usual Hermitian properties and the "bare" and "interaction" Hamiltonians are not Hermitian; however, this is not an interaction-representation theory and the bare Hamiltonian is of no physical significance. The lack of

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Hermiticity of interpolating field operators means that certain familiar theorems on positive weight functions in spectral representations of two-point functions are no longer valid, as is discussed briefly in Section VI, and doubt is thus cast on the inevitability of some assumptions commonly made in analytic S-matrix theories.

The particle and interpolating fields are related through the two expressions for the Hamiltonian, by conventional *LSZ* asymptotic conditions and by an operator $U(t)$ which is not unitary, so that the theory is not necessarily subject to Haag's theorem. The central point of the paper is the demonstration that $U(t)$ can be made to be sufficiently well-behaved in the limits $t \to \pm\infty$ for a complete development of the iteration series of the τ function and so of the general S-matrix element to be possible. The iteration series leads to conventional Feynman diagrams and rules and to the usual primitive divergents. The theory is illustrated by application to two sets of coupled fields, and it is shown that the primitive divergents in each model may be rendered convergent by suitable choices of coupling constants and masses.

The second model used to illustrate the theory contains fields that do not correspond to stable physical particles and, in order to accommodate such fields, the usual *LSZ* concept of in-fields and out-fields is extended to that of A fields and B fields, which include the in- and out-fields of the stable particles. However, this extension is not essential to the central thesis of the paper, namely that complex coupling constants can be used to avoid primitive divergents. In addition, the second model contains the electromagnetic field, which has its own peculiar difficulties of quantization, but it is considered that these difficulties are tangential to the main argument.

This paper stops at the point at which, for a suitable combination of fields, the terms of the iteration series for any S-matrix element have all been rendered finite. The convergence of these series and the process of renormalization are not discussed. Ultimately, for the theory to be successful for some combinations of fields, renormalization must lead to (1) finite values for the mass renormalization constants $\delta\kappa_j$ etc. that appear in the interpolating-field form of the Hamiltonian, (2) nonzero finite values for the constants Z_n that appear in the asymptotic relations between particle and interpolating fields, and (3) a demonstration that the S-matrix operator, as expressed by its calculated matrix elements, is indeed unitary as postulated at the beginning.

II. *LSZ* Formalism and its Extension

The theory is a collision theory; bound states are not considered. It is based on the *LSZ* formalism as presented by Bjorken and Drell (1965) and Roman (1969) (hereinafter referred to as BD and R respectively) and, in order to avoid excessive length, frequent reference will be made to their work for sections of theory that can be carried over to the present development. For one class of field couplings, including the first example considered below in which every field corresponds to a stable physical particle, we rely on the usual *LSZ* theory as given by BD and R. For another class of couplings, exemplified by the second model system below in which several fermion fields and the photon field are coupled through terms of the form $\bar{\psi}_i^U \gamma^\mu \psi_j A_\mu$, so that only the least-massive fermion and the photon are stable and can appear as asymptotic physical particles, an extension of the *LSZ* formalism is made as follows.

We postulate a positive-metric Hilbert space of time-independent state-vectors, a conserved Hermitian energy–momentum P_μ, and a unique vacuum with $\langle 0|0\rangle = 1$, in the manner elaborated by R. We postulate that there exist two sets of particle fields

$$\phi_j^A(x) \equiv (\phi_s^{\text{in}}, \phi_d^A), \qquad \phi_j^B(x) \equiv (\phi_s^{\text{out}}, \phi_d^B),$$

where the fields for stable physical particles are labelled by s and are in-fields or out-fields, while other fields are labelled by d, and two sets of orthonormal wavepacket particle states $|A\rangle$ and $|B\rangle$, all linked by equations of the usual forms for in- and out-operators and in- and out-states: for example,

$$\mathrm{i}\partial_\mu \phi_j^{A,B} = [P_\mu, \phi^{A,B}], \tag{1a}$$

$$(\partial_\nu^2 + m^2)\phi_j^{A,B} = 0, \quad (\gamma^\mu \partial_\mu + \kappa_j)\psi_j^{A,B} = 0, \quad \partial_\nu^2 A_\mu^{\text{in}} = 0, \tag{1b}$$

$$|A\rangle = N \ldots b_p^{A\dagger}(\kappa_j) \ldots b_q^{\text{in}\dagger}(\kappa_1) \ldots a_{\gamma\mu}^{\text{in}\dagger} \ldots |0\rangle, \tag{1c}$$

and similar expressions, with $|B\rangle$ a product of B- and out-operators, and N being a normalization constant. The notation used here is $x_4 = ict$ and $\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2\delta_{\mu\nu}$ in the system $\hbar = c = 1$; the index j labels a variety of scalar, spinor, and other fields. These operators are of the usual free-field forms and possess their normal Hermitian and Lorentz transformation properties. It is assumed that the sets $|A\rangle$ and $|B\rangle$ each span the Hilbert space. Stable-particle operators are labelled "in" and "out" and the in-states and out-states $|\alpha\,\text{in}\rangle$ and $|\beta\,\text{out}\rangle$ are subsets of $|A\rangle$ and $|B\rangle$. We shall deal only with collisions between particles that are free at $t \to \pm\infty$, and it is basic to the *LSZ* conception that these particles be stable. The physical states, which are collections of isolated particles carrying individual quantum numbers at $t \to \pm\infty$, are to be identified with appropriate $|\text{in}\rangle$ and $|\text{out}\rangle$ vectors.

It is postulated that all A and B operators satisfy the appropriate conventional (positive-metric, free-field) equal-time commutation relations; for example, for the two spinor fields ψ_i^A, ψ_j^A

$$[\psi_{i(\sigma)}^{\dagger A}(x,t), \psi_{j(\rho)}^A(y,t)]_+ = \delta_{ij}\delta_{\sigma\rho}\delta^3(x-y), \tag{2}$$

and similarly for the B operators. These equal-time commutation relations, together with $\langle 0|0\rangle = 1$, are sufficient to ensure that the metric is positive definite, as is shown in detail by Nagy (1966). For the electromagnetic field, in order to ensure a positive metric and for simplicity, we adopt for the second model given below the quantization method used by BD (Chs. 14, 15). Any similarities that arise between results in this theory and those in certain indefinite-metric theories, in particular negative signs before certain propagators (see Lee and Wick 1969, 1970), can raise no doubts concerning the metric here. The present work does not address the question that if the electromagnetic field is part of the system then the vacuum is the limit point of a continuum of excitations, which may be in conflict with the finite normalization $\langle 0|0\rangle = 1$ (R, Section 7.1), nor does it consider the photon infrared divergence (see Jauch and Rohrlich 1955; BD).

It is further postulated that the Hilbert space contains a subspace K that is spanned by the in-states and also by the out-states, and has most of the properties ascribed to the complete state space in the treatment by R. It is clear that a vector representing a physical collision state must be in K. It is then postulated that there exists a proper unitary S-matrix operator within K such that for each stable-particle field

$$\phi^{\text{out}}(x) = S^{-1}\,\phi^{\text{in}}(x)\,S, \qquad S^{-1} = S^{\dagger}. \tag{3}$$

(Alternatively, it could be assumed that the ϕ^{in} and ϕ^{out} each constitute an irreducible operator ring within K and then, since they satisfy the same equal-time commutation relation, they are unitarily equivalent, which gives the relation (3); see R, Section 7.2.) This is the S-matrix operator of the theory. It follows that

$$\langle\alpha\,\text{out}| = \langle\alpha\,\text{in}|\,S, \qquad S_{\alpha\beta} = \langle\alpha\,\text{out}|\beta\,\text{in}\rangle = \langle\alpha\,\text{in}|\,S\,|\beta\,\text{in}\rangle.$$

The next step in defining a physical system is to specify a form for the energy–momentum operator P_μ. In what follows we need only consider explicitly the Hamiltonian. Here H and $P_{1,2,3}$ take their usual free-field forms, which, together with the normal Hermitian-conjugate transformation properties of the A or B fields, guarantee that P_μ is a Hermitian operator in the Hilbert space. The usual steps lead to the normal picture of particle states with eigenvalues of energy–momentum etc. In order to illustrate the present theory, two models of sets of coupled fields are considered. The first consists of four spinor and two real pseudoscalar fields possessing the Hamiltonian

$$H_1 = \int \mathrm{d}^3x : \left(\sum_{j=1}^{4} \bar{\psi}_j^{\text{in}}(\gamma^k\,\partial_k + \kappa_j)\psi_j^{\text{in}} + \sum_{i=1}^{2} \{(\partial_k\,\phi_i^{\text{in}})^2 + (\dot{\phi}_i^{\text{in}})^2 + m_i^2(\phi_i^{\text{in}})^2\} \right) :, \tag{4}$$

where k is summed over $(1,2,3)$. The second model system consists of N spinor fields plus the photon field for which, following the treatment of BD, we write

$$H_2 = \int \mathrm{d}^3x : \left(\sum_{j=1}^{N} \bar{\psi}_j^{A}(\gamma^k\,\partial_k + \kappa_j)\psi_j^{A} + \tfrac{1}{2}(\partial_k\,A_l^{\text{in}} - \partial_l\,A_k^{\text{in}})^2 + \tfrac{1}{2}\dot{A}_k^{\text{in}}\,\dot{A}_k^{\text{in}} \right) : \tag{5}$$

with $k, l = 1, 2, 3$. In the first model, the interactions specified below by the Hamiltonian (13) plus mass restrictions are such that each ϕ_i or ψ_j corresponds to a stable physical particle, so that the usual *LSZ* theory applies without the proposed extension. In the second model, the interactions appearing in the Hamiltonian (14) permit only the least massive spinor field to correspond to a stable fermion. It is useful to introduce the notation $H = H_{\text{par}}(\phi^A)$ to indicate specifically the particle form of H and the operator arguments.

It follows in the usual way from the preceding assumptions that the propagators of, for example, real scalar, spinor, and photon fields are given by

$$\begin{aligned}\langle 0|T(\phi^{\text{in}}(x), \phi^{\text{in}}(y))|0\rangle &= \mathrm{i}\Delta_{\mathrm{F}}(x-y, m) \\ &= \lim_{\varepsilon\to+0} \mathrm{i} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\exp\{\mathrm{i}\boldsymbol{k}\,.\,(\boldsymbol{x}-\boldsymbol{y})\}}{k^2 - m^2 + \mathrm{i}\varepsilon},\end{aligned} \tag{6}$$

$$S_{\mathrm{F}}(x-y,\kappa) = \lim_{\varepsilon\to+0} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\exp\{\mathrm{i}\boldsymbol{p}.(\boldsymbol{x}-\boldsymbol{y})\}(\mathrm{i}\gamma^{\mu}p_{\mu}-\kappa)}{p^2-\kappa^2+\mathrm{i}\varepsilon}, \tag{7}$$

$$D_{\mathrm{F}(\mu\nu)}(x-y) = \lim_{\varepsilon\to+0,\lambda\to 0} \int \frac{\mathrm{d}^4k}{(2\pi)^4} \frac{\exp\{\mathrm{i}\boldsymbol{k}.(\boldsymbol{x}-\boldsymbol{y})\}}{k^2-\mathrm{i}\varepsilon+\lambda^2}(\delta_{\mu\nu}+X+Y). \tag{8}$$

The terms X and Y arise in the BD treatment (Section 14.6) of the electromagnetic field that is being adopted here, and X contributes zero and Y cancels the $:\rho::\rho:$ term in the Hamiltonian (14) below in all S-matrix element calculations, so that X, Y, and $:\rho::\rho:$ may in fact be omitted. The BD proofs of cancellation (Section 17.9) are valid in the present theory. In the BD treatment $A_4(x)$ is not an operator, so that in equations (9) and elsewhere below only the $A_k(x)$, $k = 1,2,3$ appear; however, the covariant propagator with X and Y omitted may be used in all calculations. The mass λ is inserted in order to avoid the infrared divergence and is ultimately to be made zero.

We now proceed to give physical content to the theory by postulating that there exist interpolating fields $\phi, \psi, A_{\mu}, \ldots$ and a scalar operator $U(t)$ such that the interpolating fields are related to the A and B fields through $U(t)$, through asymptotic conditions, and through an expression for the Hamiltonian in terms of the interpolating fields, as follows:

(i) $U(t)$

For the fields of the two examples given, it is postulated that

$$\phi_j = U^{-1}\phi_j^A U, \qquad \dot{\phi}_j = U^{-1}\dot{\phi}_j^A U, \tag{9a}$$

$$A_k = U^{-1}A_k^{\mathrm{in}} U, \qquad \dot{A}_k = U^{-1}\dot{A}_k^{\mathrm{in}} U \qquad (k = 1,2,3), \tag{9b}$$

$$\psi_j = U^{-1}\psi_j^A U, \tag{9c}$$

and we define the operators

$$\psi_j^{\dagger U} \equiv U^{-1}\psi_j^{A\dagger} U = (U^{\dagger}U)^{-1}\psi_j^{\dagger} U^{\dagger}U, \qquad \bar{\psi}_j^U \equiv \psi_j^{\dagger U}\gamma^4 = U^{-1}\overline{\psi_j^A} U. \tag{10}$$

If U were unitary, $\psi_j^{\dagger U}$ and $\bar{\psi}_j^U$ would reduce to $\psi_j^{\dagger}$ and $\bar{\psi}_j$. It may be noted that $\bar{\psi}_i^U, \psi_j$ rather than $\bar{\psi}_i, \psi_j$ satisfy the equal-time commutation relation (2) for the A operators.

(ii) *Asymptotic Conditions*

Weak asymptotic conditions of the *LSZ* type as given by BD are postulated: thus for the A operators of our examples

$$\lim_{t\to-\infty} \langle\alpha|\phi_{(f)}(t)|\beta\rangle = Z_1^{\frac{1}{2}}\langle\alpha|\phi_{(f)}^{\mathrm{in}}|\beta\rangle, \tag{11a}$$

$$\lim_{t\to-\infty} \langle\alpha|\psi_{(w)}(t)|\beta\rangle = Z_2^{\frac{1}{2}}\langle\alpha|\psi_{(w)}^{A}|\beta\rangle, \tag{11b}$$

$$\lim_{t\to-\infty} \langle\alpha|\psi_{(w)}^{\dagger U}(t)|\beta\rangle = Z_2^{\frac{1}{2}*}\langle\alpha|\psi_{(w)}^{A\dagger}|\beta\rangle, \tag{11c}$$

$$\lim_{t\to-\infty} \langle\alpha|A_{k(F)}(t)|\beta\rangle = Z_3^{\frac{1}{2}}\langle\alpha|A_{k(F)}^{\mathrm{in}}|\beta\rangle, \tag{11d}$$

and similarly for the B operators in the $t \to +\infty$ limit; where, as usual, $|\alpha\rangle$ and $|\beta\rangle$ are arbitrary normalizable states and the operators $\phi(x)$, $\psi(x)$, and $A_k(x)$ are smeared over space-like regions by normalizable scalar $f(x,t)$, spinor $w(x,t)$, and vector $F_\mu(x,t)$ functions satisfying appropriate wave equations (see BD), such as, for the jth spinor field,

$$\psi_{j(w)}(t) = \int d^3x\, w_j^\dagger(x,t)\,\psi_j(x,t), \qquad (\gamma^\mu \partial_\mu + \kappa_j)w_j = 0. \tag{12}$$

It is assumed that the constants Z_n are nonzero and finite. Ultimately, as a result of carrying renormalization through, this is one of the results that would be required to emerge for any combination of fields for which the theory were valid.

(iii) *Hamiltonian in Interpolating Fields*

Written in terms of the interpolating fields, the Hamiltonian takes a form similar to that in a normal *LSZ* theory, except that $\bar{\psi}_j$ is replaced by $\bar{\psi}_j^U$ and a similar replacement would be needed if a complex scalar field were involved. For the two examples here it is assumed that

$$H_1 = \int d^3x : \Big(\sum_{j=1}^{4} \bar{\psi}_j^U(\gamma^k \partial_k + \kappa_j + \delta\kappa_j)\psi_j + \sum_{i=1}^{2} \{(\partial_k \phi_i)^2 + \dot{\phi}_i^2 + (m_i^2 + \delta m_i^2)\phi_i^2\}$$
$$+ \sum_j \mathrm{i}g\,\bar{\psi}_j^U \gamma^5 \psi_j(\phi_1 + \mathrm{i}\phi_2)\Big) :, \tag{13}$$

$$H_2 = \int d^3x : \Big(\sum_j \bar{\psi}_j^U(\gamma^k \partial_k + \kappa_j)\psi_j + \tfrac{1}{2}(\partial_k A_l - \partial_l A_k)^2 + \tfrac{1}{2}\dot{A}_k \dot{A}_k$$
$$+ \sum_{i,j} (-\mathrm{i}\bar{\psi}_i^U J_{ij}^k \psi_j A_k + \bar{\psi}_i^U \delta\kappa_{ij}\psi_j)\Big) :$$
$$+ \iint \frac{d^3x\, d^3y}{|x-y|} :\rho(x,t)::\rho(y,t):, \tag{14}$$

where

$$J_{ij}^\mu = e_{ij}\gamma^\mu + g_{ij}\gamma^5\gamma^\mu, \qquad \rho(x) = \sum_{i,j} \bar{\psi}_i^U J_{ij}^\mu \psi_j \eta_\mu, \tag{15}$$

and $\eta = (0,0,0,1)$ in the frame used for quantization, following the BD method for the photon field. The quantities δm_i^2, $\delta\kappa_j$, and $\delta\kappa_{ij}$ are mass renormalization constants ($\delta\kappa_j$ may need to contain a γ^5 as well as a unit matrix component). As already stated, the $:\rho::\rho:$ term may be omitted from further consideration. It is useful to introduce the notation $H_{\mathrm{pol}}(\phi)$ for the interpolating-field expression for H.

The coupling constants are permitted to be complex, without restriction. Since the Hamiltonian, by construction from the A or B fields, is Hermitian, it follows that not every term in the Hamiltonian written in the interpolating fields can be separately Hermitian. If an interaction-representation theory were attempted on the basis of Hamiltonians such as (13) or (14) with complex coupling constants then either the total or the bare Hamiltonian would be non-Hermitian, with considerable problems of physical interpretation arising in either case (cf. Lee and Wick 1969, 1970). In an *LSZ* theory the bare Hamiltonian has no direct physical interpretation. It follows from the non-Hermiticity of at least some of the terms in H_{pol}, together with the relations (9) and (10), that U cannot be unitary for complex coupling constants.

The reduction formula for the S-matrix can be derived in the usual way immediately following the postulates (iii) above, but with $\bar{\psi}^U$ replacing $\bar{\psi}$ throughout and with similar changes made for any other stable-particle field that appears together with its Hermitian conjugate in the Hamiltonian. The general expression is too cumbersome to be written here usefully; examples are given in BD and R. For a single ϕ field with m incoming particles of wavepacket parameters q_j (ultimately momenta) and n outgoing particles of parameters p_i, $p_i \neq q_j$ for all i,j, we have

$$\begin{aligned} S_{pq} &= \langle p_1 \dots p_n \text{ out} \mid q_1 \dots q_m \text{ in}\rangle \\ &= (\mathrm{i}/Z^{\frac{1}{2}})^{m+n} \prod_i \int \mathrm{d}^4 x_i \\ &\quad \times \prod_j \int \mathrm{d}^4 y_j\, f_{q_i}(x_i)\,(\overrightarrow{\Box_i + m^2})\, \tau(y_1 \dots y_n, x_1 \dots x_m)\,(\overleftarrow{\Box_j + m^2})\, f^*_{p_j}(y_j), \end{aligned}$$

where the wavepackets f_p are an orthonormal set of solutions of the Klein–Gordon equation with mass m, and the τ function is

$$\tau(y_1 \dots y_n, x_1 \dots x_m) = \langle 0|T(\phi(y_1) \dots \phi(y_n), \phi(x_1) \dots \phi(x_m))|0\rangle,$$

T being the time-ordering operator. In the general case

$$\tau(x_1 \dots x_n) = \langle 0|T(\phi_1(x_1)\,\phi_2(x_2) \dots \phi_n(x_n))|0\rangle, \tag{16}$$

where each ϕ_n represents one of the interpolating-field operators $\bar{\psi}^U_s$, ψ_s, ϕ_s, A_μ, ... that correspond to the stable particles.

III. The Operator $U(t)$

From the equation of motion (21) for $U(t)$ derived below, it follows that $U(t)$ is not in general unitary for complex coupling constants, as already observed. Thus the theory may be free of Haag's theorem, which is outlined at the end of Section VI. It is commonly concluded from equations (20) and (21) that the norm of $U(t)\,|\alpha\rangle$, with $|\alpha\rangle$ a general state, would diverge to zero or infinity in the limits $t \to \pm\infty$, thus making it difficult to set up a theory that would allow calculation of the matrix elements. However, we will see below that $U(t)\,|0\rangle$ and $\langle 0|\, U^{-1}(t)$ can be made sufficiently well-behaved as $t \to \pm\infty$ by an appropriate choice of $g(t)$ in equation (21) for the usual iteration development of the τ function to be possible. Of course, the behaviour of $U(t)$ as $t \to \pm\infty$ is also governed by the relations (9) and (11).

Following BD (Section 17.2), the basic law of motion

$$\dot{\phi} = i[H, \phi] \tag{17}$$

is applied to A fields and interpolating fields and their conjugate momenta π ($\bar{\psi}^U_j$, $\dot{\phi}$, and $\dot{A}_k$ for the interpolating fields in our examples) to give

$$\dot{\phi}^A = \mathrm{i}[H_{\text{par}}(\phi^A, \pi^A), \phi^A], \qquad \dot{\phi} = \mathrm{i}[H_{\text{pol}}(\phi, \pi), \phi],$$

and similarly for $\dot{\pi}^A$ and $\dot{\pi}$. With the postulates (9), these equations lead to

$$\dot{\phi}^A = \partial[U(t)\,\phi(x,t)\,U^{-1}(t)]/\partial t = \dot{\phi}^A + [\dot{U}U^{-1} + \mathrm{i}H_\mathrm{I}(\phi^A, \pi^A),\, \phi^A], \tag{18}$$

$$\dot{\pi}^A = \dot{\pi}^A + [\dot{U}U^{-1} + \mathrm{i}H_\mathrm{I}(\phi^A, \pi^A), \pi^A], \tag{19}$$

where

$$H_\mathrm{I}(\phi^A, \pi^A) = H_\mathrm{pol}(\phi^A, \pi^A) - H_\mathrm{par}(\phi^A, \pi^A) = H_a + \mathrm{i}H_b, \tag{20}$$

in which $H_\mathrm{pol}(\phi^A, \pi^A)$ is the interpolating field form of H but with the ϕ and π replaced directly by A operators, and H_a and H_b are Hermitian. From the relations (18) and (19) it follows that

$$\mathrm{i}\,\partial U/\partial t = [H_\mathrm{I}(t) + g(t)]U, \tag{21}$$

where $g(t)$ commutes with all ϕ^A and π^A but is otherwise unrestricted, and so is an arbitrary c-number function of the time. From equations (20) and (21) it follows that U is not unitary when the coupling constants are complex, i.e. when H_b is nonzero.

On writing

$$W = UA\exp\left(\mathrm{i}\int_{t_0}^{t} g(t')\,\mathrm{d}t'\right) = G(t)\,U(t), \tag{22}$$

where A is an arbitrary c-number constant, equation (21) becomes

$$\mathrm{i}\,\partial W/\partial t = H_\mathrm{I}\,W. \tag{23}$$

Clearly $U|\alpha\rangle$ and $W|\alpha\rangle$, with $|\alpha\rangle$ arbitrary, are in the same direction in the Hilbert space and so describe the same state. Choosing

$$g = g_1 = -\mathrm{i}\langle 0|U^\dagger H_b U|0\rangle/\langle 0|U^\dagger U|0\rangle \equiv -\mathrm{i}f_1(t) \tag{24}$$

$$= -\mathrm{i}\langle 0|W^\dagger H_b W|0\rangle/\langle 0|W^\dagger W|0\rangle, \tag{25}$$

which defines the real function $f_1(t)$, it follows from equation (21) that

$$\partial_t\langle 0|U^\dagger U|0\rangle = 0.$$

Since equations (21) and (24) are unchanged by $U \to kU$ and k can be absorbed into A, we can define $U = U_1(t)$ to be the solution of

$$\mathrm{i}\,\partial U_1/\partial t = [H_\mathrm{I} - \mathrm{i}\langle 0|U_1^\dagger H_b U_1|0\rangle]U_1, \tag{26}$$

$$\langle 0|U_1^\dagger(t)\,U_1(t)|0\rangle = 1, \tag{27}$$

so that $U_1(t)|0\rangle$ is of unit norm for all t. The exponential factor in equation (22), with $g = g_1$, cancels any exponential increase or decrease in the norm of $W|0\rangle$ due to $\mathrm{i}H_b$ in (23). Equation (26) is essentially only nonlinear in the c-number function $G_1(t)$, since it transforms to equation (23), which is linear in W, and g_1 and G_1 are given explicitly by equations (22), (23), and (25).

A second well-behaved solution to equation (21) is obtained by taking

$$g = g_2 = -\mathrm{i}f_2 = -\tfrac{1}{2}\langle 0|H_\mathrm{I}\, U_2\, U_2^\dagger - U_2\, U_2^\dagger\, H_\mathrm{I}^\dagger|0\rangle / \langle 0|U_2\, U_2^\dagger|0\rangle\,, \tag{28}$$

where U_2 is the solution of (21) for $g = g_2$. As before, f_2 is unchanged if U_2 is replaced by W or U_1 in equation (28), so that $g_2(t)$ is explicit, $\langle 0|U_2\, U_2^\dagger|0\rangle$ is constant, and A is chosen to give

$$\langle 0|U_2\, U_2^\dagger|0\rangle = 1\,, \tag{29}$$

so that $U_2(t)$ satisfies equation (26) with f_1 replaced by f_2. Clearly the postulates (9) and definitions (10) are unaffected by the choice of $g(t)$.

We now show that

$$\lim_{t\to-\infty} \quad U_1(t)\,|0\rangle = \lambda_-\,|0\rangle\,, \qquad \lambda_-^*\,\lambda_- = 1\,, \tag{30}$$

and

$$\lim_{t\to+\infty} \quad \langle 0|U_1^{-1}(t) = \lambda_+^{-1}\langle 0|\,, \qquad \lambda_+^*\,\lambda_+ = 1\,. \tag{31}$$

These results enable the iteration series development of the τ function and so of the general S-matrix element to be carried through. For an A state $|p,\alpha,A\rangle$ containing a fermion of momentum p plus any other particles α, and for $|\beta\rangle$ an arbitrary normalizable state, it follows that

$$\begin{aligned}\langle p\alpha, A|U_2(t)|\beta\rangle &= \langle \alpha, A|b^A(p)\, U_2(t)|\beta\rangle \\ &= \left\langle \alpha, A \left| \int \mathrm{d}^3x\, \tilde{u}_p(x,t')\,\psi^A(x,t')\, U_2(t) \right| \beta \right\rangle \\ &= \left\langle \alpha, A \left| \int \mathrm{d}^3x\, \tilde{u}_p\, U_2(t')\,\psi(x,t')\, U_2^{-1}(t')\, U_2(t) \right| \beta \right\rangle \\ &= \left\langle \alpha, A \left| U_2(t) \int \mathrm{d}^3x\, \tilde{u}_p(x,t)\,\psi(x,t) \right| \beta \right\rangle,\end{aligned} \tag{32}$$

on using the relations (9) and then taking $t' = t$, where u_p is the spinor coefficient of the annihilation operator b_p^A in the wavepacket expansion of ψ^A (see R; BD). Use of the asymptotic condition (11b) then yields

$$\lim_{t\to-\infty} \langle p\alpha, A|U_2(t)|\beta\rangle = Z_2^{\frac{1}{2}}\Bigl(\lim_{t\to-\infty} \langle\alpha|\, U_2(t)\Bigr) b_p^A\,|\beta\rangle\,. \tag{33}$$

Similar developments follow when p is taken to be a photon or boson (BD, Section 17.3), provided there are no derivative couplings in H_I.

Consider first a one-particle state $|p,A\rangle$. By equation (29) the right-hand side and hence the left-hand side of (33) do not diverge to infinity as $t\to-\infty$. Since $|\beta\rangle$ is in an arbitrary direction, it follows that $\langle p|U_2$ possesses nondivergent components (they might oscillate finitely) along any set of basis vectors in the Hilbert space as $t\to-\infty$. Further, because the right-hand side of equation (33) cannot be zero for all $|\beta\rangle$, in view of the condition (29) and the assumption that Z_j is nonzero, the vector $\langle p|U_2(-\infty)$ cannot be the null vector for a photon or a boson, nor for a fermion unless it happened to be the case that

$$\langle 0|U_2(-\infty) = \langle p'|,$$

p' being some specific p; but the existence of such a special p' would conflict with the relativistic covariance of the basic framework and so is impossible. Taking $|\beta\rangle = |0\rangle$, it follows from equation (33) that

$$\langle p, A|U_2(-\infty)|0\rangle = 0,$$

and hence that $\langle p, A|U_2(-\infty)$, being not null, must be orthogonal to $\langle 0|$. Since $\langle p|U_1$ is parallel to $\langle p|U_2$ for all t, as discussed above, it is parallel in the limit $t \to -\infty$ and hence

$$\langle p|U_1(-\infty)|0\rangle = 0.$$

Therefore $U_1(-\infty)|0\rangle$ is orthogonal to $|p, A\rangle$ since it is of unit norm by equation (27).

The preceding argument can be repeated with $\langle p\alpha, A| = \langle p, q, A|$, a two-particle state, to show that $\langle p, q, A|U_2(-\infty)$ is a non-null vector of nondivergent norm, since now the right-hand side of (33) is again nondivergent for arbitrary $|\beta\rangle$ with $\langle\alpha| = \langle p|$, as proved above, whence by similar steps $U_1(-\infty)|0\rangle$ is orthogonal to $|p,q,A\rangle$. Repetition of the argument ultimately shows that $U_1(-\infty)|0\rangle$ is orthogonal to all A states except $|0\rangle$, whence, using the condition (27), the result (30) follows.

Following equation (32), the limit $t \to +\infty$ can be taken in place of $t \to -\infty$, to give

$$\lim_{t\to+\infty} \langle p\alpha, A|U_2(t)|\beta\rangle = \lim_{t\to+\infty} Z_2^{\frac{1}{2}}\langle\alpha, A|U_2(t)\, b_p^B|\beta\rangle.$$

The above argument may be paraphrased, working still in a sequence of A states for the $\langle p\alpha|$. It follows through because both the $|A\rangle$ and $|B\rangle$ vectors span the Hilbert space and $|0\text{ out}\rangle = |0\text{ in}\rangle = |0\rangle$. The result is

$$U_1(\infty)\,|0\rangle = \lambda_+\,|0\rangle, \qquad \lambda_+^*\,\lambda_+ = 1. \tag{34}$$

A similar argument holds for $\langle 0|\,U_1(\infty)$, beginning from, in the fermion case,

$$\begin{aligned}\langle\alpha|b_p^{A\dagger}\,U_1(t)|\beta\rangle &= \left\langle\alpha\left|\int d^3x\,\psi^{A\dagger}(x,t')\,u_p(x,t')\,U_1(t)\right|\beta\right\rangle \\ &= \left\langle\alpha\left|\int d^3x\,U_1(t')\,\psi_j^{\dagger U}(x,t')\,U_1^{-1}(t')\,u_p(x,t')\,U_1(t)\right|\beta\right\rangle,\end{aligned}$$

using the definitions (10). Taking $t = t'$ and using equations (11) for $t \to +\infty$, it follows that

$$\lim_{t\to\infty}\langle\alpha|U_1(t)|p\beta, B\rangle = \lim_{t\to\infty} Z^{-\frac{1}{2}*}\langle\alpha|b_p^{A\dagger}\,U_1(t)|\beta\rangle. \tag{35}$$

As this derivation may be carried through in reverse, it is legitimate to take $\langle\alpha| = \langle 0|$.

We first take $|\beta\rangle = |0\rangle$ and then, since $\langle\alpha|$ is in an arbitrary direction, it follows from equation (35) with (34) that $U_1(\infty)\,|p,B\rangle$ is non-null and of nondivergent norm. Taking $\langle\alpha| = \langle 0|$ in equation (35), we then have

$$\langle 0|U_1(\infty)|p,B\rangle = 0\,,$$

so that, since it is non-null, $U_1(\infty)\,|p,B\rangle$ is orthogonal to $|0\rangle$. Because $U_2|p\rangle$ is parallel to $U_1|p\rangle$, we also have

$$\langle 0|U_2(+\infty)|p,B\rangle = 0\,.$$

But $\langle 0|U_2$ is of unit norm by the condition (29) and therefore $\langle 0|U_2$ and so $\langle 0|U_1(\infty)$ are orthogonal to $|p,B\rangle$. This cycle of argument may be repeated, taking $|\beta\rangle = |q,B\rangle$ in equation (35) to show that $\langle 0|U_1(\infty)$ is orthogonal to $|p,q,B\rangle$, and so on. Similar arguments go through for boson and photon operators. We may conclude that $\langle 0|U_1(+\infty)$ is orthogonal to all states containing particles, so that

$$\langle 0|U_1(+\infty) = \mu\langle 0| = \lambda_+\langle 0|\,, \tag{36}$$

where the second equality follows from evaluating $\langle 0|U_1(+\infty)|0\rangle$ using (34). Evaluating $\langle 0|\,U(t)\,U^{-1}(t)$ in the limit of $t\to+\infty$ then gives the relations (31).

It may be noted that, from equations (24), (28), (30), and (31),

$$\lim_{t\to\pm\infty} f_1(t) = \lim_{t\to\pm\infty} f_2(t) = \lim_{t\to\pm\infty} \langle 0|H_b(t)|0\rangle\,, \tag{37}$$

and from equation (22)

$$G_{1,2}(t) = A_{1,2}\exp\left(\int_{t_0}^{t} f_{1,2}(t')\,\mathrm{d}t'\right),$$

so that G_1 and G_2 behave similarly as $t\to\pm\infty$. If, as is expected when renormalization is carried through, the constants $\delta\kappa_J$ etc. are found to be real, then contributions to H_b could come only from the interaction terms between fields. For interactions involving three field operators, as in the two models considered here, it would follow at once that

$$\langle 0|H_b(t)|0\rangle = 0\,,$$

and so in these cases f_1 and f_2 would vanish as $t\to\pm\infty$.

We now define

$$U(t) \equiv U_1(t) \tag{38}$$

so that equations (30) and (31) hold for $U(t)$.

IV. The Iteration Series and Feynman Diagrams

We paraphrase the usual (BD) heuristic development of the operator

$$V(t,t') \equiv U(t)\,U^{-1}(t')\,, \tag{39}$$

by using equation (26), into the iteration series

$$V(t,t') = T\left(\exp\left\{-\mathrm{i}\int_{t'}^{t} H_c(t_1)\,\mathrm{d}t_1\right\}\right) \tag{40}$$

where

$$H_c \equiv H_\mathrm{I} - \mathrm{i}\langle 0|U^\dagger H_b U|0\rangle \tag{41}$$

and T is the time-ordering operator.

Using the relations (9), (10), and (39), the τ function (16) can be written, as in BD,

$$\tau(x_1,\ldots,x_n) = \langle 0|\,T(U^{-1}(t)\,V(t,t_1)$$
$$\times\,\phi_1^{\mathrm{in}}(x_1)\,V(t_1,t_2)\,\phi_2^{\mathrm{in}}(x_2)\ldots\phi_n^{\mathrm{in}}(x_n)\,V(t_n,-t)\,U(-t))\,|0\rangle.$$

In the limit $t\to+\infty$, $U^{-1}(t)$ and $U(-t)$ may be extracted from the time-ordered product, and use of equations (30), (31), (39), and (40) yields

$$\tau(x_1,\ldots,x_n) = \lim_{t\to\infty}\frac{\lambda_-}{\lambda_+}\left\langle 0\left|T\left(\phi_1^{\mathrm{in}}(x_1)\ldots\phi_n^{\mathrm{in}}(x_n)\exp\left\{-\mathrm{i}\int_{-t}^{t} H_c(t')\,\mathrm{d}t'\right\}\right)\right|0\right\rangle$$

$$= \lim_{t\to\infty}\frac{\left\langle 0\left|T\left(\phi_1^{\mathrm{in}}(x_1)\ldots\phi_n^{\mathrm{in}}(x_n)\exp\left\{-\mathrm{i}\int_{-t}^{t} H_c(t')\,\mathrm{d}t'\right\}\right)\right|0\right\rangle}{\left\langle 0\left|T\left(\exp\left\{-\mathrm{i}\int_{-t}^{t} H_c(t')\,\mathrm{d}t'\right\}\right)\right|0\right\rangle}. \tag{42}$$

The exponential factors in $g_1(t)$ are now cancelled between numerator and denominator so that $H_c(t)$ is replaced by $H_\mathrm{I}(t)$. Following the usual procedure, the limit $t\to+\infty$ is taken separately in the numerator and denominator. Writing

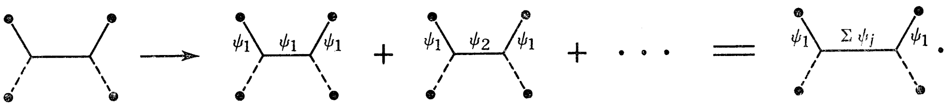
$$\int_{-\infty}^{\infty}\mathrm{d}t\,H_\mathrm{I}(t) \equiv \int \mathrm{d}^4x\,\mathscr{H}_\mathrm{I}(x),$$

the time-ordered products of in-operators in the numerator and denominator of equation (42) are developed using Wick's theorem into sums over products of propagators and normal-ordered operators in the usual way, to yield Feynman diagrams and rules of the conventional kind. It can then be shown by the standard procedure (BD, Section 17.6) that the denominator exactly cancels the disconnected vacuum bubbles that appear in the expansion of the numerator, so that finally

$$\tau = \sum_\sigma \frac{(-1)^\sigma}{\sigma!}\int\ldots\int \mathrm{d}^4y_1\ldots\mathrm{d}^4y_\sigma$$
$$\times\,\langle 0|T(\phi_1^{\mathrm{in}}(x_1)\ldots\phi_n^{\mathrm{in}}(x_n)\,\mathscr{H}_\mathrm{I}(y_1)\ldots\mathscr{H}_\mathrm{I}(y_\sigma))|0\rangle_{\mathrm{con}}, \tag{43}$$

where the subscript "con" indicates that only connected graphs are to be included.

The propagators of some fields have already been given in equations (6), (7), and (8). The external lines, terminating in $x_1,\ldots,x_n$ and associated with the $\phi_j^{\mathrm{in}}(x_j)$, represent stable particles whereas the internal lines represent all the A particles. In the second model described above the external lines represent photons or ψ_1 fermions, and each usual quantum-electrodynamics diagram is replaced by a set of diagrams in which the internal lines represent the ψ_j particles in turn; e.g. for second-order Compton scattering we have



The propagators are all essentially positive, corresponding to the positive metric. As discussed in Section VI below, the residues at the particle poles of the two-point functions $\langle 0|T(\phi(x)\,\phi(y))|0\rangle$ need not all be positive, because of the complexity of the coupling constants.

The primitive divergents of the iteration series appear in their usual forms (Schweber 1961; BD), but in the next section it is shown how they may be eliminated from the two illustrative models by appropriate choices of coupling constants and masses. The further questions of determining for what combinations of fields the resulting S-matrix series are convergent and, after renormalization to fix the mass constants, are consistent with a unitary S-matrix are not tackled in this paper.

V. Models Free of Primitive Divergents

(a) *Spinor and Pseudoscalar Fields*

For the system defined by equations (4) and (13), the interaction Hamiltonian (20) is

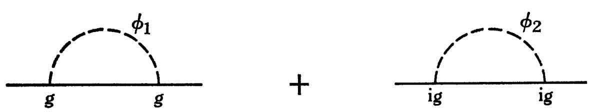
$$H_{\mathrm{I}} = \int \mathrm{d}^3x :\sum \{g_j \bar{\psi}_j^{\mathrm{in}} \gamma^5 \psi_j^{\mathrm{in}}(\phi_1^{\mathrm{in}} + \mathrm{i}\phi_2^{\mathrm{in}}) + \delta\kappa_j \bar{\psi}_j^{\mathrm{in}} \psi_j^{\mathrm{in}} + \delta m_i^2 (\phi_i^{\mathrm{in}})^2\} :. \qquad (44)$$

The topology of the diagrams plus energy–momentum conservation show that each fermion is stable. We assume that the boson masses m_1 and m_2 and the mass κ_1 of the lightest fermion satisfy

$$m_1 < 2\kappa_1\,, \qquad m_2 < 2\kappa_1\,, \qquad \tfrac{1}{2} < m_1/m_2 < 2\,,$$

so that the mesons also are stable. Thus only in- and out-operators appear in equation (4) and the extension of the *LSZ* framework to include unstable particles is not needed.

Each of the four fermion self-energy integrals is convergent, without the imposition of additional conditions. Each can be represented by

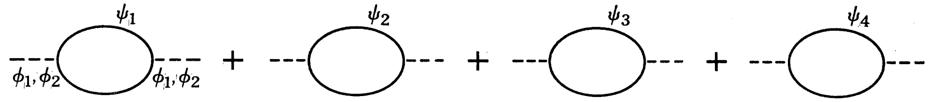


and the integral over the meson momentum takes the form

$$\int \mathrm{d}^4k\, \frac{\gamma^5[\gamma^\mu(p-k)_\mu - \kappa]\gamma^5}{(p-k)^2 + \kappa^2 + \mathrm{i}\varepsilon} \left(\frac{1}{k^2 - m_1^2 + \mathrm{i}\varepsilon_1} - \frac{1}{k^2 - m_2^2 + \mathrm{i}\varepsilon_2}\right) \sim \int \frac{k^3\,\mathrm{d}k}{k^5}$$

for large k^2.

The meson self-energy diagrams are



which represent ϕ_1 and ϕ_2 self-energies plus internal lines in which a ϕ_1 becomes a ϕ_2 or vice versa. The basic integral for a single loop is

$$L_j(k) = -g^2\alpha \int \mathrm{d}^4p\, \mathrm{Tr}\left(\gamma^5 \frac{1}{\gamma^\mu(p+k)_\mu+\kappa_j+\mathrm{i}\varepsilon_1}\gamma^5 \frac{1}{\gamma^\nu p_\nu+\kappa_j+\mathrm{i}\varepsilon_2}\right),$$

where α is 1 or i. Use of the Feynman formula, a shift in the origin of p, and the symmetry relation

$$\int \mathrm{d}^4p\, p_\mu f(p^2) = 0$$

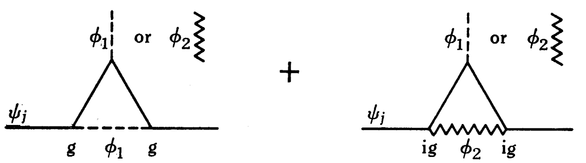
(see Jauch and Rohrlich 1955, Appendix 5; Schweber 1961, p. 577), give for the sum over the four intermediate states

$$L(k^2) = 4\alpha \int_0^1 \mathrm{d}x \int \mathrm{d}^4p \sum_j g_j^2 \frac{p^2+k^2(x^2-x)-\kappa_j^2}{\{p^2+k^2(x-x^2)-\kappa_j^2+\mathrm{i}\varepsilon\}^2},$$

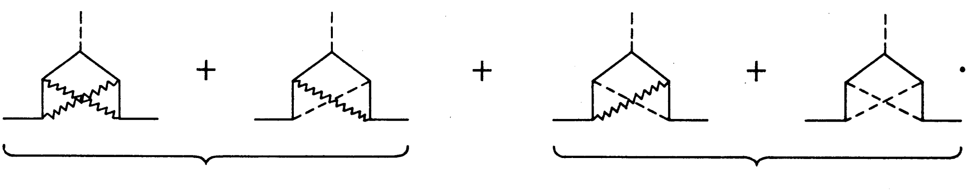
which is convergent if

$$\sum g_j^2 = 0, \qquad \sum g_j^2\kappa_j^2 = 0. \tag{45}$$

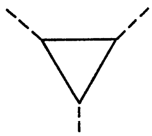
The vertex parts to all orders are convergent in the same manner as is each fermion self-energy, due to the cancellation between g^2 and $(\mathrm{i}g)^2$. Thus, in lowest order, for any of the ψ_j we have



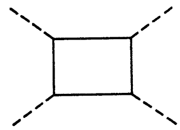
with the same ϕ_1 or ϕ_2 line at the top of each triangle. It is obvious that at high-loop momentum the integrands cancel to leading order, the only difference residing in the masses m_1 and m_2 in the denominators. Power counting shows that the combination is convergent. An examination of higher order vertex parts shows that in each case the terms group in pairs to give finite results; e.g. in next order



A fermion loop with more than four vertices gives rise to a convergent factor, so that we only need consider the two loops



and



,

where ϕ_1 or ϕ_2 lines can attach at each vertex, and where the internal loop represents any one of the ψ_j fermions. As discussed by Schweber (1961, p. 590), each triangle diagram integral is zero. Each integrand is of the form

$$\operatorname{Tr}\left(\gamma^5 \frac{1}{\gamma^\mu p_{1\mu}+\kappa_j} \gamma^5 \frac{1}{\gamma^\nu p_{2\nu}+\kappa_j} \gamma^5 \frac{1}{\gamma^\sigma p_{3\sigma}+\kappa_j}\right).$$

Rationalizing the denominator and evaluating the trace of the numerator gives zero, because the traces of γ^μ, γ^5, $\gamma^\mu\gamma^5$, $\gamma^5\gamma^\mu\gamma^\nu$, and $\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma$ are all zero. For the fermion quadrangle, the sum over the spinor fields gives the integrand

$$\alpha \sum_j g_j^4 \operatorname{Tr}\left(\gamma^5 \frac{1}{\gamma p_1+\kappa_j} \gamma^5 \frac{1}{\gamma p_2+\kappa_j} \gamma^5 \frac{1}{\gamma p_3+\kappa_j} \gamma^5 \frac{1}{\gamma p_4+\kappa_j}\right),$$

where α depends on the set of ϕ_1, ϕ_2 lines attached. On rationalization, the only nonzero term in the numerator is that in

$$\sum_j g_j^4 \operatorname{Tr}(\gamma^5\gamma^\mu\gamma^\nu\gamma^\sigma\gamma^\rho) p_1 p_2 p_3 p_4 .$$

The integral over the loop momentum is convergent if the p^4 component of this numerator vanishes, i.e. if

$$\sum_j g_j^4 = 0 . \tag{46}$$

Under the conditions (45) and (46) there are no primitive divergent integrals in the model. There are many sets of parameters satisfying these conditions for four or more fermion fields.

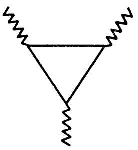
(*b*) *Spinor Fields plus the Photon Field*

In the system defined by equations (5) plus (14) only the photon and the least-massive fermion are stable particles because of the general coupling of ψ_i to ψ_j, unless the fermions fall into two or more separate families not directly linked at any vertex, each with a stable member (in this case the following discussion applies to each family). As seen in Section II, the $:\rho::\rho:$ terms in equation (14) (together with the X and Y terms in equation (8)) may be omitted, so that the interaction Hamiltonian (20) becomes

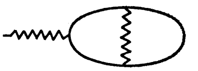
$$H_{\mathrm{I}} = \int \mathrm{d}^3x \sum_{i,j} :(\mathrm{i}\bar{\psi}_i^A J_{ij}^k \psi_j^A A_k^{\mathrm{in}} + \bar{\psi}_i^A \,\delta\kappa_{ij}\psi_j^A): . \tag{47}$$

If it is required that parity be conserved then, with the usual requirement on A_μ^{in}, the fields ψ_j must fall into two sets (Yang and Tiomno 1950), with $g_{ij} = 0$ if i,j are both in the same set, in particular $g_{jj} = 0$, in equations (15).

The primitive divergents are of the same form as in normal electrodynamics. Furry's theorem causes the divergents represented by



and



to be zero (the usual charge conjugation operator is valid for the A operators, which are the same for this purpose as in-operators). Fermion loops with more than four vertices are convergent. For a self-energy bubble in a general fermion line, α-fermion to β-fermion, we have

$$S^{(2)}_{\mathrm{F}\beta\alpha}(p) = \delta_{\beta\alpha} S_{\mathrm{F}\alpha}(p) + S_{\mathrm{F}\beta}(p)\, T_{\beta\alpha}(p)\, S_{\mathrm{F}\alpha}(p)\,, \tag{48}$$

where

$$T_{\beta\alpha}(p) = \int \mathrm{d}^4k \sum_j J^{\mu}_{\beta j}\, S_{\mathrm{F}j}(p+k)\, J^{\mu}_{j\alpha}\, D_{\mathrm{F}}(k)\,, \tag{49}$$

and for the photon self-energy, vertex parts, and four-photon (fermion loop) interaction

$$\Pi_{\mu\nu}(k) = \int \mathrm{d}^4p \;\mathrm{Tr}\Big(\sum_{i,j} J^{\mu}_{ij}\, S_{\mathrm{F}j}(p-k)\, J^{\nu}_{ji}\, S_{\mathrm{F}i}(p)\Big)\,, \tag{50}$$

$$J^{\mu(2)}_{\beta\alpha} = \int \mathrm{d}^4k \sum_{i,j} J^{\nu}_{\beta j}\, S_{\mathrm{F}j}(p_2+k)\, J^{\mu}_{ji}\, S_{\mathrm{F}i}(p_1+k)\, J^{\nu}_{i\alpha}\, D_{\mathrm{F}}(k)\,, \tag{51}$$

with similar expressions for the higher $J^{\mu(2n)}_{\beta\alpha}$, and

$$\Omega = \int \mathrm{d}^4p \sum J^{\mu}_{jl}\, S_{\mathrm{F}l}(p+k_1+k_2+k_3)\, J^{\nu}_{ln}\, S_{\mathrm{F}n}(p+k_1+k_2)\, J^{\lambda}_{ni}\, S_{\mathrm{F}i}(p+k_1)\, J^{\sigma}_{ij}\, S_{\mathrm{F}j}(p)\,. \tag{52}$$

These integrals are divergent in general, $\Pi_{\mu\nu}$ quadratically and $T_{\beta\alpha}$, Ω, and $J^{\mu(2n)}_{\beta\alpha}$ logarithmically. The notation simplifies if we define the vector $\psi = \{\psi_j\}$ and the matrices $\gamma^{\mu} = \{\delta_{ij}\gamma^{\mu}\}$, $\kappa = \{\delta_{ij}\kappa_j\}$, $E = \{e_{ij}\}$, $G = \{g_{ij}\}$, and similarly for $\delta\kappa$, J^{μ}, $S_{\mathrm{F}}(p) = \{\delta_{ij} S_{\mathrm{F}j}\}$, etc. The model then takes the form of that of a fermion possessing excited states.

The above integrals may be reduced in the standard way (see e.g. Jauch and Rohrlich 1955; Schweber 1961), especially if they are convergent. It is easily shown that $T_{\beta\alpha}$, $J^{\mu(2n)}_{\beta\alpha}$, and Ω converge if

$$EE + GG = EG + GE = 0\,, \qquad E\kappa E - G\kappa G = E\kappa G - G\kappa E = 0 \tag{53}$$

and that $\Pi_{\mu\nu}$ converges if, in addition to (53),

$$\mathrm{Tr}(E\kappa^2 E + G\kappa^2 G) = 0\,, \qquad \mathrm{Tr}(E\kappa^2 G + G\kappa^2 E) = 0\,, \tag{54}$$

although the second trace condition is unnecessary if standard quantum-electrodynamic manipulations are admitted, as is shown in the Appendix. It is found there that, with these convergence conditions,

$$\Pi_{\mu\nu}(k) = \delta_{\mu\nu}\Pi_1(k^2) + (2k_\mu k_\nu - \delta_{\mu\nu}k^2)\Pi_2(k^2), \tag{55}$$

where

$$\Pi_1(0) = \pi^2 \sum_j \kappa_j^2(3g_{jj}^2 + 8g_{jj}^2 \log \kappa_j - e_{jj}^2)$$
$$+4\pi^2 \sum_{i\neq j} \frac{\kappa_j^3}{\kappa_j^2-\kappa_i^2}\Big(e_{ij}e_{ji}(\kappa_j - 2\kappa_i) + g_{ij}g_{ji}(\kappa_j+2\kappa_i)\Big)\log\kappa_j. \tag{56}$$

This quantity is a contribution to the mass of the photon. It is necessary for such contributions coming from all orders of the interaction to sum to zero, unless a photon mass counter-term is added to the Hamiltonian (14). (The BD treatment of the quantization of A_k^{in} and A_k and their form for H_{pol}, as adopted in this model, is set in a particular gauge, namely the radiation gauge, and hence gauge invariance cannot be invoked to assert that $\Pi_1(0)$ vanishes nor to disallow a photon mass counter-term in H_{pol} (cf. Jauch and Rohrlich 1955, Section 9–5; Schweber 1961, pp. 558–9).) Renormalization is not pursued here. The strong condition $\Pi_1(0) = 0$ might be imposed.

Sets of coupling constants and masses exist that satisfy the above convergence conditions and, if imposed, $\Pi_1(0) = 0$. Two examples are provided by four fermion fields with

$$E = \begin{bmatrix} 1 & a & b & c \\ a & a^2 & ab & ac \\ b & ab & b^2 & bc \\ c & ca & cb & c^2 \end{bmatrix}, \qquad G = 0, \tag{57}$$

where

$$1+a^2+b^2+c^2 = 0, \qquad \kappa_1 + a^2\kappa_2 + b^2\kappa_3 + c^2\kappa_4 = 0, \tag{58}$$

or with

$$E = \begin{bmatrix} 1 & 0 & \mathrm{i}a & 0 \\ 0 & c & 0 & \mathrm{i}b \\ \mathrm{i}/a & 0 & -1 & 0 \\ \mathrm{i}c/b & \mathrm{i}c^2/b & 0 & -c \end{bmatrix}, \; G = \begin{bmatrix} 0 & c/f & 0 & \mathrm{i}b/f \\ f & 0 & \mathrm{i}af & 0 \\ 0 & \mathrm{i}c/af & 0 & -b/af \\ \mathrm{i}fc/b & 0 & -afc/b & 0 \end{bmatrix}, \tag{59}$$

where

$$\kappa_3 - \kappa_1 = c(\kappa_4 - \kappa_2). \tag{60}$$

Parity is conserved in both examples. If $\Pi_1(0)$ is not required to vanish then there are many solutions in either case; indeed the number of fermion fields may be decreased to three in the first example and the masses chosen arbitrarily. On the other hand, if $\Pi_1(0)$ is required to vanish then equations (56) and (58) or (60) lead to a quadratic equation for c (or a or b in the first example); thus for the first example there is still a solution eliminating all the primitive divergents for any mass spectrum, whereas for the second example, in which equation (60) requires c to be real, there may not be a solution to the convergence conditions.

It can be readily seen from the $\bar{\psi}_i^A \dots \psi_j^A$ products in the Hamiltonian (47) and the topology of the Feynman diagrams that charge is conserved from the initial state to the final state. Any fermion line running from initial to final state starts and finishes as a ψ_1 line (which is conceptually necessary in *LSZ* theory since this is the only stable fermion, as discussed above). The line retains its particle–antiparticle character at each vertex even if its j-label (mass) changes, while particle pairs must ultimately appear in the initial or final state, if at all, as ψ_1 pairs. This conservation of charge is not formally distinct from conservation of one-fermion number. Inside each Feynman diagram the conservation of charge is not apparent, but of course these diagrams cannot be construed as an infinitude of simultaneously observable space–time sequences of (off-mass-shell) trajectories; indeed in the *LSZ* concept it is only the resolution of the state vector into in-state or out-state components that enables contact to be made with a physical picture, and this contact is for $t \to \pm\infty$. A discussion is given by R (Section 9.1) of the definition of incoming and outgoing current densities $J_\mu^{\text{in}}(x)$ and $J_\mu^{\text{out}}(x)$ and the problem of defining $J_\mu(x)$ and relating it to the interpolating fields.

VI. Some Other Aspects

The non-Hermiticity of the interpolating field operators or their products vitiates some conventional results on non-negative weight functions $\rho(\sigma^2)$ in the spectral representations of two-point functions, and similarly makes inapplicable here a demonstration by R, based on analytic S-matrix theory assumptions, that coupling constants must be real.

The spectral representations in question are

$$\langle 0|T(\phi(x)\,\phi(y))|0\rangle = \mathrm{i}\Delta_{\mathrm{f}}'(x-y) = \mathrm{i}\int_0^\infty \mathrm{d}\sigma^2\, \rho(\sigma^2)\,\Delta_{\mathrm{F}}(x-y,\sigma),$$

or in momentum space

$$\Delta_{\mathrm{F}}'(p) = \int_0^\infty \mathrm{d}\sigma^2\, \rho(\sigma^2)\,(p^2-\sigma^2+\mathrm{i}\varepsilon)^{-1},$$

and similar representations for other fields and for the vacuum expectation values of the product and commutator (BD; R). Typically the weight function ρ enters in the following way:

$$\begin{aligned}\langle 0|\phi(x)\,\phi(y)|0\rangle &= \sum_{p,\alpha} \langle 0|\phi(x)|p\alpha\ \mathrm{in}\rangle\langle p\alpha\ \mathrm{in}|\phi(y)|0\rangle \\ &= \exp\{-\mathrm{i}p(x-y)\} \sum_{p,\alpha} \langle 0|\phi(0)|p\alpha\ \mathrm{in}\rangle\langle p\alpha\ \mathrm{in}|\phi(0)|0\rangle \\ &= \exp\{-\mathrm{i}p(x-y)\}\, \rho(p^2),\end{aligned}$$

where the $|p\alpha\ \mathrm{in}\rangle$ ($|p\alpha, A\rangle$) are a complete set of momentum etc. eigenstates, and translation invariance has been used. Thus in the usual theory

$$\rho(p^2) = \sum_{p,\alpha} |\langle 0|\phi(0)|p\alpha\rangle|^2 \geqslant 0 \qquad \text{(wrong)} \tag{61}$$

but this relation does not hold in the present theory because ϕ is not Hermitian. Further, from the relation (BD, p. 141)

$$1 = Z_1 + \int_{m^2}^{\infty} \rho(\sigma^2)\, d\sigma^2 ,$$

it would not follow in the present theory that Z_1 is less than unity. It appears to be widely believed that, because of equation (61), $\rho(\sigma^2) \geqslant 0$ can be violated only if the metric of the Hilbert space is not positive definite (Barton 1963; R, p. 472), but we now see that this is not so.

In analytic S-matrix theory, of which there are several versions, it is a basic relation that the residue of the s-channel scattering amplitude M_s be positive at particle poles. This is essentially the same proposition as that $\rho(\sigma^2)$ be real and positive, as is shown by R (Sections 10–2 and 10–4), relying on his demonstration from the unitarity of the S-matrix that the coupling constant in pseudoscalar $N\pi$–$N\pi$ scattering is real, which rests on analytic S-matrix assumptions. Of course, since the various versions of analytic S-matrix theory are separate domains with their own assumptions on analyticity etc., there is no reason why the present field theory should conform with them. Nevertheless, it is interesting to note that the reality of the $N\pi$ coupling constant g in the demonstration by R depends ultimately on the two expressions (R, equations (10–53) and (10–62))

$$\mathscr{C} \cdot \langle N_k|j(0)|N_q\rangle = ig\,\bar{u}(k)\gamma^5 u(q), \qquad j(x) \equiv (\partial_\mu^2 + m^2)\phi(x),$$

where $\mathscr{C}$ is a real constant. Thus again the reality of g is a consequence of the Hermiticity of $\phi(x)$ in the usual theory.

The Wightman theory, which is a more rigorous alternative to the LSZ formalism, leads to Haag's theorem (Barton 1963; R): this theorem, under rather general assumptions, states that any (for example) neutral scalar field theory defined by a $\phi(x)$ which is related at any instant to a free neutral scalaı theory defined by $\phi_f(x)$ through

$$\phi_f(x) = V\phi(x)V^{-1},$$

where V is unitary, and through $\phi(x)$ and $\phi_f(x)$ satisfying the same equal-time commutation relations, is itself a free field theory. Then the interaction picture of canonical field theory could not exist unless there were no interactions, nor could the BD operator $U(t)$. However, $U(t)$ in equations (9) is not unitary.

VII. Acknowledgments

The author wishes to thank Professor C. A. Hurst and Professor H. S. Green for valuable discussions and encouragement and Professor K. J. Le Couteur and Dr. L. J. Tassie for helpful discussions.

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Appendix

Equation (50) is

$$\Pi_{\mu\nu}(k) = \lim_{\varepsilon_i,\varepsilon_j\to+0} \theta \int \mathrm{d}^4p$$

$$\times \operatorname{Tr}\left(\sum_{i,j} \frac{(c_{ijji}+c_{ijji}^{(5)}\gamma^5)\mathrm{i}\gamma^\mu\gamma^\sigma\gamma^\nu(p-k)_\sigma - \frac{1}{4}d_{ijji}\gamma^\mu\gamma^\nu}{(p-k)^2+\kappa_j^2-\mathrm{i}\varepsilon_j}\left[\frac{\mathrm{i}\gamma^\lambda p_\lambda-\kappa_i}{p^2+\kappa_i^2-\mathrm{i}\varepsilon_i}\right]\right), \tag{A1}$$

where θ is a constant and

$$c_{ijji} = e_{ij}e_{ji} + g_{ij}g_{ji} = c_{ij},$$

$$c_{ijji}^{(5)} = e_{ij}g_{ji} + g_{ij}e_{ji} = c_{ij}^{(5)},$$

$$d_{ijji} = e_{ij}\kappa_j e_{ji} - g_{ij}\kappa_j g_{ji} = d_{ij}.$$

Imposing the conditions (54), which are

$$\sum_{i,j} c_{ijji}\kappa_j^2 = 0, \qquad \sum_{i,j} c_{ijji}^{(5)}\kappa_j^2 = 0,$$

together with (53) renders the integral (A1) convergent. We may then evaluate the trace and use the standard Feynman integral to obtain

$$\Pi_{\mu\nu}(k) = \lim_{\varepsilon_i,\varepsilon_j\to+0} 4\theta \int \mathrm{d}^4p \int_0^1 \mathrm{d}x$$

$$\times \sum_{i,j}\Bigl(c_{ij}\{\delta_{\mu\nu}\boldsymbol{p}\cdot(\boldsymbol{p}-\boldsymbol{k}) - p_\mu(p-k)_\nu - p_\nu(p-k)_\mu\}$$

$$+ c_{ij}^{(5)}\sum_{\sigma\lambda}\varepsilon_{\mu\nu\sigma\lambda}\,p_\sigma k_\lambda + \delta_{\mu\nu}\,d_{ij}\,\kappa_i\Bigr)\Bigl((p-kx)^2+B_{ij}\Bigr)^{-2}, \tag{A2}$$

where

$$B_{ij} = (\kappa_i^2 - \mathrm{i}\varepsilon_i)(1-x) + (\kappa_j^2 - \mathrm{i}\varepsilon_j)x + k^2 x(1-x)$$

and where $\varepsilon_{\mu\nu\sigma\lambda}$ is zero if the $\mu\nu\sigma\lambda$ are not all different and takes the value $+1$ (-1) if $\mu\nu\sigma\lambda$ is obtained from 1234 by an even (odd) number of exchanges.

The origin in p-space is now shifted in the usual way, $p - kx \to p$, and the symmetrical integration procedure is applied (Jauch and Rohrlich 1955, Appendix 5). Shifting the origin gives

$$\begin{aligned} \Pi_{\mu\nu} = & \lim_{\varepsilon_i,\varepsilon_j \to +0} 4\theta \iint \mathrm{d}^4 p \,\mathrm{d}x \\ & \times \sum_{i,j} \Big(c_{ij}\{\delta_{\mu\nu} p^2 - 2p_\mu p_\nu + (2x-1)(\delta_{\mu\nu}\, \boldsymbol{p}\cdot\boldsymbol{k} - p_\mu k_\nu - p_\nu k_\mu) - x(1-x)(\delta_{\mu\nu} k^2 - 2k_\mu k_\nu)\} \\ & \qquad + c_{ij}^{(5)} \sum_{\sigma\lambda} \varepsilon_{\mu\nu\sigma\lambda}\, p_\sigma k_\lambda + \delta_{\mu\nu} d_{ij} \kappa_i \Big) \Big(p^2 + B_{ij} \Big)^{-2}. \end{aligned} \tag{A3}$$

The path of integration in the p_4 plane ($p_4 = \mathrm{i}p_0$) is rotated from the imaginary to the real axis, which can be done without crossing any of the singularities of the integrand, and a transformation is made to polar coordinates in the resulting Euclidean momentum four-space. It is then apparent that the terms odd in p_μ, p_ν, and p_σ make a zero contribution to the integral, as does $p_\mu p_\nu$ unless $\mu = \nu$, so that

$$\begin{aligned} \Pi_{\mu\nu}(k) &= \lim_{\varepsilon_i,\varepsilon_j \to +0} 2\mathrm{i}\theta \iint_{\mathrm{I}} \mathrm{d}^4 p \,\mathrm{d}x \sum_{i,j} \left(\frac{\delta_{\mu\nu}(c_{ij} p^2 + 2d_{ij}\kappa_i) + (2k_\mu k_\nu - \delta_{\mu\nu} k^2) 2x(1-x)}{(p^2 + B_{ij})^2} \right) \\ &= \delta_{\mu\nu} \Pi_1(k^2) + (2k_\mu k_\nu - \delta_{\mu\nu} k^2) \Pi_2(k^2). \end{aligned} \tag{A4}$$

Since the $c_{ij}^{(5)}$ term has disappeared, it can be seen that, if the preceding manipulations were performed without first requiring the integral to be convergent, it would be sufficient to take the conditions (53) plus only the first trace condition (54) at this point to secure convergence. Evaluation of $\Pi_1(0)$ gives equation (56).