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Elementary Particle Scattering Lengths and Resonances

D. C. Peaslee

Laboratory for Nuclear Science and Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139, U.S.A.; on leave from the Australian National University, Canberra, A.C.T.

Abstract

An investigation is made of the relation between scattering lengths and resonances in a two-nucleon system. For resonances E_0 near zero energy the usual determination of an effective optical potential in mesonic atoms is limited, and it is shown here that the scattering length can only be represented usefully as a sum over resonances of the compound system if the condition $|E_0/D| \gg (\rho\Omega_0)^{\frac{1}{2}}$ is satisfied, where D is the average spacing of s-wave states, ρ the density of target nucleons and Ω_0 the Compton volume of the reduced system. This condition is seen to be valid for $\pi\pi$, πK and K$\overline{\text{K}}$ interactions and these systems are considered in some detail. It is shown that knowledge of the level shifts of each of these examples can help resolve present uncertainties in associated boson structure.

Introduction

Scattering lengths of π^- N, K$^-$ N or $\bar{\text{p}}$ N interactions are generally used to estimate an effective nuclear optical potential that produces level shifts in the corresponding atomic levels. Ericson and Sheck (1970) discuss the method and cite references to earlier work, while Bardeen and Torigoe (1971) give a very straightforward derivation of the effective optical potential, or 'pseudopotential'. Since the heavy particle in a Coulomb orbit has a very low momentum on a nuclear scale, s-wave scattering lengths are sufficient, although p-wave effects have been considered for scattering of π mesons from complex nuclei. Difficulties arise with this treatment in practice, however. The $\overline{\text{K}}$N system with $I = 0$ has a resonance at 1405 MeV, only 27 MeV below threshold, which severely complicates the analysis. Some doubts are also expressed at the end of a $\bar{\text{p}}$–atomic study about the efficacy of the scattering-length approach for this case (Backenstoss *et al.* 1972).

The present work investigates the relation between scattering lengths and resonances in a two-body system. It concludes that the scattering length can be represented as a sum over resonances, as in any standard non-relativistic treatment of nuclear reactions, with the inclusion of a constant term. From this it follows that for a given density ρ of scattering centres the effective potential approach can be used in a simple linear way only if the s-wave resonance energy E_0 nearest to zero satisfies

$$|E_0|/D \gg (\rho\Omega_0)^{\frac{1}{2}}, \tag{1}$$

where D is the average spacing of s-wave resonances and Ω_0 is of the order of the Compton volume of the reduced two-body system. For nuclear densities, equation (1) is not even approached for $\overline{\text{K}}$N with $I = 0$; the inequality is not attained for $\bar{\text{p}}$N with $I = 1$, if the reported state at $E_0 \approx -80$ MeV (Gray *et al.* 1971) is an s-state; and the inequality may be barely satisfied for πN with $I = 1$, although the parameters of the S_{11} resonances are somewhat uncertain.

As instances where the condition (1) is sure to be satisfied, since $\rho^{\frac{1}{2}}$ is about 10^{-3} that of nuclear matter, we consider level shifts in $\pi\pi$, πK and K$\bar{\text{K}}$ atoms (L. Rosenson, personal communication).

Scattering Length Formulation

For easy comparison with standard Breit–Wigner forms it is convenient to deal with the complex conjugate of the scattering amplitude,

$$A^* = \{1-\exp(-2i\delta)\}/2ik \underset{k\to 0}{\to} -a, \tag{2}$$

where a is the s-wave scattering length. Using the expansion

$$k\cot\delta = -a^{-1}+\tfrac{1}{2}k^2 r_0 + \ldots,$$

we have

$$A^* = \frac{1}{k\cot\delta + ik} = \frac{1}{ik}\frac{\frac{1}{2}i\Gamma}{E-E_0+\frac{1}{2}i\Gamma}, \tag{3}$$

where

$$E = k^2/2\mu, \qquad E_0 = \gamma^2/2a, \qquad \gamma^2 = 2/\mu r_0, \qquad \Gamma = k\gamma^2. \tag{3a}$$

Here μ is the reduced mass of the two-body system and $\hbar = c = 1$ throughout. The relations (3a) show how a resonance close to $E = 0$ dominates the scattering length, since $a \to \pm\infty$ for $E_0 \to 0^{\pm}$ where $E = 0$. Equations (2) and (3) imply the correct imaginary part for the amplitude:

$$-a = A^* \underset{k\to 0}{\to} \frac{\Gamma}{2k}\frac{-E_0-\frac{1}{2}i\Gamma}{E_0^2+(\frac{1}{2}\Gamma)^2} = -a_0 - ib. \tag{4}$$

This shows a necessarily positive imaginary part for the scattering length as well as for the scattering amplitude,

$$\operatorname{Im} A = \operatorname{Im} a = \frac{1}{4k}\frac{\Gamma^2}{E_0^2+(\frac{1}{2}\Gamma)^2} = \frac{k\sigma}{4\pi}. \tag{5}$$

For pure scattering we take $\Gamma = \Gamma_s$; in the case where the resonance has other channels that lead to effective absorption, $E_0 \to E_0 - \frac{1}{2}i\Gamma_a$, so that as $k \to 0$

$$\sigma = \frac{\pi}{k^2}\frac{\Gamma_s(\Gamma_s+\Gamma_a)}{E_0^2+\frac{1}{4}(\Gamma_s+\Gamma_a)^2} = \frac{\pi}{k^2}\frac{\Gamma_s\Gamma}{E_0^2+(\frac{1}{2}\Gamma)^2}, \tag{6}$$

which is the usual Breit–Wigner form. For bound levels, if $E < 0$ then $k \to i\kappa$ and $\frac{1}{2}i\Gamma_s \to -\frac{1}{2}\kappa\gamma^2$, which represents a level shift: $E_0 \to E_0 + \frac{1}{2}\kappa\gamma^2$.

A nuclear reaction amplitude can be expanded as a sum over resonances, so that a generalization of equations (2) and (3) is (Feshbach *et al.* 1947)

$$-A^* = a = R - (ik)^{-1}\sum_j \frac{\frac{1}{2}i\Gamma_j}{E-E_j+\frac{1}{2}i\Gamma_j}. \tag{7}$$

Here R is a remanent "hard-sphere" term when no resonances are present. We now adopt the suggestion of equation (7), that the scattering length for any angular momentum can be expressed in terms of resonances in the two-body system. As an example, for p-waves, we begin with the expansion

$$k^3 \cot\delta = -a^{-3} + k^2/r_1 + \ldots$$

and obtain

$$-A^* \underset{k\to 0}{\to} k^2 a^3 = \tfrac{1}{3}k^2 R^3 - (\mathrm{i}k)^{-1} \sum_j \frac{\tfrac{1}{2}\mathrm{i}\Gamma_j}{E - E_j + \tfrac{1}{2}\mathrm{i}\Gamma_j}, \tag{7a}$$

with

$$\Gamma_j = k\gamma_j^2 (kr_1)^2, \qquad E_j = (\gamma^2/2a)(r_1/a)^2. \tag{7b}$$

Effective Optical Potential

The effective optical potential is derived by comparing a many-centre scattering amplitude with the Born approximation. If there are N scattering centres in a volume Ω, the first-order scattering amplitude as $k \to 0$ is

$$a = \sum_{n=1}^{N} A_n \to \int \bar{A} N \, \mathrm{d}^3 r/\Omega = \int \bar{A}\rho \, \mathrm{d}^3 r, \tag{8a}$$

where $\bar{A}$ is the average amplitude and $\rho = N/\Omega$ is the density. The Born approximation, or first-order scattering amplitude for an optical potential V at zero energy, is

$$a = -(4\pi)^{-1} \int 2MV \, \mathrm{d}^3 r, \tag{8b}$$

where M is the particle–nucleus reduced mass. Comparison between equations (8a) and (8b) yields the standard relation

$$V = -2\pi\rho \bar{A}/M = 2\pi\rho \, a/M \tag{9}$$

for homogeneous scatterers with $A = \bar{A} = -a$.

It is always desirable at this point to verify signs by checking that the imaginary potential is correctly given to first order by equation (9): For s-waves, $\mathrm{Im}\,\bar{A} = k\sigma/4\pi$ and if we write equation (9) as $V \to V_0 - \mathrm{i}W$ then

$$W = k(\rho\sigma)/2M = (k/\lambda)/2M, \tag{10}$$

where $\lambda = 1/\rho\sigma$ is the mean free path for removal from the beam by scattering centres. For nonzero energy,

$$k^2 = 2M(E - V) \to 2M(E - V_0 + \mathrm{i}W) = k_0^2 + \mathrm{i}k/\lambda,$$

or hence

$$k \approx k_0 + \mathrm{i}/2\lambda. \tag{11}$$

The plane wave $\psi \sim \exp(\mathrm{i}kz)$ then has $|\psi|^2 \exp(-z/\lambda)$, representing correctly the loss of beam intensity in passing through the scatterers.

Returning now to (approximately) real V, the validity of the first-order optical approximation depends on the implicit assumption that equation (3) is not substantially disturbed by putting $E \to E - V_0$. Since the application to atomic systems involves $E \approx 0$, the requirement is that

$$|V_0| \ll |E_0|, \tag{12}$$

or hence

$$E_0^2 \gg \pi\rho\gamma^2/M. \tag{13}$$

If condition (13) is not satisfied, the scattering-length approach will require a number of iterations with an uncertain rate of convergence, and thus an alternative approach may be better.

For low energies the states of a meson–nucleon system appear to be adequately counted by simple, even non-relativistic, quark models. For any such model it will be true that $\gamma^2 = D/\pi K$, where D is the mean spacing between resonance levels and K some momentum internal to the two-body system. The above condition then becomes

$$|E_0/D| \gg (\rho/DKM)^{\frac{1}{2}}. \tag{14a}$$

In terms of the reduced mass μ of the two-body system, elementary resonances have $D \sim \mu$, $M \sim \mu$ and $K \lesssim \mu$; then with the Compton volume $\Omega_0 = (4/3)\pi\mu^{-3}$

$$(\rho/DKM)^{\frac{1}{2}} \sim (\rho\Omega_0)^{\frac{1}{2}} \sim (\mu_\pi/\mu)^{3/2}, \tag{14b}$$

where μ_π is the pion Compton wavelength, and the last form is for nuclear densities ρ.

In almost all elementary two-body systems with Coulomb binding where one particle is a nucleon, at least one isospin seems hardly to satisfy the conditions (14). The most obvious instance is $\bar{\mathrm{K}}\mathrm{N}$ ($I = 0$) where the Y*(1405) has $|E_0/D| \lesssim 0\cdot1$ which is less than $(\mu_\pi/\mu_\mathrm{K})^{3/2} \approx 0\cdot3$; for $\bar{\mathrm{p}}\mathrm{N}$ ($I = 1$), the bound state (Gray *et al.* 1971) has $|E_0/D| \sim 0\cdot15 \approx (\mu_\pi/\mu)^{3/2}$, which does not satisfy the much greater than condition. Even for $\pi\mathrm{N}$ scattering, the $I = \frac{1}{2}$ system seems to have s-wave resonances at $|E_0/D| \sim 1 \approx (\mu_\pi/\mu)^{3/2}$.

The p-wave potential condition turns out to resemble equations (14). Here the potential (Kisslinger 1955) is energy-dependent:

$$V' = (2\pi/\mu)\rho a(ka)^2 = 4\pi\rho a^3 E. \tag{15}$$

A condition for applicability is surely that

$$1 \gg |\mathrm{d}V'/\mathrm{d}E| = |4\pi\rho a^3| = 2\pi\rho\gamma^2 r_1^2/|E_0|. \tag{16a}$$

Again taking $\gamma^2 = D/\pi K$, we have

$$|E_0/D| \gg 2\rho r_1^2/K \sim \rho\Omega_0, \tag{16b}$$

which is the same as equations (14) except for an insignificant change of power.

For pion–nucleon scattering, $\rho\Omega_0$ is of order unity and the $I = 3/2$ resonance $\Delta(1236)$ violates equation (16b).

Equivalence with Second-order Perturbation

The effective optical potential provides a means to check the statement above that the scattering length is expressible entirely in terms of a sum over resonances. Suppose again that we have N identical, independent scattering centres in a volume Ω; a particle impinges on this collection, and we compute its second-order energy shift ΔE due to excitation of resonant states. For a single resonant state at energy E_0 this is

$$\Delta E = N|H_{\mathrm{int}}|^2/(E-E_0). \tag{17}$$

Here the interaction energy H_{int} induces the transitions between a compound resonant state at energy E_0 and two free particles. Hence, for the resonance level,

$$\Gamma = 2\pi\,|H_{\text{int}}|^2 \frac{\Omega\,\mathrm{d}^3k}{(2\pi)^3\,\mathrm{d}E} = \frac{|H_{\text{int}}|^2}{2\pi} 2\mu k\Omega \tag{18}$$

so that

$$\Delta E = \left(\frac{(N/\Omega)(1/2\mu)\,2\pi(\Gamma/k)}{E-E_0}\right) = \frac{2\pi\rho}{\mu}\left(\frac{\frac{1}{2}\gamma^2}{E-E_0}\right). \tag{19}$$

The last factor on the right-hand side of (19) immediately generalizes to a sum over many levels,

$$\sum_{j=1}^{n}\left(\frac{\frac{1}{2}\gamma_j^2}{E-E_j+\frac{1}{2}\mathrm{i}\Gamma_j}\right) + \int_{E_n}^{\infty}\frac{\mathrm{d}E'}{D(E')}\left(\frac{\frac{1}{2}\gamma^2(E')}{E-E'+\frac{1}{2}\mathrm{i}\Gamma(E')}\right). \tag{19a}$$

Here the integral is a sum over far-away resonances. It is essentially a constant for variations in the neighbourhood of $E = 0$ and stands in place of hard-sphere terms in equation (7).

It only remains to note that since V and E appear in the Schrödinger equations with opposite signs, the equivalent $\Delta V = -\Delta E$. One then reverses the multiple-scattering versus Born approximation argument to obtain

$$a = -\frac{\mu}{2\pi}\frac{\Delta E}{\rho} = -\sum_{j=1}^{n}\left(\frac{\frac{1}{2}\gamma_j^2}{E-E_j+\frac{1}{2}\mathrm{i}\Gamma_j}\right) + R'(E_n). \tag{20}$$

This is just equation (7) with the hard-sphere term $R'(E_n)$ representing the integral in (19a). On a full relativistic treatment of elementary-particle interactions this term contains all the contributions of physical cuts as distinct from poles. Although there is no deductive proof for the magnitude of this term, we assume throughout that it is quite small; all the examples considered in the next section indicate that it modifies any resonance contributions by $|\Delta a| < 0{\cdot}1$ fm in the scattering length.

Meson–Meson Potentials

If quasi-atomic states of positive and negative mesons are considered, similar energy shifts will arise from the optical potential associated with meson–meson scattering lengths. In that case, however, the densities are not nuclear but are instead associated with Coulomb orbits, so that the limits of equations (14) on $|E_0/D|$ are lower by factors of order 10^3 and hence certain to be satisfied. Accordingly, we reverse the approach used above and try to estimate some meson–meson scattering lengths from what is known of boson resonance structure. The effective optical potential is then as accurate and reliable as the scattering lengths. The general conclusion will be that 'nearby' resonances induce scattering lengths $|a|$ of order μ_π^{-1}; conversely, $|a| \lesssim 0{\cdot}1\,\mu_\pi^{-1}$ will indicate the absence of nearby resonances.

For s-waves we take the scattering length as

$$a_0 = \Gamma/2E_0 k = \gamma^2/2E_0, \tag{21}$$

where E_0 is the energy of the nearest resonance relative to the two-meson threshold, and k is the momentum for decay of the resonance into two mesons. For a bound state, E_0 is negative and γ^2 must be estimated independently.

Consider first the $\pi\pi$ system for $I = 0$. If there were really a broad 'resonance' (Banaigs *et al.* 1972) at around $E_0+2\mu_\pi \sim 400$ MeV with $\frac{1}{2}\Gamma \sim 100$ MeV, the scattering length from equation (21) would have the relatively large value

$$a_0(\text{'DEF meson'}) \sim 0\cdot 8\,\mu_\pi^{-1}. \tag{22a}$$

For an ε resonance with the extreme parameters $E_0+2\mu_\pi = 700$ MeV, $\frac{1}{2}\Gamma = 300$ MeV, we obtain

$$a_0(\varepsilon) \sim 0\cdot 3\,\mu_\pi^{-1}. \tag{22b}$$

This is within the range of values, $a_0 \approx (0\cdot 16\text{–}0\cdot 5)\mu_\pi^{-1}$, that is currently being considered as a result of K_{l4} measurements (Ely *et al.* 1969; Basile *et al.* 1971; Auvil 1972; Sirlin and Weinstein 1972).

The values in equations (22) can be inserted in the model-independent relationship (Weinberg 1966)

$$a_2 = \tfrac{2}{5}a_0 - \tfrac{3}{20}\pi^{-1}\,m_\pi/F^2 \approx \tfrac{2}{5}a_0 - 0\cdot 1\,\mu_\pi^{-1} \tag{23}$$

to yield

$$a_2 \approx 0\cdot 2\,\mu_\pi^{-1} \qquad \text{or} \qquad a_2 \approx 0. \tag{24a, b}$$

The first implies suspicion of an $I = 2$ resonance only slightly weaker than the very strong ε, while the second implies absence of any nearby $I = 2$ resonance. This is of course another argument for rejecting the DEF meson as a $\pi\pi$ resonance.

For the $\mathrm{K}^+\pi^-$ or $\mathrm{K}^-\pi^+$ system there are no resonances expected in $I = 3/2$ states, so that

$$a_{3/2} \approx 0. \tag{25}$$

For $I = 1/2$, if there is a κ resonance at $E_0+\mu_\mathrm{K}+\mu_\pi \approx 1200$ MeV with $\frac{1}{2}\Gamma \sim 200$ MeV, then

$$a_{1/2} \sim 0\cdot 1\,\mu_\pi^{-1}. \tag{26a}$$

If there is an s-wave $\pi\mathrm{K}$ resonance under the K*(890) with a width $\frac{1}{2}\Gamma \sim 100$ MeV, we would have

$$a_{1/2} \sim (0\cdot 2\text{–}0\cdot 3)\mu_\pi^{-1}. \tag{26b}$$

This suggests a possible opportunity to investigate the presence of a K resonance near the $\mathrm{K}\pi$ threshold.

The most interesting and intensely studied meson system with regard to scattering lengths is the $\mathrm{K}\bar{\mathrm{K}}$, which is often produced near threshold in meson reactions. Both the $I = 1$ and $I = 0$ systems appear to have resonances near the threshold, $E_0+2\mu_\mathrm{K} = 990$ MeV. The decays of the isoscalar mesons $\mathrm{D}, \mathrm{E} \to \mathrm{K}\bar{\mathrm{K}}$ assure the production of an $I = 1$ $\mathrm{K}\bar{\mathrm{K}}$ state, which has been exhaustively analysed in terms of the $\mathrm{K}\bar{\mathrm{K}}$ scattering length (Astier *et al.* 1967) with the result

$$|a_1| \approx (2\cdot 5 \pm 1)\mu_\pi^{-1}. \tag{27}$$

The sign was not determined. Since the associated resonance seems almost certain to be the δ or $\pi_\mathrm{N}(975)$, the scattering length is therefore negative in sign and, if we take the δ^0 mass as 975 MeV, we obtain $\gamma^2(\delta \to \mathrm{K}\bar{\mathrm{K}}) = 2E_0\,a_0 = (75\pm 30)\mathrm{MeV}\,\mu_\pi^{-1}$.

Comparison with equation (22b) yields a ratio of coupling constants

$$\gamma^2(\delta \rightarrow \mathrm{K\bar{K}})/\gamma^2(\varepsilon \rightarrow \pi\pi) \approx 0\cdot3 \pm 0\cdot1 \tag{28}$$

which seems satisfactory.

The widths reported for $\delta \rightarrow \pi\eta$ (Particle Data Group 1972) can be summarized as $\Gamma = 45 \pm 25$ MeV, whence $\gamma^2(\delta \rightarrow \pi\eta) \approx (20 \pm 10)\mathrm{MeV}\,\mu_\pi^{-1}$. This suggests for the δ resonance a ratio

$$\gamma^2(\pi\eta)/\gamma^2(\mathrm{K\bar{K}}) = R \approx 0\cdot3 \pm 0\cdot2.$$

The $I = 0$ s-wave $\mathrm{K\bar{K}}$ system, signalized by $\mathrm{K_s^0 K_s^0}$ decays, has been analysed in terms of both scattering lengths and resonances (Hoang *et al.* 1969; Protopopescu *et al.* 1972). In each analysis it was found that either description seemed adequate—which should of course be true if they are equivalent as maintained here. Hoang *et al.* obtained a resonance at 1·02–1·06 GeV, well above the $\mathrm{K\bar{K}}$ threshold so that a is positive,

$$a_0 = (0\cdot8 \pm 0\cdot2)\mu_\pi^{-1}. \tag{29}$$

Protopopescu *et al.*, on the other hand, obtained an S* resonance slightly below the $\mathrm{K\bar{K}}$ threshold and, consistent with this, a negative scattering length†

$$a_0 = -(0\cdot4 \pm 0\cdot1)\mu_\pi^{-1}. \tag{30}$$

The difference between equations (29) and (30) is so large that it is tempting to suggest the possibility of a decision between them by measurements on the $\mathrm{K^+K^-}$ system.

Conclusions

This analysis of the relation between scattering lengths and resonances in two-nucleon interactions has shown that an effective optical model approach of representing scattering length as a simple sum over resonances can be used only if the s-wave resonance energy satisfies the condition (1). The standard linear treatment is then not valid for $\mathrm{\bar{K}N}$ interactions and is problematic for $\mathrm{\bar{p}N}$. However, for such systems as $\pi\pi$, πK and $\mathrm{K\bar{K}}$, where the density due to Coulomb attraction is some 10^{-3} below the level at which the strong forces producing the scattering lengths are distorted, the present analyses have indicated that useful information can be obtained from a knowledge of the level shifts in each of these mesonic systems. In particular, the small $\pi\pi$ scattering length for $I = 0$ argues against a DEF meson but is compatible with the ε; an s-wave Kπ resonance closer to threshold than the K*(890) should reveal itself in a sizable scattering length; and the very sign of the $\mathrm{\bar{K}K}$ scattering length will help to determine the position of a nearby resonance.

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† Note that their sign convention for scattering lengths is opposite to that used here (cf. Barbaro-Galtieri 1968).

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