Aust. J. Phys., 1974, **27**, 53–7

On the Kelvin–Helmholtz Discontinuity in Two Superposed Plasmas

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Abstract

An investigation has been made of the effects of collisions with neutral particles on the stability of the plane interface separating two streaming superposed plasmas of uniform densities. It has been found that, whereas the ambient magnetic field has a stabilizing influence, a collision frequency has a stabilizing effect when it is small and a destabilizing effect when it exceeds a certain value.

Perturbation Equations for Conducting Fluid–Neutral Gas Mixture

The effects of collisions with neutral particles on the instability of the plane interface which separates two uniform superposed composite hydromagnetic streaming systems have been studied previously by Hans (1968), for transverse propagation, and by Bhatia (1970), for longitudinal propagation. Sharma and Srivastava (1968) have studied this stability problem for general perturbations in the absence of the effects of neutral gas friction, but it may be of some interest to investigate the influence of these collisional effects for general perturbations, and this is the object of the present paper.

We consider the motion of the *mixture* of an infinitely conducting, incompressible and inviscid, hydromagnetic fluid and a neutral gas. Assuming that (i) both the conducting fluid and the neutral gas behave like continuum fluids and (ii) the effects on the neutrals resulting from the presence of a magnetic field and the fields of gravity and pressure are not included, we finally obtain the linearized perturbation equations:

$$\rho\,\partial\boldsymbol{v}/\partial t+\rho(\boldsymbol{U}.\nabla)\boldsymbol{v} = -\nabla(\delta p)+\tfrac{1}{4}\pi^{-1}(\nabla\times\boldsymbol{h})\times\boldsymbol{H}+\boldsymbol{g}\,\delta\rho+\rho_{\mathrm{d}}\nu_{\mathrm{c}}(\boldsymbol{v}_{\mathrm{d}}-\boldsymbol{v}), \tag{1}$$

$$\partial\boldsymbol{v}_{\mathrm{d}}/\partial t+(\boldsymbol{U}.\nabla)\boldsymbol{v}_{\mathrm{d}} = -\nu_{\mathrm{c}}(\boldsymbol{v}_{\mathrm{d}}-\boldsymbol{v}), \tag{2}$$

$$\partial\boldsymbol{h}/\partial t+(\boldsymbol{U}.\nabla)\boldsymbol{h} = \mathrm{curl}(\boldsymbol{v}\times\boldsymbol{H}), \tag{3}$$

$$\partial(\delta\rho)/\partial t+(\boldsymbol{U}.\nabla)\delta\rho+(\boldsymbol{v}.\nabla)\rho = 0, \tag{4}$$

$$\nabla.\boldsymbol{h}=0 \qquad \text{and} \qquad \nabla.\boldsymbol{v}=0. \tag{5}$$

Here $\boldsymbol{v}(u,v,w)$, $\boldsymbol{h}(h_x,h_y,h_z)$, $\delta\rho$ and δp denote respectively the perturbations in velocity $\boldsymbol{v}$, magnetic field $\boldsymbol{H}$, density ρ and pressure p of the ionized fluid while ρ_{d} and $\boldsymbol{v}_{\mathrm{d}}$ are the density and velocity of the neutrals in the presence of a downward gravitational field $\boldsymbol{g}=(0,0,-g)$. Of the remaining quantities $\boldsymbol{U}(U,0,0)$ is the streaming velocity of the composite medium and ν_{c} is the collision frequency which represents the mutual collisional (frictional) effects between the ionized fluid and the neutral gas.

Taking the ambient magnetic field to be $\boldsymbol{H} = (H_x, H_y, 0)$ we analyse the disturbance into normal modes by seeking solutions, of the above equations, whose dependence on x, y and t is of the form

$$\exp(\mathrm{i}k_x x + \mathrm{i}k_y y + nt), \tag{6}$$

where k_x and k_y ($k^2 = k_x^2 + k_y^2$) are the horizontal wave numbers and n is the frequency of the harmonic disturbance. Eliminating $\boldsymbol{v}_{\mathrm{d}}$ from equations (1) and (2) and using the form (6), we obtain

$$\rho\left(n + \mathrm{i}k_x U + \frac{(n + \mathrm{i}k_x U)\beta\nu_{\mathrm{c}}}{n + \mathrm{i}k_x U + \nu_{\mathrm{c}}}\right)u = -\mathrm{i}k_x \delta p + \frac{H_y}{4\pi}\left(\mathrm{i}k_y h_x - \mathrm{i}k_x h_y\right), \tag{7}$$

$$\rho\left(n + \mathrm{i}k_x U + \frac{(n + \mathrm{i}k_x U)\beta\nu_{\mathrm{c}}}{n + \mathrm{i}k_x U + \nu_{\mathrm{c}}}\right)v = -\mathrm{i}k_y \delta p + \frac{H_x}{4\pi}\left(\mathrm{i}k_x h_y - \mathrm{i}k_y h_x\right), \tag{8}$$

$$\begin{aligned}\rho\left(n + \mathrm{i}k_x U + \frac{(n + \mathrm{i}k_x U)\beta\nu_{\mathrm{c}}}{n + \mathrm{i}k_x U + \nu_{\mathrm{c}}}\right)w &= -\mathrm{D}\,\delta p - g\,\delta\rho \\ &\quad + \frac{H_x}{4\pi}\left(\mathrm{i}k_x h_z - \mathrm{D}h_x\right) + \frac{H_y}{4\pi}\left(\mathrm{i}k_y h_z - \mathrm{D}h_y\right),\end{aligned} \tag{9}$$

$$(n + \mathrm{i}k_x U)\boldsymbol{h} = (\mathrm{i}k_x H_x + \mathrm{i}k_y H_y)\boldsymbol{v}, \tag{10}$$

$$(n + \mathrm{i}k_x U)\delta\rho = -w(\mathrm{D}\rho), \tag{11}$$

where we have written $\beta = \rho_{\mathrm{d}}/\rho$ and $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$. Multiplying equations (7) and (8) by $-\mathrm{i}k_x$ and $-\mathrm{i}k_y$ and adding the results, and also using equations (5), (9), (10) and (11), we finally obtain the following equation in w

$$\begin{aligned}&\left(\mathrm{D}(\rho\,\mathrm{D}w) - k^2\rho w\right)\left(n + \mathrm{i}k_x U + \frac{(n + \mathrm{i}k_x U)\beta\nu_{\mathrm{c}}}{n + \mathrm{i}k_x U + \nu_{\mathrm{c}}}\right) \\ &\qquad + \frac{(k_x H_x + k_y H_y)^2}{4\pi(n + \mathrm{i}k_x U)}(\mathrm{D}^2 - k^2)w + \frac{gk^2(\mathrm{D}\rho)}{n + \mathrm{i}k_x U}\,w = 0.\end{aligned} \tag{12}$$

Two Uniform Superposed Composite Media

We suppose that the two superposed composite media, in which the densities ρ_1 and ρ_2 (and also ρ_{d}) are assumed to be uniform, are streaming past each other with uniform streaming velocities U_1 and U_2 and are separated by a horizontal boundary at $z = 0$. Then, in each region of constant ρ, equation (12) becomes

$$(\mathrm{D}^2 - k^2)w = 0. \tag{13}$$

Since w must be bounded both when $z \to -\infty$ (in the lower medium) and $z \to +\infty$ (in the upper medium), the appropriate solutions for w can be written as

$$w_1 = A(n + \mathrm{i}k_x U_1)\exp(kz), \qquad z < 0, \tag{14}$$

$$w_2 = A(n + \mathrm{i}k_x U_2)\exp(-kz), \qquad z > 0, \tag{15}$$

where the constant A has been chosen to be the same to ensure the continuity of $w/(n + \mathrm{i}k_x U)$.

Integrating equation (12) across the interface at $z = 0$, we obtain

$$\Delta_0\left(\left\{n+ik_xU+\frac{(n+ik_xU)\beta\nu_c}{n+ik_xU+\nu_c}\right\}\rho\,\mathrm{D}w\right)+\frac{H_x^2k_x^2}{4\pi}\Delta_0\left(\frac{\mathrm{D}w}{n+ik_xU}\right)+\frac{H_y^2k_y^2}{4\pi}\Delta_0\left(\frac{\mathrm{D}w}{n+ik_xU}\right)$$
$$+\frac{k_xk_yH_xH_y}{2\pi}\Delta_0\left(\frac{\mathrm{D}w}{n+ik_xU}\right)+gk^2\Delta_0(\rho)\left(\frac{w}{n+ik_xU}\right)_0=0,\qquad(16)$$

where $\Delta_0(f)$ is the jump that a quantity f experiences at $z = 0$ and $(w/(n+ik_xU))_0$ is the unique value that this quantity has at $z = 0$. Using in equation (16) the values of w_1 and w_2 from equations (14) and (15), we obtain the dispersion relation

$$(n+ik_xU_2)^2\alpha_2+\frac{\alpha_2\beta_2\nu_c(n+ik_xU_2)^2}{n+ik_xU_2+\nu_c}+(n+ik_xU_1)^2\alpha_1+\frac{\alpha_1\beta_1\nu_c(n+ik_xU_1)^2}{n+ik_xU_1+\nu_c}$$
$$-gk(\alpha_2-\alpha_1)+2(k_xV_A+k_yV_B)^2=0,\qquad(17)$$

where

$$\alpha_{1,2}=\frac{\rho_{1,2}}{\rho_1+\rho_2},\qquad\beta_{1,2}=\frac{\rho_{\mathrm{d}}}{\rho_{1,2}},\qquad V_{A,B}^2=\frac{H_{x,y}^2}{4\pi(\rho_1+\rho_2)}.\qquad(18)$$

Here the subscripts 1 and 2 distinguish the quantities for the lower ($z < 0$) and upper ($z > 0$) media respectively.

Discussion

In its present form the dispersion relation (17) is quite complex. We therefore consider a simple model in which the two media of the same density ($\alpha_1 = \alpha_2$) are flowing across each other with streaming velocities $U, -U$. The same model was previously investigated by Hans (1968). Then, putting

$$\bar{n}=n/kU,\quad\bar{\nu}_c=\nu_c/kU,\quad\bar{V}_A=V_A/U,\quad\bar{V}_B=V_B/U\qquad(19)$$

in equation (17), we obtain the dimensionless form of the dispersion relation

$$\bar{n}^4+\bar{n}^3\bar{\nu}_c(2+\beta)+\bar{n}^2\{\bar{\nu}_c^2(1+\beta)+2(\bar{V}_A\sin\theta+\bar{V}_B\cos\theta)^2\}$$
$$+\bar{n}\,\bar{\nu}_c\{(\beta-2)\sin^2\theta+4(\bar{V}_A\sin\theta+\bar{V}_B\cos\theta)^2\}$$
$$+\{2(\bar{\nu}_c^2+\sin^2\theta)(\bar{V}_A\sin\theta+\bar{V}_B\cos\theta)^2-(\bar{\nu}_c^2+\beta\,\bar{\nu}_c^2+\sin^2\theta)\sin^2\theta\}=0,\qquad(20)$$

where θ is the angle between the directions of $\boldsymbol{k}$ and H_y.

If we put $\bar{V}_A = 0$, $\bar{V}_B \neq 0$ and $\theta = 90°$ in equation (20) we recover the dispersion relation obtained by Hans (1968) for the transverse mode of propagation (namely his equation (19), with $\nu = 0$ and n and ν_c measured in units of kU). It can then be easily seen from equation (20) that the growth rate is independent of the strength of the ambient magnetic field. Although the equations for $\bar{n}$ and $\bar{\nu}_c$ in (19) become singular when $k \to 0$, equation (20) still remains meaningful because, for $k = 0$, the value of the growth rate is otherwise easily seen to be zero.

Numerical calculations were performed to locate the roots of $\bar{n}$ from equation (20) for several values of the parameters $\bar{\nu}_c$, $\bar{V}_A$, $\bar{V}_B$, β and θ. The results are presented in Table 1, where the growth rate (positive real value of $\bar{n}$) is given as a function of $\bar{\nu}_c$, for $\bar{V}_A = 0{\cdot}20$ and $0{\cdot}25$ with $\bar{V}_B = 0{\cdot}20$, $0{\cdot}25$, taking $\beta = 0{\cdot}5$ and $\theta = 30°$, 45° and 60°. These calculations give an idea of the behaviour of the effects of collisions and the magnetic field on the instability of the considered configuration. The variation of the growth rate with θ has also been included to show the influence of the orientation of the magnetic field with respect to the wave vector on the unstable configuration.

Table 1. Growth rate $\bar{n}$ as a function of $\bar{\nu}_c$

The values are the positive real roots of $\bar{n}$ from equation (20) for $\beta = 0\cdot5$

$\bar{\nu}_c$	$\theta = 30°$ $\bar{V}_A = 0\cdot20$ $\bar{V}_B = 0\cdot20$	0·20 0·25	$\theta = 45°$ 0·20 0·20	0·20 0·25	$\theta = 60°$ 0·20 0·20	0·20 0·25	$\theta = 90°$ $\bar{V}_A = 0$ $\bar{V}_B \neq 0$
0	0·3174	0·2228	0·5831	0·5454	0·7751	0·7564	1·0000
0·1	0·3004	0·2091	0·5626	0·5252	0·7535	0·7349	0·9777
0·2	0·2953	0·2112	0·5497	0·5132	0·7380	0·7196	0·9602
0·3	0·2966	0·2196	0·5424	0·5069	0·7273	0·7093	0·9468
0·4	0·3006	0·2294	0·5389	0·5046	0·7203	0·7026	0·9367
0·5	0·3055	0·2387	0·5379	0·5047	0·7160	0·6987	0·9292
0·6	0·3105	0·2470	0·5385	0·5063	0·7137	0·6968	0·9238
0·7	0·3153	0·2543	0·5400	0·5087	0·7128	0·6962	0·9200
0·8	0·3197	0·2606	0·5421	0·5115	0·7128	0·6967	0·9174
0·9	0·3237	0·2662	0·5445	0·5146	0·7136	0·6977	0·9159
1·0	0·3274	0·2710	0·5470	0·5177	0·7148	0·6992	0·9151
1·1	0·3306	0·2753	0·5495	0·5207	0·7164	0·7010	0·9148
1·2	0·3336	0·2792	0·5520	0·5237	0·7181	0·7030	0·9150
1·3	0·3363	0·2826	0·5545	0·5265	0·7200	0·7051	0·9154
1·4	0·3388	0·2856	0·5568	0·5292	0·7219	0·7072	0·9162
1·5	0·3410	0·2884	0·5591	0·5318	0·7236	0·7093	0·9171

$\bar{\nu}_c$	$\theta = 30°$ $\bar{V}_A = 0\cdot25$ $\bar{V}_B = 0\cdot20$	0·25 0·25	$\theta = 45°$ 0·25 0·20	0·25 0·25	$\theta = 60°$ 0·25 0·20	0·25 0·25
0	0·2686	0·1294	0·5454	0·5000	0·7414	0·7189
0·1	0·2530	0·1233	0·5252	0·4803	0·7200	0·6976
0·2	0·2511	0·1385	0·5132	0·4693	0·7049	0·6828
0·3	0·2557	0·1572	0·5069	0·4645	0·6948	0·6731
0·4	0·2625	0·1740	0·5046	0·4638	0·6884	0·6672
0·5	0·2695	0·1879	0·5047	0·4653	0·6849	0·6642
0·6	0·2762	0·1995	0·5063	0·4682	0·6833	0·6631
0·7	0·2823	0·2091	0·5087	0·4718	0·6830	0·6633
0·8	0·2877	0·2172	0·5115	0·4756	0·6837	0·6644
0·9	0·2925	0·2241	0·5146	0·4795	0·6850	0·6661
1·0	0·2968	0·2300	0·5177	0·4834	0·6868	0·6682
1·1	0·3006	0·2351	0·5207	0·4871	0·6888	0·6706
1·2	0·3040	0·2396	0·5237	0·4906	0·6909	0·6730
1·3	0·3071	0·2436	0·5265	0·4939	0·6932	0·6755
1·4	0·3099	0·2471	0·5292	0·4971	0·6954	0·6780
1·5	0·3124	0·2503	0·5318	0·5000	0·6977	0·6804

It can be seen from Table 1 that the growth rate decreases as $\bar{V}_A$ and $\bar{V}_B$ increase, thereby depicting the stabilizing influence of the magnetic field. The growth rate is also seen to be suppressed for small collision frequencies, although it increases thereafter with increase in collisions. We conclude that a small collision frequency renders the configuration more stable but that as the collision frequency increases beyond a critical value, the stable configuration becomes unstable. Finally Table 1 shows that the growth rate increases as θ increases, for the same $\bar{V}_A$, $\bar{V}_B$ and β.

For the sake of completeness, the important special case of transverse propagation, $\theta = 90°$, is also included in Table 1. The values of the growth rate obtained for this case agree with those of Hans (1968), as expected.

Acknowledgments

We are grateful to the referee for his valuable comments and to Barbara for her careful typing of the manuscript.

References

Bhatia, P. K. (1970). *Cosmic Electrodynamics* **1**, 269.
Hans, H. K. (1968). *Nucl. Fusion* **8**, 89.
Sharma, R. C., and Srivastava, K. M. (1968). *Aust. J. Phys.* **21**, 917.

Manuscript received 15 December 1972, revised 19 July 1973