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Equivalence of the Dicke Maser Model and the Ising Model at Equilibrium

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Abstract

The equilibrium properties of a system of N two-level atoms interacting with a quantized field, the so-called Dicke model of superradiance, are evaluated exactly by the 'thermodynamically equivalent Hamiltonian' method of Bogoliubov *et al.* (1957). The results are identical with those obtained by Hepp and Lieb (1973) and Wang and Hioe (1973). The thermodynamically equivalent Hamiltonian is shown to be equivalent to the mean field Hamiltonian of the Ising model, which indicates why the transition from the 'normal' to the 'superradiant' state of the Dicke model is similar to the transition from the disordered to the ordered state of the Ising model.

1. Introduction

The time-dependent properties of a system of N two-level atoms interacting with a radiation field (i.e. the Dicke maser model) have been studied by a number of authors. In the Dicke model the direct interaction between the atoms is ignored, but the atoms are presumed to interact via the common radiation field. Dicke (1954) first introduced the concept of coherence in the spontaneous radiative emission from such a system. The main feature of these coherent spontaneous radiation processes is the possibility, in certain configurations, of having the radiation rate proportional to N^2 rather than to N, as would be expected if the atoms were independent of each other.

Recently, Hepp and Lieb (1973) and Wang and Hioe (1973) have evaluated exactly the equilibrium properties of the Dicke model. As shall be shown in Section 3, the equilibrium properties of the model indicate that the system of N atoms can be in either a coherent or an incoherent state, thus supporting the non-equilibrium results of Dicke (1954), Tavis and Cummings (1968) and Bonifacio and Preparata (1970).

In this paper, we shall solve the Dicke model by using the 'thermodynamically equivalent Hamiltonian' method of Bogoliubov *et al.* (1957; hereinafter referred to as BZT). This method was originally used to solve the model Hamiltonian of the Bardeen *et al.* (1957) theory of superconductivity and has subsequently been extended by Wentzel (1960) to a wider class of model Hamiltonians. The application of the BZT technique to the Dicke model yields the same results as were first obtained by Hepp and Lieb (1973) in their mathematically complicated but rigorous derivation, and which were subsequently rederived by Wang and Hioe (1973) using Glauber's (1963) coherent state formalism. The advantage of the present formalism, besides its greater simplicity, is that it illustrates the equivalence between the Dicke model and the mean-field Ising model.

In Section 3, the 'thermodynamically equivalent Hamiltonian' is shown to be equivalent to the mean-field Ising model Hamiltonian, the normal state of the Dicke model being equivalent to the disordered state of the Ising model and the superradiant state being equivalent to the ordered state of the Ising model. In Section 4, the Dicke model is generalized to include a finite number of photon modes, and this model is solved using the same approach.

2. Dicke Model of Superradiance

The Hamiltonian of the Dicke model is given by

$$H = a^+a + \sum_{j=1}^{N} \varepsilon\sigma_j^z + \lambda N^{-\frac{1}{2}} \sum_{j=1}^{N} a\sigma_j^+ + a^+\sigma_j^- , \tag{1}$$

where the Pauli spin matrices $\boldsymbol{\sigma}_j$ describe the two-level atoms, a^+ and a are the boson creation and annihilation operators for a single mode of the electromagnetic field, ε is the energy associated with the excited state of an atom, and λ measures the coupling between the atoms and the electromagnetic field. To calculate the partition function Z, given by

$$Z = \mathrm{Tr}(\exp -\beta H), \tag{2}$$

we linearize the interaction terms $a\sigma_j^+$ through the following transformation of the boson operators,

$$a^+ = b^+ + N^{\frac{1}{2}}\alpha^*, \qquad a = b + N^{\frac{1}{2}}\alpha, \tag{3}$$

where α^* and α are c-number parameters to be determined. Substituting equations (3) into (1) we obtain

$$H = H_0 + H_1,$$

with

$$H_0 = b^+b + N\alpha^*\alpha + \sum_{j=1}^{N} \varepsilon\sigma_j^z + \lambda\alpha^*\sigma_j^- + \lambda\alpha\sigma_j^+ , \tag{4a}$$

$$H_1 = b^+\Big(N^{\frac{1}{2}}\alpha + \lambda N^{-\frac{1}{2}} \sum_j \sigma_j^-\Big) + b\Big(N^{\frac{1}{2}}\alpha^* + \lambda N^{-\frac{1}{2}} \sum_j \sigma_j^+\Big). \tag{4b}$$

The partition function Z can be calculated if the term H_1 in the Hamiltonian is ignored. The aim of the BZT method is to select the constants α and α^* in such a way that the corrections arising from the neglected term H_1 are of order one ($O(1)$) and hence do not contribute in the thermodynamic limit ($N \to \infty$). H_0 is then called the thermodynamically equivalent Hamiltonian of H. The evaluation of the partition function, using only H_0, is as follows. We have

$$\mathrm{Tr}(\exp -\beta H_0) = \exp(-\beta\alpha^*\alpha N)\,\mathrm{Tr}(\exp -\beta b^+b) \prod_{j=1}^{N} \mathrm{Tr}(\exp -\beta M_j), \tag{5}$$

where

$$M_j = \varepsilon\sigma_j^z + \lambda\alpha^*\sigma_j^- + \lambda\alpha\sigma_j^+ = \begin{pmatrix} \varepsilon & \lambda\alpha \\ \lambda\alpha^* & -\varepsilon \end{pmatrix}.$$

The eignenvalues of M_j are $\pm\mu$ where

$$\mu = (\varepsilon^2 + \lambda^2\alpha^*\alpha)^{\frac{1}{2}}. \tag{6}$$

Substituting equation (6) into (5) gives

$$Z = \exp(-\beta N\alpha^*\alpha)(1-\exp-\beta)^{-1}(2\cosh\beta\mu)^N,$$

and the free energy per atom, F, in the thermodynamic limit is given by

$$F = \lim_{N\to\infty} -\ln(Z)/N\beta = \alpha^*\alpha - \ln(2\cosh\beta\mu)/\beta. \tag{7}$$

At this stage we have neglected H_1, and it is now necessary to show that H_1 does not contribute to F for an appropriately chosen value of α. If we expand $\mathrm{Tr}\{\exp-\beta(H_0+H_1)\}$ as a perturbation series in H_1 then we obtain terms of the form $\mathrm{Tr}(H_1\exp-\beta H_0)$ and $\mathrm{Tr}(H_1^2\exp-\beta H_0)$, where the integrations with respect to β' etc. have been omitted. The first term is trivially zero since

$$\mathrm{Tr}(b^+\exp-\beta b^+b) = 0.$$

The first nonzero term comes from the second term of the perturbation series and is given by

$$\mathrm{Tr}(b^+b\exp-\beta b^+b)\,\mathrm{Tr}\left\{\left(N^{\frac{1}{2}}\alpha+\lambda N^{-\frac{1}{2}}\sum_j\sigma_j^-\right)\left(N^{\frac{1}{2}}\alpha^*+\lambda N^{-\frac{1}{2}}\sum_j\sigma_j^+\right)\exp-\beta H_0\right\},$$

which is $O(N)$ unless we choose α to satisfy equation (8), in which case it is $O(1)$. Hence α is determined by

$$\mathrm{Tr}\left\{\left(N^{\frac{1}{2}}\alpha+\lambda N^{-\frac{1}{2}}\sum_j\sigma_j^-\right)\exp-\beta H_0\right\} = 0, \tag{8}$$

which reduces to

$$\alpha = -\lambda\,\mathrm{Tr}(\sigma_j^-\exp-\beta M_j)/\mathrm{Tr}(\exp-\beta M_j). \tag{9}$$

Since σ_j^- and M_j are 2×2 matrices the right-hand side of equation (9) can be evaluated quite simply. The result is

$$\alpha = (\lambda^2\alpha/2\mu)\tanh\beta\mu. \tag{10}$$

This equation has two solutions for α, either $\alpha = 0$ or $\alpha \neq 0$, and in the latter case α is determined by

$$2\mu = \lambda^2\tanh\beta\mu. \tag{11}$$

As will become clear in the next section, $\alpha = 0$ corresponds to the normal state and $\alpha \neq 0$ to the superradiant state. One of the conditions for there to be a solution to equation (11) is that $2\mu/\lambda^2 < 1$, which implies that $\lambda^2 > 2\varepsilon$ is a necessary condition for a superradiant state to exist at some temperature. Further, the temperature at which the transition from superradiant to normal occurs is obtained from equation (11), when $\mu = \varepsilon$. Thus β_c is given by

$$2\varepsilon = \lambda^2\tanh\beta_c\varepsilon. \tag{12}$$

3. Values of Thermodynamic Variables

Besides using the thermodynamically equivalent Hamiltonian H_0 to calculate the free energy, it can also be used to evaluate the thermodynamic variables, such as the average number of excited atoms and the number of photons. If n_+ denotes the number of excited atoms and n_- the number of unexcited atoms then

$$n_+ - n_- = \mathrm{Tr}\Big(Z^{-1} \sum_j \sigma_j^z \exp -\beta H_0\Big). \tag{13}$$

The right-hand side of equation (13) can be evaluated to yield

$$\begin{aligned}(n_+ - n_-)/N &= -\tanh \beta\varepsilon, \qquad \alpha = 0,\\ &= -2\varepsilon/\lambda^2, \qquad \alpha \neq 0.\end{aligned}$$

Hence the average number of excited atoms is given by

$$n_+/N = \tfrac{1}{2}(1 - \tanh \beta\varepsilon), \qquad \alpha = 0, \tag{14a}$$

$$= \tfrac{1}{2}(1 - 2\varepsilon/\lambda^2), \qquad \alpha \neq 0, \tag{14b}$$

The form of n_+ as a function of temperature is illustrated in Fig. 1.

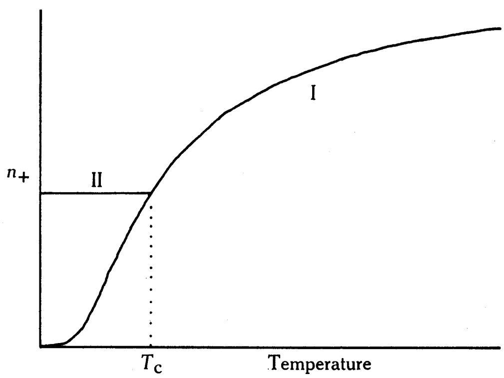

Fig. 1. Number n_+ of excited atoms as a function of temperature, from the derived equations (14). T_c is the critical temperature for transition between the normal state (I) and the superradiant state (II).

The average number of photons per atom is given by

$$\begin{aligned}(NZ)^{-1}\mathrm{Tr}(a^+ a \exp -\beta H_0) &= (NZ)^{-1}\mathrm{Tr}(b^+ b \exp -\beta H_0) + \alpha^*\alpha\\ &= \alpha^*\alpha \qquad \text{in} \qquad \lim N \to \infty. \end{aligned} \tag{15}$$

Thus in the normal state the number of photons is of order one while in the superradiant state it is of order N. This is analogous to the difference in the radiation rates of the normal and superradiant states mentioned in the Introduction.

Another interesting quantity to calculate from the point of view of establishing the equivalence between the Dicke and Ising models is

$$\langle \sigma^x \rangle = (NZ)^{-1} \sum_{j=1}^{N} \sigma_j^x \exp -\beta H_0,$$

which can be verified to be equal to $-2\alpha/\lambda$. The quantity $\langle\sigma^x\rangle$ can be regarded as the average magnetization in the x direction, where the atom is now regarded as a quantum mechanical spin. This then gives us a physical interpretation of the parameter α,

$$\alpha = -\tfrac{1}{2}\lambda\langle\sigma^x\rangle. \tag{16}$$

Using the fact that α is real, we can substitute equation (16) into (4a) to obtain the following form for the thermodynamically equivalent Hamiltonian

$$H_0 = b^+b + \tfrac{1}{4}N\lambda^2\langle\sigma^x\rangle^2 + \sum_{j=1}^{N} \varepsilon\sigma_j^z - \tfrac{1}{2}\lambda^2\sigma_j^x\langle\sigma^x\rangle, \tag{17}$$

which, on ignoring the unimportant terms, is the thermodynamically equivalent Hamiltonian of the long-range interaction Ising model, given by

$$H = -(\lambda^2/2N)\sum_{j=1}^{N}\sum_{j'=1}^{N}\sigma_j^x\sigma_{j'}^x + \varepsilon\sum_{j=1}^{N}\sigma_j^z. \tag{18}$$

In the case $\alpha = 0$ the Hamiltonian H_0 can be written as

$$H_0 = \varepsilon\sum_{j=1}^{N}\sigma_j^z.$$

Thus, in the normal state, the system behaves as N non-interacting spins but, in the superradiant phase, an effective Ising-model type of interaction is set up between the spins by means of their interaction with the common electromagnetic field.

For temperatures near but below the critical temperature T_c it can be verified that $\langle\sigma^x\rangle$ behaves as $(T-T_c)^{\frac{1}{2}}$, which corresponds to the classical value of $\beta = \frac{1}{2}$. Similarly the number of photons $\alpha^*\alpha$ goes to zero like $(T-T_c)$ as $T\to T_c$.

4. Generalization to Finite Mode Case

The generalization to the finite mode system is trivial in this formalism. The Hamiltonian with m radiation modes of frequencies $\nu_1, \nu_2, \ldots, \nu_m$ is given by (Wang and Hioe 1973)

$$H = \sum_{s=1}^{m}\nu_s a_s^+ a_s + \sum_{j=1}^{N}\left\{\varepsilon\sigma_j^z + N^{-\frac{1}{2}}\left(\sum_{s=1}^{m}\lambda_s a_s\right)\sigma_j^+ + N^{-\frac{1}{2}}\left(\sum_{s=1}^{m}\lambda_s a_s^+\right)\sigma_j^-\right\},$$

where $\lambda_1, \lambda_2, \ldots, \lambda_m$ are the coupling constants. The BZT technique applies equally well to this Hamiltonian; the boson operators are transformed as

$$a_s = b_s + \alpha_s N^{\frac{1}{2}}, \qquad a_s^+ = b_s^+ + \alpha_s^* N^{\frac{1}{2}}.$$

The thermodynamically equivalent Hamiltonian is

$$H_0 = N\sum_s \nu_s\alpha_s^*\alpha_s + \sum_s \nu_s b_s^+ b_s + \sum_j \varepsilon\sigma_j^z + \Delta\sigma_j^+ + \Delta^*\sigma_j^-,$$

where $\Delta = \sum\lambda_s\alpha_s$.

The free energy per atom can be evaluated to give in the thermodynamic limit

$$F = \sum_s \nu_s \alpha_s^* \alpha_s - \ln(2\cosh\beta\mu)/\beta,$$

where $\mu = (\varepsilon^2 + \Delta^2)^{\frac{1}{2}}$.

The equation for determining α_s is equivalent to (8) and gives the following equation for Δ

$$\Delta = (\Lambda^2 \Delta / 2\mu)\tanh\beta\mu,$$

where

$$\Lambda^2 = \sum_{s=1}^{m} \lambda_s^2 / \nu_s.$$

Hence, there are again two solutions: $\Delta = 0$ or $\Delta \neq 0$ corresponding to the natural and superradiant states. This result was also obtained by Wang and Hioe (1973). The thermodynamic variables can be calculated in a similar manner to those calculated in Section 3, giving

$$\begin{aligned} n_+/N &= \tfrac{1}{2}(1 - \tanh\beta\varepsilon), && \Delta = 0, \\ &= \tfrac{1}{2}(1 - 2\varepsilon/\Lambda^2), && \Delta \neq 0. \end{aligned}$$

The average number of photons of the sth mode per atom is given by

$$\begin{aligned} N^{-1}\langle a_s^+ a_s \rangle &= 0, && \Delta = 0, \\ &= \lambda_s^2 \Delta^* \Delta / \Lambda^4, && \Delta \neq 0, \end{aligned}$$

while the thermodynamically equivalent Hamiltonian can be written

$$H_0 = \varepsilon \sum_j \sigma_j^z - (\Lambda^2/2N) \sum_j \sum_{j'} \sigma_j^x \sigma_{j'}^x.$$

As can be seen by comparing the results in this section with those obtained in Section 3, the generalized model has the same properties as the Dicke model.

5. Conclusions

Wentzel (1960) has shown that the BZT method allows an exact evaluation of thermodynamic functions for a certain class of Hamiltonians. Although Wentzel did not consider the type of Hamiltonians investigated above, his proof can presumably be generalized to include Hamiltonians of the Dicke type. The BZT solution given here is not only a considerable simplification of Hepp and Lieb's (1973) original solution but also provides greater insight into the nature of the coupling between the atoms when the system is in the superradiant state. The similarity between the Dicke model and the mean-field approximation of the Ising model is explained in terms of the similarity of the thermodynamically equivalent Hamiltonians. It is interesting to ask whether the equivalence between the Dicke model and the Ising model at equilibrium is also valid for some non-equilibrium states, and whether the dynamical properties of the Ising model can be related to those of the Dicke model.

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