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A Proposed Experimental Investigation of Transverse Stress Wave Propagation in Plasmas at Microwave Frequencies

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Abstract

The propagation properties of transverse electromagnetic waves in plasmas are discussed from the point of view of the 13-moment approximation. It is shown that additional modes, termed stress modes, should be able to propagate in a plasma even below the plasma frequency. The properties of these modes are examined, particularly with regard to energy transmission, and the conditions for their experimental detection are considered. Practical parameters for possible experiments are given.

Introduction

Propagation properties of waves in plasmas have been calculated for various cases of magnetized and inhomogeneous media, both of low and high frequencies with respect to the ion plasma frequency (Spitzer 1962; Stix 1962; Denisse and Delcroix 1963; Liley 1963; Ginzburg 1964; Bydder 1967). In the present investigation, an interest in microwave propagation restricts the discussion to electronic modes, while for simplicity only a non-magnetized homogeneous plasma is considered. When the dispersion equations for such a plasma are calculated using expressions derived with the 13-moment approximation of Grad (1949; Herdan and Liley 1960), certain modes of transverse electromagnetic propagation not predicted by less detailed treatments are found. Since the 13-moment equations are a more accurate description of the plasma dynamics in physically significant variables than the hydromagnetic equations (which only predict the 'normal' electromagnetic modes), there can be little doubt that the additional modes exist. That these modes have not been found experimentally is probably due to the relatively low energy that they can carry through a plasma.

The above additional modes allow the propagation (in a non-magnetized plasma) of transverse waves at velocities of the order of the electron thermal speed, normally much smaller than the group velocity of light in the medium. They have been termed stress waves (Liley 1963; Bydder 1967) because they are associated with the non-diagonal terms of the plasma stress tensor and the plasma heat flux vector. The resemblance to longitudinal plasma waves is superficial, since the propagating stress modes are incompressible and involve transverse oscillating electric and magnetic fields with the same phase relations as in the transverse electromagnetic waves. Since the boundary conditions which allow coupling of transverse electromagnetic waves into the plasma also match coupling into stress modes, incident microwaves should also propagate through the plasma as stress waves. Accordingly the conditions under which their propagation could be measured may be obtained from the propagation characteristics of these waves. Their propagation could then be tested experimentally.

13-moment Equations

As is well known (Chapman and Cowling 1960), when the Enskog equations of change are derived from the Boltzmann equation they do not form a closed set. The equation of change or *moment equation* for a general function $\psi(\boldsymbol{r}, \boldsymbol{w}_i, t)$ of the particle velocity $\boldsymbol{w}_i$ is

$$\frac{\mathrm{d}(n_i\langle\psi_i\rangle)}{\mathrm{d}t} + n_i\langle\psi_i\rangle\nabla\cdot\boldsymbol{V} + \nabla\cdot(n_i\langle\psi_i\boldsymbol{w}_i\rangle)$$

$$-n_i\left(\left\langle\frac{\mathrm{d}\psi_i}{\mathrm{d}t}\right\rangle + \left\langle\boldsymbol{w}_i\cdot\nabla\psi_i\right\rangle + \boldsymbol{F}_i\cdot\left\langle\frac{\partial\psi_i}{\partial\boldsymbol{w}_i}\right\rangle + \boldsymbol{b}_i\cdot\left\langle\frac{\partial\psi_i}{\partial\boldsymbol{w}_i}\times\boldsymbol{w}_i\right\rangle - \left\langle\frac{\partial\psi_i}{\partial\boldsymbol{w}_i}\boldsymbol{w}_i\right\rangle : \nabla\boldsymbol{V}\right)$$

$$= \tfrac{1}{2}\sum_{i,j} I_{jk}(\psi_i), \qquad (1)$$

with

$$\mathrm{d}/\mathrm{d}t \equiv \partial/\partial t + \boldsymbol{V}\cdot\nabla, \qquad \boldsymbol{F}_i = \boldsymbol{a}_i + \boldsymbol{V}\times\boldsymbol{b}_i - \mathrm{d}\boldsymbol{V}/\mathrm{d}t.$$

In equation (1), $\boldsymbol{w}_i = \boldsymbol{c}_i - \boldsymbol{V}$, where $\boldsymbol{c}_i$ is the particle velocity relative to a rest frame and $\boldsymbol{V}$ is the velocity of the frame to which the equations are referred; $\boldsymbol{b}_i = (e_i/m_i)\boldsymbol{B}$, where e_i and m_i are the charge and mass of particles of type i and $\boldsymbol{B}$ is the macroscopic magnetic field; $\boldsymbol{a}_i + \boldsymbol{V}\times\boldsymbol{b}_i$ is the acceleration due to the macroscopic fields (e.g. for an electric field $\boldsymbol{E}$, $\boldsymbol{a}_i = (e_i/m_i)\boldsymbol{E}$); and n_i is the number density of particles of type i. The average of ψ_i over velocity space, $\langle\psi_i\rangle$, is

$$\langle\psi_i\rangle = n_i^{-1}\int_{w_i} f_i\psi_i\,\mathrm{d}\boldsymbol{w}_i$$

and the moment of ψ_i is $n_i\langle\psi_i\rangle$, where f_i is the distribution function of type i particles. The right-hand side of equation (1) is the collision integral.

For the 13-moment equations, a description is sought in terms of the physically significant variables: density ρ_i, mean velocity $\boldsymbol{u}_i$, temperature T_i, stress tensor $\mathbf{S}_i$ and heat flux $\boldsymbol{R}_i$. The equations for these parameters are obtained by taking ψ_i respectively equal to m_i, $m_i\boldsymbol{w}_i$, $\frac{1}{2}m_i w_i^2$, $m_i\boldsymbol{w}_i\boldsymbol{w}_i$ and $\frac{1}{2}m_i w_i^2\boldsymbol{w}_i - \frac{5}{2}kT_i\boldsymbol{w}_i$. The equations for the stress tensor and heat flux vector involve, however, moments of higher order in $\boldsymbol{w}_i$ than those listed and therefore the system of equations must be truncated. Grad (1949) achieved this by expressing the distribution function as a linear function of the above moments for the purpose of calculating the higher order moments in terms of them. Using a three-dimensional Hermite polynomial expansion, the distribution function is written

$$f(\boldsymbol{r},\boldsymbol{w},t) = \exp(-mw^2/2kT)\sum_{n=1}^{4} a_n(\boldsymbol{r},t)\,H^n(\boldsymbol{w}).$$

By making use of the orthogonality properties of the Hermite polynomials H^n, the coefficients a_n are readily obtained in terms of the moments. The expression becomes

$$f = n(\alpha/\pi)^{3/2}\exp(-\alpha w^2)\{1 + 2\alpha\boldsymbol{u}\cdot\boldsymbol{w} + (\alpha/p)\mathbf{P}:\boldsymbol{w}\boldsymbol{w} + (\tfrac{4}{5}\alpha^2/p)\boldsymbol{R}\cdot\boldsymbol{w}(w^2 - \tfrac{5}{2}\alpha^{-1})\},$$

where $\mathbf{P}$ is the non-hydrostatic component of the stress tensor (see below), p is the hydrostatic pressure, and $\alpha = m/2kT$, k being Boltzmann's constant. When the stress tensor and heat flux vector equations are written in terms of the 13 independent moments, namely ρ (1 component), $\boldsymbol{u}$ (3 components), T (1 component), $\mathbf{S}$ (5 independent components) and $\boldsymbol{R}$ (3 components), with the higher moments calculated from this distribution function, the equations are known as the 13-moment equations (Grad 1949; Herdan and Liley 1960). There is, of course, one set of equations for each type of particle present in the plasma. However, the moment equations for the electron component of the plasma separate, apart from the collision terms, from those of the heavy components provided that (Bydder 1967)

$$m_e/T_e \ll m_i/T_i \qquad \text{and} \qquad m_e T_e \ll m_i T_i \,.$$

Neglecting the subscripts and considering only elastic collisions, the 13-moment equations for the electrons in the plasma, with the collision terms written in terms of the momentum transfer collision frequency ν, are as follows.

(i) The continuity equation is

$$\partial\rho/\partial t + \nabla.(\rho\boldsymbol{u}+\rho\boldsymbol{V}) = 0\,.$$

(ii) Writing $\boldsymbol{J} = ne\boldsymbol{u}$, the momentum equation becomes a generalized Ohm's Law:

$$\frac{m}{e}\frac{\mathrm{d}\boldsymbol{J}}{\mathrm{d}t} + \frac{m}{e}\boldsymbol{J}\nabla.\boldsymbol{V} + \nabla.\mathbf{P} + \nabla p - \frac{\rho e}{m}(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B})$$
$$-\boldsymbol{J}\times\boldsymbol{B} + \frac{m}{e}\boldsymbol{J}.\nabla\boldsymbol{V} + \frac{m}{e}\nu\boldsymbol{J} - \frac{3p}{5\rho}\varepsilon\nu\boldsymbol{R} = 0\,.$$

(iii) Since $p = nkT$, the thermal energy equation is

$$\frac{\mathrm{d}(\frac{3}{2}p)}{\mathrm{d}t} + \tfrac{3}{2}p\nabla.\boldsymbol{V} + \nabla.\boldsymbol{R} + \frac{5m}{2e}\nabla.\left(\frac{p\boldsymbol{J}}{\rho}\right) + \frac{m}{e}\boldsymbol{J}.\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}$$
$$-\boldsymbol{J}.(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B}) + \mathbf{P}:\nabla\boldsymbol{V} + p\nabla.\boldsymbol{V} = 0\,.$$

(iv) The stress tensor equation is

$$\frac{\mathrm{d}\mathbf{P}}{\mathrm{d}t} + \mathbf{P}\nabla.\boldsymbol{V} + \tfrac{4}{5}\{\nabla\boldsymbol{R}\} + \frac{2m}{e}\left\{\nabla\left(\frac{p\boldsymbol{J}}{\rho}\right)\right\} - 2\{(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B})\boldsymbol{J}\}$$
$$+\frac{2m}{e}\left\{\boldsymbol{J}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right\} + 2\{\mathbf{P}.\nabla\boldsymbol{V}\} + 2p\{\nabla\boldsymbol{V}\} - \frac{2e}{m}\{\mathbf{P}\times\boldsymbol{B}\} + \beta\nu\mathbf{P} = 0\,,$$

where $\mathbf{P} \equiv \{\mathbf{S}\}$, the symbol $\{\,\}$ denoting the non-hydrostatic component of the tensor (the explicit form of $\{\,\}$ as an operator is given by Bydder and Liley 1968).

(v) Finally, the heat flux equation is

$$\frac{\mathrm{d}\boldsymbol{R}}{\mathrm{d}t} + \tfrac{7}{2}\mathbf{P}.\nabla\left(\frac{p}{\rho}\right) + \tfrac{5}{2}p\nabla\left(\frac{p}{\rho}\right) + \frac{p}{\rho}\nabla.\mathbf{P} + \frac{5m}{2e}\boldsymbol{J}\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{p}{\rho}\right)$$
$$-\frac{e}{m}\mathbf{P}.\left(\boldsymbol{E}+\boldsymbol{V}\times\boldsymbol{B} - \frac{m}{e}\frac{\mathrm{d}\boldsymbol{V}}{\mathrm{d}t}\right) - \frac{e}{m}\boldsymbol{R}\times\boldsymbol{B} + \tfrac{7}{5}\boldsymbol{R}\nabla.\boldsymbol{V} + \tfrac{7}{5}\boldsymbol{R}.\nabla\boldsymbol{V} + \tfrac{2}{5}\nabla\boldsymbol{V}.\boldsymbol{R}$$
$$+\frac{mp}{e\rho}\boldsymbol{J}\nabla.\boldsymbol{V} + \frac{mp}{e\rho}\nabla\boldsymbol{V}.\boldsymbol{J} + \frac{mp}{e\rho}\boldsymbol{J}.\nabla\boldsymbol{V} - \frac{3mp}{2e\rho}\theta\nu\boldsymbol{J} + \zeta\nu\boldsymbol{R} = 0\,.$$

The coefficients β, ζ, ε and θ depend on the nature of the collisions with the heavy particles, but they are all of order unity.

Wave Equations

At sufficiently high frequencies, it is apparent that the ions and neutrals, because of their large inertia, will be effectively stationary and the electrons will determine the wave properties. This is the case when

$$\omega \gg \omega_{\mathrm{bi}} \quad \text{and} \quad \omega \gg \tfrac{5}{3}(p_{\mathrm{e0}}/\rho_0)^{\frac{1}{2}}\kappa\,, \tag{2a, b}$$

where ω_{bi} is the ion cyclotron frequency, p_{e0} the average electron pressure, ρ_0 the plasma density and κ the wave number corresponding to the angular frequency ω. The restriction (2b), taken in conjunction with the decoupling conditions, is equivalent to requiring that the wave speed is much greater than the heavy particle thermal speeds.

The steady state solution for a homogeneous plasma is obviously $\boldsymbol{J} = 0$, $\mathbf{P} = 0$, $\boldsymbol{R} = 0$, and $p = p_0$, $\rho = \rho_0$. From Maxwell's equations, $\boldsymbol{E} = 0$ and $\boldsymbol{B} = \boldsymbol{B}_0$. Referring the equations to a frame moving with the plasma mean mass velocity further simplifies the equations. For small amplitude wave motion in the homogeneous plasma, the equations may be linearized for the perturbations. With the zero subscripts designating steady state parameters, and non-subscripted parameters the perturbed variables, the linearized equations for wave motion in the homogeneous plasma are:

$$\frac{\partial\rho}{\partial t} + \frac{m}{e}\nabla.\boldsymbol{J} = 0,$$

$$\frac{\partial\boldsymbol{J}}{\partial t} + \frac{e}{m}\nabla.\mathbf{P} + \frac{e}{m}\nabla p - \frac{\rho_0 e^2}{m^2}\boldsymbol{E} - \frac{e}{m}\boldsymbol{J}\times\boldsymbol{B}_0 + \nu\boldsymbol{J} - \frac{3ep_0}{5m\rho_0}\varepsilon\nu\boldsymbol{R} = 0,$$

$$\frac{\partial p}{\partial t} + \tfrac{2}{3}\nabla.\boldsymbol{R} + \frac{5mp_0}{3e\rho_0}\nabla.\boldsymbol{J} = 0,$$

$$\frac{\partial\mathbf{P}}{\partial t} + \tfrac{4}{5}\{\nabla\boldsymbol{R}\} + \frac{2mp_0}{e\rho_0}\{\nabla\boldsymbol{J}\} - \frac{2e}{m}\{\mathbf{P}\times\boldsymbol{B}_0\} + \beta\nu\mathbf{P} = 0,$$

$$\frac{\partial\boldsymbol{R}}{\partial t} + \frac{5p_0}{2\rho_0}\nabla p - \frac{5p_0^2}{2\rho_0^2}\nabla\rho + \frac{p_0}{\rho_0}\nabla.\mathbf{P} - \frac{e}{m}\boldsymbol{R}\times\boldsymbol{B}_0 - \frac{3mp_0}{2e\rho_0}\theta\nu\boldsymbol{J} + \zeta\nu\boldsymbol{R} = 0.$$

Maxwell's equations, already linear, are necessary to complete the set:

$$\nabla\times\boldsymbol{B} - \mu_0\boldsymbol{J} - \frac{1}{c^2}\frac{\partial\boldsymbol{E}}{\partial t} = 0, \qquad \nabla\times\boldsymbol{E} + \frac{\partial\boldsymbol{B}}{\partial t} = 0.$$

These equations separate into a number of independent sets, depending on the orientation of the steady magnetic field with respect to the oscillating electric field components. In the absence of a magnetic field two similar sets of equations describe transverse wave motion and a third describes longitudinal motion.

Since this paper is concerned with establishing the propagation properties of transverse modes for experimental study, only the simplest case of no magnetic field is considered. Using a cartesian frame with propagation in the z direction, and the x axis aligned with the wave polarization direction, the wave motion can be described in terms of the five components B_y, J_x, E_x, P_{xz} and R_x. The linearized equations relating to these are

$$-\frac{\partial B_y}{\partial z}-\mu_0 J_x-\frac{1}{c^2}\frac{\partial E_x}{\partial t}=0,$$

$$\frac{\partial E_x}{\partial z}+\frac{\partial B_y}{\partial t}=0,$$

$$\frac{\partial J_x}{\partial t}+\frac{e}{m}\frac{\partial P_{xz}}{\partial z}-\frac{e^2}{m^2}\rho_0 E_x+\nu J_x-\frac{3ep_0}{5m\rho_0}\varepsilon\nu R_x=0,$$

$$\frac{\partial P_{xz}}{\partial t}+\frac{2}{5}\frac{\partial R_x}{\partial z}+\frac{mp_0}{e\rho_0}\frac{\partial J_x}{\partial z}+\beta\nu P_{xz}=0,$$

$$\frac{\partial R_x}{\partial t}+\frac{p_0}{\rho_0}\frac{\partial P_{xz}}{\partial z}-\frac{3mp_0}{2e\rho_0}\theta\nu J_x+\zeta\nu R_x=0.$$

Assuming plane steady waves of angular frequency ω and wave number κ, the dispersion relation for this set of equations is

$$\begin{aligned}
&\kappa^4u^2c^2(\tfrac{7}{5}\mathrm{i}\omega+\tfrac{2}{5}\nu+\zeta\nu+\tfrac{3}{5}\theta\nu+\tfrac{3}{5}\varepsilon\nu)\\
&+\kappa^2c^2(\mathrm{i}\omega\beta\zeta\nu^2-\mathrm{i}\omega^3-\beta\omega^2\nu-\zeta\omega^2\nu+\beta\zeta\nu^3-\omega^2\nu+\mathrm{i}\omega\beta\nu^2\\
&\qquad+\mathrm{i}\omega\zeta\nu^2-\tfrac{9}{10}\varepsilon\theta\beta\nu^3-\tfrac{9}{10}\mathrm{i}\omega\varepsilon\theta\nu^2)\\
&+\kappa^2u^2(-\tfrac{7}{5}\mathrm{i}\omega^3-\tfrac{2}{5}\omega^2\nu-\omega^2\zeta\nu-\tfrac{3}{5}\omega^2\theta\nu-\tfrac{3}{5}\omega^2\varepsilon\nu+\tfrac{2}{5}\mathrm{i}\omega\omega_\mathrm{p}^2)\\
&+(-\mathrm{i}\omega^3\beta\zeta\nu^2+\mathrm{i}\omega^5+\omega^4\beta\nu+\omega^4\zeta\nu-\omega^2\beta\zeta\nu^3+\omega^4\nu-\mathrm{i}\omega^3\beta\nu^2-\mathrm{i}\omega^3\zeta\nu^2\\
&\qquad+\tfrac{9}{10}\omega^2\varepsilon\theta\beta\nu^3+\tfrac{9}{10}\mathrm{i}\omega^3\varepsilon\theta\nu^2+\mathrm{i}\omega\omega_\mathrm{p}^2\beta\zeta\nu^2-\omega^2\omega_\mathrm{p}^2\beta\nu-\omega^2\omega_\mathrm{p}^2\zeta\nu-\mathrm{i}\omega^3\omega_\mathrm{p}^2)=0,
\end{aligned}\tag{3}$$

where $\omega_\mathrm{p}=(ne^2/m\varepsilon_0)^{\frac{1}{2}}$ is the plasma frequency and $u=(kT_\mathrm{e}/m_\mathrm{e})^{\frac{1}{2}}$ is the electron thermal speed.

Neglecting collisions, the solutions of the dispersion equation (3) are

$$\kappa^2=(\omega^2-\omega_\mathrm{p}^2)/c^2 \quad\text{and}\quad \kappa^2=5\omega^2/7u^2. \tag{4a, b}$$

The solution (4a) is the well-known collisionless result for microwave propagation in a homogeneous plasma, giving a refractive index

$$\eta=(1-\omega_\mathrm{p}^2/\omega^2)^{\frac{1}{2}}=\kappa/\kappa_0,$$

where κ_0 is the vacuum region wave number. The second solution (4b) is the stress mode. Although this result is probably contained in direct solutions of the Boltzmann equation, including the Chapman–Enskog solution, it has not been recognized because these methods do not give results in a readily interpretable physical form. It is a dispersionless mode, the phase and group velocity being given by $\sqrt{\frac{7}{5}}\,u$. The ratio of the wavelength λ to the Debye length λ_D is

$$\lambda/\lambda_D = 2\pi\sqrt{\tfrac{7}{5}}\,\omega_p/\omega\,.$$

When the collision terms are included, it is found that the damping lengths are of the order of a few electron mean free paths.

It is difficult to resolve the question as to whether the transverse electron stress mode can propagate when the wavelength is less than the Debye length (the electromagnetic modes do propagate in such circumstances), but at frequencies less than the plasma frequency there appears to be no impediment to propagation. It is in this frequency region that the normal electromagnetic mode does not propagate and therefore this is the region in which stress propagation should be experimentally studied. Furthermore, it is in this frequency region that practical use could possibly be made of stress-mode propagation.

Energy Considerations

The transverse modes are incompressible, so that the plasma energy density $\frac{3}{2}p$ does not fluctuate (heating effects due to collisions are not being considered). Hence, in obtaining the Poynting vector, only the electromagnetic energy need be considered. The energy density input is

$$\boldsymbol{J}.\boldsymbol{E} = \mu_0^{-1}\nabla\times\boldsymbol{B}.\boldsymbol{E} - \varepsilon_0\boldsymbol{E}.\frac{\partial\boldsymbol{E}}{\partial t}$$

$$= -\varepsilon_0\boldsymbol{E}.\frac{\partial\boldsymbol{E}}{\partial t} - \mu_0^{-1}\nabla.\boldsymbol{E}\times\boldsymbol{B} - \mu_0^{-1}\boldsymbol{B}.\frac{\partial\boldsymbol{B}}{\partial t}.$$

Integrating over a volume v, we find

$$\int_v\left(\boldsymbol{J}.\boldsymbol{E} + \varepsilon_0\boldsymbol{E}.\frac{\partial\boldsymbol{E}}{\partial t} + \mu_0^{-1}\boldsymbol{B}.\frac{\partial\boldsymbol{B}}{\partial t}\right)\mathrm{d}v = \mu_0^{-1}\int\boldsymbol{E}\times\boldsymbol{B}.\,\mathrm{d}\boldsymbol{s},$$

showing that the energy flux for transverse waves in the plasma is given by this form of the Poynting vector, namely $\mu_0^{-1}\boldsymbol{E}\times\boldsymbol{B}$.

For wave motion of the general form

$$\boldsymbol{E} = \tilde{\boldsymbol{E}}\exp[\mathrm{i}(\omega t - \boldsymbol{\kappa}.z)]\,,$$

where $\tilde{\boldsymbol{E}}$ is the peak electric field, the Maxwell equations give

$$\boldsymbol{\kappa}\times\boldsymbol{E} = \omega\boldsymbol{B}\,.$$

The instantaneous energy flux $\boldsymbol{S}$ can be written in terms of the electric field:

$$\boldsymbol{S} = \mu^{-1}\boldsymbol{E}\times(\boldsymbol{\kappa}\times\boldsymbol{E})/\omega = \boldsymbol{\kappa}E^2/\mu\omega\,.$$

The averaged flux is simply

$$\bar{S} = \kappa \bar{E}^2/2\mu\omega .$$

In terms of the electric field amplitude in the plasma for the various modes, the electromagnetic energy flux is

$$\bar{S}_e = \bar{E}_e^2(1-\omega_p^2/\omega^2)^{\frac{1}{2}}/2\mu c$$

and the stress flux is

$$\bar{S}_s = \bar{E}_s^2/2\mu u .$$

However, in order to compare the energy transmitted by different modes, the Poynting flux must be expressed in terms of the field external to the plasma. This is readily calculated for a uniform plasma slab with sharp boundaries normal to the wave vector. Denoting the incident and reflected vacuum fields by $\boldsymbol{E}_i, \boldsymbol{B}_i$ and $\boldsymbol{E}_r, \boldsymbol{B}_r$ respectively and the transmitted plasma fields by $\boldsymbol{E}_t, \boldsymbol{B}_t$, these boundary fields can be matched. As there are no surface fluctuating charges or currents, $\boldsymbol{E}$ and $\boldsymbol{B}$ are continuous across the boundary and hence

$$\boldsymbol{E}_i + \boldsymbol{E}_r = \boldsymbol{E}_t \qquad \text{and} \qquad \boldsymbol{B}_i + \boldsymbol{B}_r = \boldsymbol{B}_t .$$

Writing $\boldsymbol{B}$ in terms of $\boldsymbol{E}$, the second condition becomes

$$\boldsymbol{\kappa}_0 \times \boldsymbol{E}_i/\omega - \boldsymbol{\kappa}_0 \times \boldsymbol{E}_r/\omega = \boldsymbol{\kappa} \times \boldsymbol{E}_t/\omega .$$

Since $\boldsymbol{\kappa}$ and $\boldsymbol{\kappa}_0$ are perpendicular to the electric fields,

$$E_i - E_r = (\kappa/\kappa_0)E_t ,$$

which gives

$$E_t = 2E_i/(1+\kappa/\kappa_0) .$$

In terms of the incident (vacuum) electric field the averaged energy flux is (see Appendix 1)

$$\bar{S} = \frac{\kappa}{2\mu_0\,\omega}\,\frac{4\bar{E}_i^2}{(1+\kappa/\kappa_0)^2} .$$

The electromagnetic energy flux is

$$\bar{S}_e = \frac{2\bar{E}_i^2}{\mu_0\, c}\left(1-\frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}\left[1+\left(1-\frac{\omega_p^2}{\omega^2}\right)^{\frac{1}{2}}\right]^{-2}$$

while the stress energy flux is

$$\bar{S}_s = \frac{2\bar{E}_i^2}{\mu_0}\,\frac{u}{(u+c)^2} \sim \frac{2\bar{E}_i^2}{\mu_0}\,\frac{u}{c^2} .$$

For $\omega_p \ll \omega$ the electromagnetic energy flux is essentially equal to the vacuum flux, while in all cases the stress energy flux is $\sim u/c$ smaller than the free space flux.

In order to measure the propagation of stress waves through a slab of plasma in the electromagnetic wave cutoff region, the evanescent energy flux penetrating the slab should be less than the energy transmitted by the stress wave. This is readily calculated using the same boundary conditions. Again let the incident and reflected fields on the vacuum side of the interface be E_i, E_r etc., while the evanescent fields in the plasma at the boundary are $E_1, \ldots$. It follows at once that

$$E_i + E_r = E_1, \qquad B_i + B_r = B_1$$

and, as for the propagating case, that

$$E_1 = 2E_i/(1+\kappa/\kappa_0).$$

The intensity or energy transmission coefficient for the boundary is

$$T = |2/(1+\kappa/\kappa_0)|^2.$$

As shown in Appendix 2, collisions allow the cutoff mode to transmit energy to the second boundary, at which

$$E_{e2i} = E_{e1}\exp(-\kappa_i z_0),$$

where $i\kappa_i$ is the complex part of the wave number and z_0 the slab thickness. Since for cases of interest we have $|E_{e2i}/E_{e1}| \ll 1$, the effect of the wave reflected from the second boundary has been neglected. The phase difference between the boundaries has also been ignored for this reason.

At the second interface the boundary conditions are

$$E_{e2i} + E_{e2r} = E_{e2}, \qquad B_{e2i} + B_{e2r} = B_{e2},$$

from which follows

$$E_{e2} = 2\kappa E_{e2i}/(\kappa+\kappa_0), \qquad B_{e2} = 2\kappa_0\kappa E_{2i}/\omega(\kappa+\kappa_0).$$

In terms of the incident vacuum field,

$$E_{e2} = [4\kappa\kappa_0 \exp(-\kappa_i z_0)/(\kappa+\kappa_0)^2]E_i.$$

The mean energy flux out of the plasma slab from the evanescent mode is

$$\bar{S}_{e2} = \frac{\mathrm{Re}(E_{e2}B_{e2}^*)}{2\mu_0} = \left(\frac{16\kappa_0^3\exp(-2\kappa_i z_0)(\kappa_r^2+\kappa_i^2)^2}{2\mu_0\,\omega[(\kappa_r+\kappa_0)^2+\kappa_i^2]^2}\right)E_i^2.$$

Since we have $|\kappa_i| \gg |\kappa_r|$,

$$\bar{S}_{e2} \sim \frac{16\kappa_0\exp(-2\kappa_i z_0)}{2\mu_0\,\omega}\left(\frac{\kappa_0}{\kappa_i}\right)^2 E_i^2$$

$$= 16(\kappa_0/\kappa_i)^2\exp(-2\kappa_i z_0)\,\bar{S}_i,$$

where κ_i for the evanescent wave is algebraically κ_r for the same mode in the propagation region. Taking

$$\kappa_i = \omega/c(1-\omega_p^2/\omega^2)^{\frac{1}{2}} \sim (\omega/c)(1+\omega_p^2/2\omega^2),$$

we have

$$\bar{S}_{e2}/\bar{S}_i \sim 16(1+\omega_p^2/2\omega^2)^{-2}\exp(-2z_0/d),$$

where $d = c(1-\omega_p^2/\omega^2)^{\frac{1}{2}}/\omega$.

The energy transmitted through the slab by the stress wave is calculated similarly. Neglecting losses due to collisions, the vacuum electric field from the stress wave is

$$E_{s2} = 2E_{s1}/(1+\kappa_0/\kappa).$$

The energy flux is

$$S_{s2} = \left(\frac{2E_i}{1+\kappa/\kappa_0}\right)^2\left(\frac{2}{1+\kappa_0/\kappa}\right)^2\frac{\kappa_0}{\mu_0\,\omega}$$

$$= S_i[16\,\kappa_0^2\,\kappa^2/(\kappa_0+\kappa)^4].$$

Substituting for κ, and changing to averaged fluxes,

$$\bar{S}_{s2} = \bar{S}_i\left(\frac{4\omega^2/uc}{(\omega/u+\omega/c)^2}\right)^2 \sim \bar{S}_i\left(\frac{16u^2}{c^2}\right).$$

It is apparent that both the process of converting incident electromagnetic waves into stress waves and its converse have weak coupling. The Fabry–Perot effect of producing high transmittance because of many reflections would not contribute significantly to the transmitted stress energy unless the electron mean free path were very long, apart from the physical difficulties of maintaining a sharp boundary.

Plasma Parameters for Experimental Propagation of Stress Waves in the Cutoff Region

In order to measure the propagation of stress waves through a plasma slab, the plasma frequency should be above cutoff. The slab should be as homogeneous as possible to ensure uniform opaqueness in all regions to the incident microwaves, and to reduce diffraction effects which could result in microwaves being scattered into the receiving antenna. The dimensions and parameters of the plasma must ensure that the propagated microwave flux is much less than the calculated stress energy flux. To this end, stress wave losses should be kept low, corresponding to achieving electron mean free paths at least of the order of the slab thickness. For a reasonably ionized plasma this requires the electron temperature to be as high as possible. Again, since the stress energy flux is proportional to T_e, the received energy should scale as T_e.

The proposed experiment could be carried out using R band microwave components at a frequency of 37 GHz and a power input of 20 mW. The plasma density for cutoff at this frequency is $\sim 1\cdot 8\times 10^{19}$ m^{-3}, and a practical minimum density is therefore likely to be about twice this, e.g. 3–4$\times 10^{19}$ m^{-3}. The attenuation distance for the microwaves is then of the order of 0·3 mm. A suitable method of producing this plasma may be with an r.f. discharge. The electron temperature would possibly be in the range 5–10 eV. Taking the latter value, the ratio of stress wave velocity to free space wave velocity is $u/c \sim 5\times 10^{-3}$. The electron mean free path for electron–ion collisions,

$$\lambda_i = \left(\frac{4\pi m_e\,\varepsilon_0}{e^2}\right)^2\left(\frac{kT_e}{m_e}\right)^2\frac{1}{4\pi n\ln\Lambda}$$

(where $\Lambda = 12\pi(\varepsilon_0\,kT/e^2)^{3/2}/n^{\frac{1}{2}}$), would be $\sim$30 cm. Since the mean free path for electron–neutral particle collisions is

$$\lambda_n = \left(\frac{kT_e}{m_e}\right)^{\frac{1}{2}} \frac{1}{7\times 10^{-14} n_n},$$

the neutral density should be less than 10^{20} for a similar value for λ_n.

Maximum coupling between microwave horns is limited to about -6 dB. With optimum horn coupling and using a 20 mW input, the detected power would be of the order

$$\bar{S}_2 \sim 20\times 10^{-3}\times\tfrac{1}{4}\times 16\times 25\times 10^{-6} = 2\times 10^{-6}\ \mathrm{W}.$$

The plasma thickness should be sufficient to reduce the microwave power to $\sim 10^{-7}$ W. This requires roughly

$$\exp(-2z_0/d) \sim 5\times 10^{-6} \qquad \text{or} \qquad z_0/d \sim 6,$$

which is equivalent to a plasma thickness of about 2 cm.

A 'typical' detector crystal has a sensitivity of 300 μV mW^{-1} (Heald and Wharton 1965) and figure of merit $\sim$20. The threshold sensitivity with a 100 kHz bandwidth is about -55 dB with respect to 1 mW, i.e. better than 10^{-8} W. Clearly there is in principle no difficulty in detecting 1 μW, which would in fact give an output of about 0·3 μV.

Overall the experiment appears feasible. Possibly the greatest difficulty would be the elimination of stray microwave signals scattered or refracted into the receiving antennas. Provided sufficient plasma of adequate density could be generated, a propagation experiment would obviously be worth attempting.

Stress Wave Properties

Assuming that microwave propagation of the predicted intensity for stress waves was detected in a cutoff plasma, it would be necessary to show that it was due to stress modes rather than stray microwave scattering or other effects. The most obvious property of the stress wave is its slow speed. The wave speed could be measured by modulating the incident microwave radiation in some fashion and then comparing the phase difference of the incident and detected wave forms. A currently available *R* band modulator (Hitachi Electronics Co. Ltd 1973) has a maximum frequency of approximately 200 kHz. For a 10 cm path in a plasma at 10 eV electron temperature, the propagation delay for the stress wave is

$$\tau = 0\cdot 1(7kT_e/5m_e)^{-\frac{1}{2}} \sim 10^{-7}\ \mathrm{s}.$$

This corresponds to a phase shift of 7° at 200 kHz. Such a small phase shift may be difficult to measure accurately, although with larger plasma dimensions, e.g. 30 cm, the phase shift would increase to $\sim$ 20° and would be convincing if measured. Additionally, if the propagation distance was varied, a corresponding variation in phase shift could be shown to correspond to slow wave propagation.

Further effects that could be exploited are the variation in wave speed and attenuation with electron temperature and the general variation in attenuation due to changes in fractional ionization or number density. More subtle would be the introduction of a magnetic field. As with electromagnetic waves in plasmas, stress waves are split into ordinary and extraordinary rays by a magnetic field (Bydder 1967). While such considerations are beyond the scope of the present paper, the Faraday rotation, for example, of stress waves in a magnetized plasma is different from that of microwaves, and could be used to distinguish stress waves. Again, the differences in propagation in inhomogeneous plasmas, which have been considered by Bydder (1967), may be exploitable. Finally, it may also be possible to detect the stress mode inside the plasma, where the energy flux would be greater by a factor $\sim c/u$ than that transmitted through the slab.

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Appendix 1. Wave Transmission through Interface with Multirefringent Medium

In discussing the coupling of microwaves into electromagnetic waves and stress waves at a plasma boundary, it was assumed that the transmission coefficients were given by

$$T_{1,2} = |E_{1,2}/E_0|^2 = |2\kappa_0/(\kappa_0+\kappa_{1,2})|^2,$$

which corresponds to $S_1/S_0, S_2/S_0$ where S_0 and S_1, S_2 are respectively the incident and the two transmitted mode (e.g. stress and electromagnetic) Poynting vectors. As in the body of the paper, the zero subscript indicates the incident (vacuum) wave and the subscripts 1, 2 the two possible modes in the plasma.

To show that this is so, consider a boundary in which there is an incident and a reflected wave on the vacuum side and two transmitted waves on the birefringent (plasma) side. Then the transmitted energy density is given by the total Poynting vector $\boldsymbol{S}$, as

$$\boldsymbol{S} = \frac{\boldsymbol{E}\times\boldsymbol{B}}{\mu_0} = \frac{(\boldsymbol{E}_1+\boldsymbol{E}_2)\times(\boldsymbol{B}_1+\boldsymbol{B}_2)}{\mu_0} = \boldsymbol{S}_1+\boldsymbol{S}_2+\left(\frac{\boldsymbol{E}_2\times\boldsymbol{B}_1+\boldsymbol{E}_1\times\boldsymbol{B}_2}{\mu_0}\right),$$

where $\boldsymbol{S}_1 = (\boldsymbol{E}_1\times\boldsymbol{B}_1)/\mu_0$ and similarly for $\boldsymbol{S}_2$, these being the transmitted energy fluxes in modes 1 and 2 respectively.

The energy flux, as with the field vectors, by convention in this notation, is the real part of the complex term. Hence the energy in the term

$$S_{12} = (E_2 \times B_1 + E_1 \times B_2)/\mu_0$$

is in fact the real part. Since $\kappa \times E = \omega B$,

$$S_{12} = \frac{E_2 \times (\kappa_1 \times E_1) + E_1 \times (\kappa_2 \times E_2)}{\omega\mu_0} = \frac{(\kappa_1 + \kappa_2)E_1 \cdot E_2}{\omega\mu_0},$$

and κ_1 and κ_2 are assumed real, the energy flux has magnitude

$$\varepsilon = \mathrm{Re}(S_{12}) = (\kappa_1 + \kappa_2)\,\mathrm{Re}(E_1 E_2)/\omega\mu_0 .$$

As normal incidence is assumed and the waves are transverse, the vectors have been replaced by their scalar amplitudes. Using additional subscripts r and i to denote the real and complex components of the field vectors, the energy term becomes

$$\varepsilon = \frac{\kappa_1 + \kappa_2}{\omega\mu_0}\,\mathrm{Re}\Big((E_{1\mathrm{r}} + \mathrm{i}E_{1\mathrm{i}})(\cos\phi_1 + \mathrm{i}\sin\phi_1)(E_{2\mathrm{r}} + \mathrm{i}E_{2\mathrm{i}})(\cos\phi_2 + \mathrm{i}\sin\phi_2)\Big),$$

where $\phi_1 = \omega t - \kappa_1 z$ and similarly for ϕ_2. On multiplying out, the real part gives

$$\begin{aligned}\varepsilon = \frac{\kappa_1 + \kappa_2}{\omega\mu_0}\Big(&(E_{1\mathrm{r}}E_{2\mathrm{r}} - E_{1\mathrm{i}}E_{2\mathrm{i}})(\cos\phi_1\cos\phi_2 - \sin\phi_1\sin\phi_2) \\ &- (E_{1\mathrm{r}}E_{2\mathrm{i}} + E_{2\mathrm{r}}E_{1\mathrm{i}})(\sin\phi_1\cos\phi_2 + \sin\phi_2\cos\phi_1)\Big) \\ = \frac{\kappa_1 + \kappa_2}{\omega\mu_0}\Big(&(E_{1\mathrm{r}}E_{2\mathrm{r}} - E_{1\mathrm{i}}E_{2\mathrm{i}})\cos(\phi_1 + \phi_2) - (E_{1\mathrm{r}}E_{2\mathrm{i}} + E_{2\mathrm{r}}E_{1\mathrm{i}})\sin(\phi_1 + \phi_2)\Big).\end{aligned}$$

The time dependence is in the terms $\cos(\phi_1 + \phi_2)$ and $\sin(\phi_1 + \phi_2)$. Averaging these over a complete period gives

$$\mathrm{Av}[\cos(\phi_1 + \phi_2)] = (\omega/2\pi)\int_0^{2\pi/\omega} \cos(2\omega t - \kappa_1 z - \kappa_2 z)\,\mathrm{d}t = 0 .$$

Similarly $\mathrm{Av}[\sin(\phi_1 + \phi_2)] = 0$, whence $\varepsilon = 0$. That is, the total transmitted averaged flux for just two permitted modes is

$$\bar{S} = \bar{S}_1 + \bar{S}_2 .$$

Appendix 2. Energy Propagation in Cutoff Modes

It is well known that electromagnetic energy propagates through thin slabs, e.g. metals, in the cutoff region for which $\omega < \omega_{\mathrm{pe}}$. The propagation can be calculated phenomenologically in terms of the conductivity or the transport coefficients, particularly the collision frequency ν. The details depend on what parameters are appropriate to the approximations.

When the collision frequency is small compared with the wave frequency, the electromagnetic wave dispersion solution to first order in ν/ω is (Bydder 1967)

$$\kappa = \pm\left[\left(\frac{\omega^2 - \omega_{\mathrm{p}}^2}{c^2}\right)^{\frac{1}{2}} - \frac{\mathrm{i}\,\omega_{\mathrm{p}}^2\,\nu}{2\omega c(\omega^2 - \omega_{\mathrm{p}}^2)^{\frac{1}{2}}}\right].$$

In the cutoff region $\omega < \omega_p$, and writing $(\omega_p^2 - \omega^2)/c^2 = \alpha^2$ with α real, the wave solution for this case is

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp[\mathrm{i}(\omega t - \omega_p^2 \, v z/2\omega c^2 \alpha)] \exp(-\alpha z).$$

The propagation is in the positive z direction. Using Maxwell's equations

$$B = (-\mathrm{i}\alpha/\omega + \omega_p^2 \, v/2\omega^2 c^2 \alpha) E,$$

the energy propagation is

$$\bar{S} = (\omega_p^2 \, v/4\omega^2 c^2 \alpha \mu_0) E_0^2 \exp(-2\alpha z).$$

An alternative approach is to assume a relationship of the form

$$\boldsymbol{J} = \sigma \boldsymbol{E}.$$

This is appropriate (σ being taken as real) for low wave frequencies and high collision frequencies. The conductivity σ, for a singly ionized plasma, is given by $\sigma = ne^2/m_e \, v$. Since

$$\nabla \times \boldsymbol{B} = \mu_0 \sigma \boldsymbol{E} + (\mathrm{i}\omega/c^2) \boldsymbol{E},$$

we have

$$\nabla^2 \boldsymbol{E} = (\mathrm{i}\omega\mu_0 \sigma - \omega^2/c^2) \boldsymbol{E}$$

and, provided $\mu_0 \sigma c \gg 1$,

$$\boldsymbol{E} = \boldsymbol{E}_0 \exp[\mathrm{i}(\omega t - \gamma z)] \exp(-\gamma z),$$

with $\gamma^2 = \frac{1}{2}\mu_0 \sigma \omega$. This gives

$$B = (\gamma/\omega)(1 - \mathrm{i}) E,$$

and so the energy flux is

$$\begin{aligned} \bar{S} &= E_0^2 (\gamma/2\mu_0 \omega) \exp(-2\gamma z) \\ &= E_0^2 (\omega\omega_p^2/2vc^2)^{\frac{1}{2}} (\omega\mu_0)^{-1} \exp[-2(\omega\omega_p^2/2vc^2)^{\frac{1}{2}} z]. \end{aligned}$$

The latter result may be compared with more formal calculations (Bydder 1967) which show that for $v \gg \omega$ the wave penetrates in the form

$$E = E_0 \exp[\mathrm{i}(\omega t - \omega z/c)] \exp(-\omega_p^2 \, z/vc),$$

that is,

$$\bar{S} = (E_0^2/2\mu_0 c) \exp(-2\omega_p^2 \, z/vc).$$

The difference between the various cases is due to the approximations used, and of course the results are for different parameter regions. The common conclusion, however, is that, in the presence of collisions, energy penetrates the plasma even during wave cutoff, the propagated energy flux decreasing exponentially with distance.

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