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Ising Model on a Triangular Lattice with Three-spin Interactions. II* Free Energy and Correlation Length

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Abstract

Following the demonstration in Part I that the Ising model with three-spin interactions on a triangular lattice is equivalent to a colouring problem on a hexagonal lattice, and that a generalized Bethe ansatz can be used to obtain equations for the eigenvalues of the transfer matrix of this colouring problem, the resulting equations are solved here to obtain the largest and next-largest eigenvalues in the limit of a large lattice. This gives the free energy and correlation length. The free energy is obtained as a simple algebraic relation and critical exponents $\alpha = \alpha' = \nu = \nu' = 2/3$ are derived. The scaling relation $d\nu = 2-\alpha$ is satisfied.

1. Introduction

In the preceding Part I (Baxter and Wu 1974; present issue pp. 357–67) we considered an Ising model on a two-dimensional triangular lattice of N sites, with solely a three-spin interaction. If $\sigma_i = +1$ or -1 is the spin on the site i then the Hamiltonian is

$$\mathscr{H} = -J\sum \sigma_i \sigma_j \sigma_k, \tag{1}$$

where i, j, k are the vertices of a triangular face of the lattice and the summation is over all such faces. Our aim is to calculate the partition function

$$Z = \sum \exp(-\mathscr{H}/kT), \tag{2}$$

the summation here being over all possible spin configurations of the lattice.

We saw in Part I that the above problem is equivalent to a colouring problem on a related hexagonal lattice, and that it can be solved by a generalized Bethe ansatz for the eigenvectors of the transfer matrix. Thus we found that for N large

$$Z^{1/N} = (2\sinh 4K)^{1/3}\,\Lambda_0^{2/3M}, \tag{3}$$

where

$$K = |J|/kT \tag{4}$$

and Λ_0 is the maximum eigenvalue of the transfer matrix for a hexagonal lattice of M sites per row.

In this paper we calculate Λ_0 and the next-largest eigenvalue Λ_1 for M large. This gives us the free energy f and the correlation length ξ. Our principal results are that $Z^{1/N}$ is an algebraic function of e^K, given by equations (6), (56) and (61)

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below, and that the critical exponents α and ν satisfy

$$\alpha = \alpha' = \nu = \nu' = 2/3, \tag{5}$$

This agrees with the scaling prediction $d\nu = 2-\alpha$ (Fisher 1967, equation 9.3.15). The results for f, α and α' have been reported earlier (Baxter and Wu 1973). The derivation of these and of ξ, ν and ν' are given here.

2. Equations for Eigenvalues

Putting

$$t = \sinh 2K, \tag{6}$$

the results of Part I can be summarized as follows. Consider n wave numbers $k_1, ..., k_n$ and define E_j, $B_{j,l}$ and λ_j in terms of them by

$$\cosh 2E_j = \cos 2k_j + t + t^{-1}, \tag{7}$$

$$B_{j,l} = -\cosh(E_j + \mathrm{i}k_l)/\cosh(E_l + \mathrm{i}k_j), \tag{8}$$

$$\lambda_j = \exp(E_j - \mathrm{i}k_j), \tag{9}$$

for j and $l = 1, ..., n$. Let $k_1, ..., k_n$ be distinct (modulo π) and given by the n equations

$$\exp(\mathrm{i}Mk_j) = -\prod_{l=1}^{n} B_{j,l}, \qquad j = 1, ..., n. \tag{10}$$

Then the eigenvalue Λ of the transfer matrix of the colouring problem is given by

$$\Lambda = \lambda_1 \dots \lambda_n. \tag{11}$$

There will be many solutions of equation (10), corresponding to the various eigenvalues. In addition, the integer n can take any even value from 0 to $2M$. Out of all these possibilities we must choose the numerically largest eigenvalue Λ_0 to substitute into equation (3). Fortunately we are interested only in the limit $M \to \infty$, when we expect $\Lambda_0^{1/M}$ to tend to a limit. It should be noted that the above equations are invariant with respect to inversion of t. The critical temperature T_c occurs when $t = 1$.

3. Low Temperature Limit

Largest Eigenvalue

We first consider the low temperature limit, when K and t are large and positive. (From the duality relation $t \to t^{-1}$, this is also equivalent to the high temperature limit.) This enables us to explicitly locate the largest eigenvalue Λ_0.

We look for a solution of equation (10) such that $k_1, ..., k_n$ are real. It was remarked in Part I that such a solution is expected to exist, since the B_{jl} are then unimodular. From equation (7) we can choose $E_1, ..., E_n$ to be real and positive, and for t large

$$\exp(2E_j) \simeq 2t. \tag{12}$$

Thus to first order $E_1, ..., E_n$ are the same. Substituting into equations (8) and (10) and retaining only dominant terms, we obtain

$$\exp(\mathrm{i}Mk_j) = (-1)^{n-1}\exp\left(\mathrm{i}\sum_{l=1}^{n}(k_l-k_j)\right). \tag{13}$$

Let

$$z_j = \exp(2\mathrm{i}k_j). \tag{14}$$

Equation (13) can then be written

$$z_j^{\frac{1}{2}(M+n)}+(-1)^n(z_1 \dots z_n)^{\frac{1}{2}} = 0. \tag{15}$$

This is a polynomial equation of degree $\frac{1}{2}(M+n)$ for each z. There must be at least n distinct roots $z_1, ..., z_n$, since if any two z_j's coincide it follows from equations (30) and (48) of Part I that $f(m, x) \equiv 0$ and our eigenvalue equations become meaningless. Thus we must have

$$\tfrac{1}{2}(M+n) \geqslant n, \qquad \text{that is,} \qquad n \leqslant M. \tag{16}$$

Provided this is so, there will be solutions of equation (15). From equations (9), (11) and (12) it follows that

$$|\Lambda| = (2t)^{n/2}. \tag{17}$$

Thus to maximize Λ we must choose n as large as possible, consistent with the condition (16), i.e.

$$n = M. \tag{18}$$

Substituting equation (17) into (3), we obtain in this limit that

$$N^{-1}\ln Z = 2K, \tag{19}$$

which is the correct low temperature value. This suggests that we have indeed found the maximum eigenvalue Λ_0. Inspection of the transfer matrix as defined in Part I shows that in this limit there should be two equally large maximum eigenvalues, one corresponding to eigenvectors which are symmetric with respect to incrementing all colours m by 4, and another which is antisymmetric. Since we have assumed symmetry, we only find the former, but for finite t and M we expect this to be the larger, so all is well.

When $n = M$, $z_1, ..., z_n$ are the complete set of roots of the polynomial equation (15) for z_j, so we must have

$$z_1 \dots z_n = (z_1 \dots z_n)^{\frac{1}{2}} = 1. \tag{20}$$

Hence from equation (14) we can choose $k_1, ..., k_n$ to be

$$k_j = \pi(2j-n-1)/2n, \qquad j = 1, ..., n. \tag{21}$$

Thus $k_1, ..., k_n$ are distributed throughout the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$ and tend to a continuous distribution as $n \to \infty$.

In subsequent sections we shall use corresponding properties of $k_1, ..., k_n$ to obtain Λ_0 as an analytic function of t in the domain $1 \leqslant t < \infty$ (for n large). We assume that Λ_0 remains the maximum eigenvalue throughout this domain, i.e. that no other eigenvalue crosses it.

Next-largest Eigenvalue

To obtain the correlation length ξ and the critical exponents ν and ν' we need the next-largest eigenvalue Λ_1 in the $n = M$ subspace (other than the antisymmetric eigenvalue referred to above, which is asymptotically degenerate with Λ_0 for M large). Some inspection suggests that this corresponds to $k_1, ..., k_n$ again being real but one of the E_j's, say E_1, being negative (the rest positive). In the low temperature limit, the equations (10) become

$$z_j^{n-1}+(-1)^n(z_1 \dots z_n)^{\frac{1}{2}} = 0\,, \qquad j = 2, ..., n\,, \tag{22a}$$

$$z_1^{n-1}+(-1)^n(z_1 \dots z_n)^{-\frac{1}{2}} = 0\,. \tag{22b}$$

Since $z_2, ..., z_n$ are the $n-1$ distinct roots of the equation (22a) for z_j, it follows that

$$z_2 \dots z_n = (z_1 \dots z_n)^{\frac{1}{2}} = z_1\,, \tag{23}$$

and so from equation (22b)

$$z_1^n = (-1)^{n-1}\,. \tag{24}$$

Thus in this limit there are n possible values for k_1 which are real and lying in the interval $(-\frac{1}{2}\pi, \frac{1}{2}\pi)$; $k_2, ..., k_n$ are distributed uniformly throughout the interval and

$$|\Lambda_1/\Lambda_0| = (2t)^{-1}\,. \tag{25}$$

4. Change of Variables

We now return to considering finite values of t. To the author's knowledge, in every previous application of the Bethe ansatz (Yang and Yang 1966; Lieb 1967*a*, 1967*b*, 1967*c*; Baxter 1969, 1970, 1971, 1972) it has been found possible to transform from the wave numbers $k_1, ..., k_n$ to some new variables $u_1, ..., u_n$ so as to make B_{jl} a function only of $u_j - u_l$ (and parameters such as t). The present model is no exception. Suppose there exists a function $u(k)$ such that

$$B_{1,2} = F\{u(k_1)-u(k_2)\}\,. \tag{26}$$

Differentiating with respect to k_1 and k_2 and taking ratios, we find

$$-\frac{\partial B_{1,2}/\partial k_1}{\partial B_{1,2}/\partial k_2} = \frac{u'(k_1)}{u'(k_2)}\,. \tag{27}$$

Thus for the required transformation to exist it is a necessary condition that the left-hand side of equation (27) be a function of k_1 divided by the same function of k_2. This condition is also sufficient, since it ensures that the Jacobian of $B_{1,2}$ and $u(k_1)-u(k_2)$ vanishes.

Differentiating equation (8), using (7), we find after some manipulation that the left-hand side of equation (27) is

$$(\sinh 2E_2)/(\sinh 2E_1)\,. \tag{28}$$

Remembering that E_j is a function $E(k_j)$ of k_j, we see that the condition (27) is indeed satisfied and that

$$u'(k) \propto \operatorname{cosech} 2E(k)\,. \tag{29}$$

Introduction of Elliptic Functions

We can integrate equation (29), using equation (7), and obtain k and E as functions $k(u)$ and $E(u)$ of the new variable u. We find that we are forced to introduce elliptic functions. Since the direct derivation is not particularly illuminating, the results are only quoted here and are justified by using certain identities between elliptic functions, as given in the Appendix. We use the notation of Gradshteyn and Ryzhik (1965; hereinafter designated GR), except that we denote the modulus of the elliptic functions by m and the conjugate modulus by $m' = (1-m^2)^{\frac{1}{2}}$. As usual, we write the complete elliptic integrals of the first kind of these moduli as K and K' respectively.

Define m by

$$m = \min(t^2, t^{-2}) \tag{30}$$

(so $0 < m \leqslant 1$) and set

$$\eta = \tfrac{1}{4}K'. \tag{31}$$

Then we find that $k(u)$ and $E(u)$ are given by

$$\exp\{2\mathrm{i}k(u)\} = \mathrm{sn}(\mathrm{i}\eta - u)/\mathrm{sn}(\mathrm{i}\eta + u), \tag{32}$$

$$\exp\{-2E(u)\} = m\,\mathrm{sn}(u-\mathrm{i}\eta)\,\mathrm{sn}(u+\mathrm{i}\eta). \tag{33}$$

The relation (7) is now satisfied identically for all complex numbers u, as we can verify by substituting equations (31) and (32) into (7) and using (30) and the identity (A3) of the Appendix.

To summarize: we are transforming from $k_1, ..., k_n$ to new variables $u_1, ..., u_n$ such that

$$k_j = k(u_j), \qquad E_j = E(u_j), \qquad j = 1, ..., n, \tag{34}$$

where the functions $k(u)$ and $E(u)$ are defined by equations (30)–(33). If u is real then $k(u)$ and $E(u)$ are real and $E(u)$ is positive. We shall find that $u_1, ..., u_n$ are real for the maximum eigenvalue Λ_0.

Substituting equations (32) and (33) into (8), we obtain (using $\mathrm{sn}(-u) = -\mathrm{sn}\,u$)

$$B_{j,l} = -\frac{\mathrm{sn}(u_l-\mathrm{i}\eta) - m\,\mathrm{sn}(u_j-\mathrm{i}\eta)\,\mathrm{sn}(u_j+\mathrm{i}\eta)\,\mathrm{sn}(u_l+\mathrm{i}\eta)}{\mathrm{sn}(u_j-\mathrm{i}\eta) - m\,\mathrm{sn}(u_l-\mathrm{i}\eta)\,\mathrm{sn}(u_l+\mathrm{i}\eta)\,\mathrm{sn}(u_j+\mathrm{i}\eta)}. \tag{35}$$

Multiplying the numerator and denominator by $m\,\mathrm{sn}(u_j+3\mathrm{i}\eta)$, and using equations (31) and (A1b), this becomes

$$B_{j,l} = -\frac{m\{\mathrm{sn}(u_j+3\mathrm{i}\eta)\,\mathrm{sn}(u_l-\mathrm{i}\eta) - \mathrm{sn}(u_j+\mathrm{i}\eta)\,\mathrm{sn}(u_j+\mathrm{i}\eta)\}}{1 - m^2\,\mathrm{sn}(u_j+3\mathrm{i}\eta)\,\mathrm{sn}(u_l-\mathrm{i}\eta)\,\mathrm{sn}(u_j+\mathrm{i}\eta)\,\mathrm{sn}(u_l+\mathrm{i}\eta)}. \tag{36}$$

From equation (A4) it follows that

$$B_{j,l} = -m\,\mathrm{sn}(2\mathrm{i}\eta)\,\mathrm{sn}(u_l-u_j-2\mathrm{i}\eta) = -\mathrm{i}m^{\frac{1}{2}}\,\mathrm{sn}(u_l-u_j-2\mathrm{i}\eta), \tag{37}$$

using equation (A2). Thus $B_{j,l}$ is a function only of $u_j - u_l$ (and t), as was required.

Following Yang and Yang (1966), we define one further function $\Theta(u)$ by equation (37) and

$$B_{j,l} = -\exp\{-\mathrm{i}\,\Theta(u_j-u_l)\}. \tag{38}$$

This $\Theta(u)$ is *not* the elliptic theta function.

Fourier Expansions

Using equation (A5b), we obtain the Fourier expansions of $k(u)$, $E(u)$ and $\Theta(u)$, which are convergent in a strip containing the real axis:

$$k(u) = \frac{\pi u}{2K} + \sum_{r=1}^{\infty} \frac{q^{r/4}(1+q^{r/2})}{r(1+q^r)} \sin\left(\frac{\pi r u}{K}\right), \tag{39}$$

$$E(u) = \frac{\pi K'}{8K} + \sum_{r=1}^{\infty} \frac{(q^{r/4}-q^{3r/4})}{r(1+q^r)} \cos\left(\frac{\pi r u}{K}\right), \tag{40}$$

$$\Theta(u) = \frac{\pi u}{2K} + 2\sum_{r=1}^{\infty} \frac{q^{r/2}}{r(1+q^r)} \sin\left(\frac{\pi r u}{K}\right), \tag{41}$$

where

$$q = \exp(-\pi K'/K). \tag{42}$$

Note that $\Theta(u)$ is a real function and is odd. This verifies our observation that $B_{j,l}$ is unimodular when k_j and k_l are real.

5. Integral Equation for Distribution of u_j's

We can now employ standard methods (Yang and Yang 1966) to solve the equations (10) for Λ_0 when n and M are large. First we substitute equation (38) into (10) and take logarithms, choosing the branches so as to obtain the equations

$$k(u_j) + M^{-1} \sum_{l=1}^{n} \Theta(u_j - u_l) = \pi(2j-n-1)/M, \qquad j = 1, \dots, n. \tag{43}$$

Suppose that for every u_l there is a $-u_l$. Then as u_j increases from $-K$ to K (a period of the elliptic functions), the left-hand side of equation (43) increases from $-\frac{1}{2}\pi(1+n/M)$ to $\frac{1}{2}\pi(1+n/M)$. Correspondingly, as j increases from 1 to n the right-hand side increases from $-\pi(n-1)/M$ to $\pi(n-1)/M$. Thus there will be real solutions to equation (43) if $(n-1)/M < \frac{1}{2}(1+n/M)$, that is, if $n \leqslant M$ (n must be even). Further, to every u_j there will be a $-u_j$, as assumed.

We restrict attention to the case $n = M$, which we know contains the maximum eigenvalue in the low and high temperature limits. Then from the above reasoning we expect $u_1, \dots, u_n$ to be distributed throughout the interval $(-K, K)$ and to tend to a continuous distribution as $n \to \infty$. Let $M\rho(u)\,\mathrm{d}u$ be the number of u_j's between u and $u+\mathrm{d}u$. Then we require that the total number of u_j's be

$$M \int_{-K}^{K} \rho(u)\,\mathrm{d}u = M. \tag{44}$$

The equation (43) becomes, for $-K < u < K$,

$$k(u) + \int_{-K}^{K} \Theta(u-v)\rho(v)\,\mathrm{d}v = 2\pi \int_{0}^{u} \rho(v)\,\mathrm{d}v. \tag{45}$$

Differentiating with respect to u, the resulting linear integral equation for $\rho(u)$ can be solved by Fourier transforms. Using equations (39) and (41), we find

$$\rho(u) = \tfrac{1}{2}K^{-1} + \tfrac{1}{2}K^{-1} \sum_{r=1}^{\infty} \frac{q^{r/4}(1+q^{r/2})}{(1-q^{r/2}+q^r)} \cos\left(\frac{\pi r u}{K}\right). \tag{46}$$

This does indeed satisfy equation (44).

6. Maximum Eigenvalue and Free Energy

Noting that for every k_j there is a $-k_j$, we see from equations (9) and (11) that

$$\begin{aligned} I = M^{-1}\ln \Lambda_0 &= M^{-1}\sum_{j=1}^{n} E_j = \int_{-K}^{K} E(u)\,\rho(u)\,du \\ &= \frac{\pi K'}{8K} + \tfrac{1}{2}\sum_{r=1}^{\infty} \frac{q^{\frac{1}{2}r}(1-q^r)}{r(1+q^r)(1-q^{\frac{1}{2}r}+q^r)}, \end{aligned} \tag{47}$$

using equations (40) and (46). At low temperatures t is large while m and q are small and $q \approx m^2/16$, so equation (47) becomes

$$M^{-1}\ln \Lambda_0 \approx -\tfrac{1}{8}\ln q \approx -\tfrac{1}{4}\ln \tfrac{1}{4}m \approx \tfrac{1}{2}\ln 2t. \tag{48}$$

This agrees with equation (17), verifying that we have found the maximum eigenvalue at low temperatures, and by duality at high temperatures. As t decreases from ∞ to 1, m and q increase from 0 to 1, so the expression (47) is analytic throughout this interval (and through $0 < t < 1$). We conjecture that the result is the maximum eigenvalue throughout these ranges, i.e. no other eigenvalue crosses it.

The problem is now solved, since for a given t we can calculate m, q, Λ_0 and Z from equations (30), (42), (47) and (3) respectively. However, it is a remarkable fact that we can eliminate the elliptic parameters m and q and express equation (47) directly in terms of t by an algebraic equation, as is now shown.

Set

$$p = q^{\frac{1}{2}} \tag{49}$$

and multiply the numerator and denominator of the summand in equation (47) by $1+p^r$. The summand can then be re-arranged to give

$$I = \frac{\pi K'}{8K} + \tfrac{1}{2}\sum_{r=1}^{\infty} \frac{1}{r}\left(\frac{p^r}{1+p^r} + \frac{2p^{2r}}{1+p^{2r}} - \frac{3p^{3r}}{1+p^{3r}}\right). \tag{50}$$

Writing this summation out and collecting terms proportional to $p^r/(1+p^r)$ for $r = 1, 2, 3, \ldots$, we find

$$I = \frac{\pi K'}{8K} + \sum_{r=1}^{\infty} \frac{p^r}{r(1+p^r)}\left(\cos(\pi r) - 3\cos(\tfrac{2}{3}\pi r)\right). \tag{51}$$

Comparing this with the identity (A5a), we see that we can write I in terms of sn functions of modulus m_1 such that if K_1 and K_1' are the corresponding elliptic integrals then

$$\exp(-\pi K_1'/K_1) = p, \qquad \text{that is,} \qquad K_1'/K_1 = K'/2K. \tag{52}$$

From equations 8.126.1, 3 of GR it follows that, using equation (30),

$$m_1 = 2m^{\frac{1}{2}}/(1+m) = 2/(t+t^{-1}). \tag{53}$$

Evaluating the sums in equation (51), using the identity (A5a) and the result $\mathrm{sn}(K_1, m_1) = 1$, we obtain

$$I = \tfrac{1}{4}\ln(27\,m_1\,y^3/4), \tag{54}$$

where

$$y = \{m_1^{\frac{1}{2}}\,\mathrm{sn}(\tfrac{2}{3}K_1, m_1)\}^{-2}. \tag{55}$$

With m replaced by m_1 in the identity (A6) and using equation (53), we see that y is the root of the equation

$$(y-1)^3(1+3y)/y^3 = 2(1-t)^4/(t+t^3), \tag{56}$$

lying in the interval $1 \leqslant y < \infty$. There is one and only one such root.

Noting from equations (3) and (47) that

$$Z^{1/N} = (2\sinh 4K)^{1/3}\exp\tfrac{2}{3}I, \tag{57}$$

it follows from equations (53), (54) and (6) that in the limit of N large

$$Z^{1/N} = (6ty)^{\frac{1}{2}}. \tag{58}$$

This is the result reported earlier by Baxter and Wu (1973). From equation (56) we can verify that y is an analytic nonzero function of t, and hence of temperature T, except at $t = 1$ where

$$y = 1+4^{-1/3}|\,t-1\,|^{4/3} + \text{smaller terms}. \tag{59}$$

Since t is an analytic monotonic decreasing function of T for all real T, it follows that a phase transition occurs at (and only at) the temperature T_c given by $t = 1$, that is,

$$\sinh(2J/kT_c) = 1 \qquad \text{or} \qquad kT_c/J = 2/\ln(\sqrt{2}+1) = 2{\cdot}269185\ldots. \tag{60}$$

The free energy per site of the three-spin Ising model is, using equation (58),

$$f = -N^{-1}kT\ln Z = -\tfrac{1}{2}kT\ln(6ty). \tag{61}$$

Near T_c we see from equation (59) that this has a dominant singularity proportional to $|\,T-T_c\,|^{4/3}$. Differentiating twice to obtain the specific heat, it follows that the critical exponents α and α' are

$$\alpha = \alpha' = 2/3, \tag{62}$$

which is an unusually large value.

We note that the present problem is similar to the three colourings of the square lattice (Baxter 1970), in that with both problems one is led to introduce elliptic functions to solve the Bethe ansatz equations, but these functions can finally be eliminated to express $Z^{1/N}$ as a simple algebraic function of the Boltzmann weights.

7. Next-largest Eigenvalue: Exponents ν and ν′

We now evaluate the next-largest eigenvalue Λ_1 in the $n = M$ subspace of the transfer matrix. Guided by the results of Section 3, we assume that $u_2, ..., u_n$ are real but $u_1 = \mathrm{i}K' + v$ with v real. This ensures that $k_1, ..., k_n$ and $E_1, ..., E_n$ are real, $E_2, ..., E_n$ are positive and E_1 is negative. The equations (34) and (38) remain true and can be used as they stand for $j, l = 2, ..., n$. From equations (32), (33), (37) and (A1b) we see also that

$$k_1 = \pi - k(v), \qquad E_1 = -E(v), \tag{63a}$$

$$B_{1,l} = \exp\{\mathrm{i}\Theta(v-u_l)\}, \qquad B_{j,1} = \exp\{\mathrm{i}\Theta(u_j-v)\}, \tag{63b}$$

for $j, l = 2, ..., n$. (The inclusion of π in k_1 ensures consistency with equation (8).) Substituting into equation (10), taking logarithms and choosing appropriate branches, we obtain

$$-Mk(v) = 2\pi s + \sum_{l=2}^{n} \Theta(v-u_l), \tag{64}$$

$$Mk(u_j) = \pi(2j-n-2) + \Theta(u_j - v) - \sum_{l=2}^{n} \Theta(u_j - u_l), \tag{65}$$

where s is an arbitrary integer and $j = 2, ..., n$ in equation (65). There will be n distinct solutions of equations (64) and (65), corresponding to different values of s, all with v and $u_1, ..., u_n$ real. The values of v are distributed throughout the interval $(-k, k)$. It follows that in the limit of n, M large we can regard v as an arbitrary real parameter and s/M as defined by equation (64). Thus we need only consider equation (65).

Let $M\rho(u)\,\mathrm{d}u$ be the number of u_j's (for $j = 2, ..., n$) between u and $u+\mathrm{d}u$. Proceeding as in Section 5, taking $n = M$, we obtain

$$k'(u) = 2\pi\rho(u) + n^{-1}\,\Theta'(u-v) - \int_{-K}^{K} \Theta'(u-w)\,\rho(w)\,\mathrm{d}w. \tag{66}$$

(At first sight the replacement of sums by integrals would seem suspect here, since we wish to retain terms of relative order n^{-1}. However, because $u_2, ..., u_n$ tend to a continuous distribution over a period $2K$ of the elliptic functions, the error involved in replacing sums by integrals is of order $\exp(-\mu n)$, where μ is some positive number. Thus these errors are still relatively negligible and all is well. The author has obtained the results of this section independently by a 'perturbation expansion' method (similar to that used by Baxter 1971, 1972) that makes these points clearer. However, it involves developing more formalism than seems worth while for this section.)

Solving equation (66) by Fourier analysis, using (3) and (41), we obtain

$$\rho(u) = \rho_0(u) - n^{-1}\,\rho_1(u-v), \tag{67}$$

where $\rho_0(u)$ is the maximum eigenvalue distribution given by equation (46) and

$$\rho_1(u) = \tfrac{1}{2}K^{-1} + K^{-1} \sum_{r=1}^{\infty} \frac{q^{\frac{1}{2}r}}{(1-q^{\frac{1}{2}r}+q^r)} \cos\left(\frac{\pi r u}{K}\right). \tag{68}$$

From this we can readily deduce that

$$M \int_{-K}^{K} \rho(u)\, \mathrm{d}u = n-1, \tag{69}$$

which is indeed the total number of u_j's for $j = 2, \ldots, n$.

From equations (9), (11) and (63a) we see that

$$\ln \Lambda_1 = -E(v) - \mathrm{i}\pi + \mathrm{i}k(v) + \sum_{j=2}^{n} (E_j - \mathrm{i}k_j). \tag{70}$$

Adding equation (64) to the sum of the equations (65) for $j = 2, \ldots, n$, using $\Theta(-u) = -\Theta(u)$, we obtain

$$M\{-k(v) + k_2 + \ldots + k_n\} = 2\pi s. \tag{71}$$

Thus equation (70) can be written

$$\ln \Lambda_1 = -\mathrm{i}\pi(1+2s/M) - E(v) + n \int_{-K}^{K} E(u)\, \rho(u)\, \mathrm{d}u. \tag{72}$$

Using equations (40), (47), (67) and (68), it follows that

$$\ln |\Lambda_1/\Lambda_0| = -\frac{\pi K'}{4K} - \sum_{r=1}^{\infty} \frac{q^{r/4} - q^{3r/4}}{r(1-q^{r/2}+q^r)} \cos\left(\frac{\pi r v}{K}\right). \tag{73}$$

Since equation (73) represents a band of n eigenvalues Λ_1, and remembering that, for n sufficiently large, v can approach as close as desired to any real value, the largest eigenvalue is obtained by choosing $v = K$, to give

$$\ln |\Lambda_1/\Lambda_0| = -\frac{\pi K'}{4K} - \sum_{r=1}^{\infty} (-1)^r \frac{q^{r/4} - q^{5r/4}}{r(1+q^{3r/2})} \tag{74}$$

(multiplying the numerator and denominator of the summand by $1+q^{r/2}$). Comparing the result (74) with equation (A5b), we are led to define a third elliptic modulus m_2, with associated elliptic integrals K_2 and K_2' such that

$$K_2'/K_2 = 3K'/2K, \tag{75}$$

and consequently find that

$$\ln |\Lambda_1/\Lambda_0| = \ln\{m_2^{\frac{1}{2}} \operatorname{sn}(K_2 + \tfrac{1}{6}\mathrm{i}K_2', m_2)\}. \tag{76}$$

The author has used methods similar to those employed above to derive the identity (A6) in order to eliminate the elliptic functions and write $|\Lambda_1/\Lambda_0|$ as an algebraic function of t, involving the roots of equation (56). However, the result seems rather complicated and not particularly illuminating. We content ourselves here with obtaining the behaviour of Λ_1/Λ_0 near T_c, that is, when t, m and m_2 are close to unity.

Successively applying the formulae 8.151.2, 8.153.2,3 and 8.146.25 of GR to equation (76), we have

$$\begin{aligned}\ln|\Lambda_1/\Lambda_0| &= \ln\{m_2^{\frac{1}{2}}\operatorname{cn}(\tfrac{1}{6}\mathrm{i}K_2', m_2)/\operatorname{dn}(\tfrac{1}{6}\mathrm{i}K_2', m_2)\} \\ &= \ln\{m_2^{\frac{1}{2}}/\operatorname{dn}(\tfrac{1}{6}K_2', m_2')\} \\ &= -4\sum_{r=1}^{\infty}\frac{1}{2r-1}\,\frac{(q_2')^{2r-1}\cos\{\frac{1}{6}\pi(2r-1)\}}{1-(q_2')^{2r-1}}, \end{aligned} \tag{77}$$

where

$$q_2' = \exp(-\pi K_2/K_2') = \exp(-2\pi K/3K') = (q')^{2/3}. \tag{78}$$

Near T_c the quantities q' and q_2' are small, so from equation (77)

$$\ln|\Lambda_1/\Lambda_0| \approx -2\sqrt{3}\,q_2' = -2\sqrt{3}\,(q')^{2/3}. \tag{79}$$

Replacing k by m' in equation 8.197.3 of GR we have that near T_c

$$4(q')^{\frac{1}{2}} \approx m',$$

that is, using equation (30),

$$q' \approx \tfrac{1}{16}(1-m^2) \approx \tfrac{1}{4}|\ln t|. \tag{80}$$

As $\ln t$ has a simple zero at $T = T_c$, it follows that

$$\ln|\Lambda_1/\Lambda_0| \propto |T-T_c|^{2/3}. \tag{81}$$

Since correlations decay to their asymptotic values at large distances r as $(\Lambda_1/\Lambda_0)^r$, the correlation length ξ is given by

$$\xi^{-1} = \ln|\Lambda_1/\Lambda_0|, \tag{82}$$

(Fisher and Burford 1967, equation 5.21). Hence near T_c

$$\xi \propto |T-T_c|^{-2/3}, \tag{83}$$

so that the critical exponents ν and ν' describing the singularity in ξ are both equal to 2/3.

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Appendix. Elliptic Function Densities

We use the notation of Gradshteyn and Ryzhik (1965; designated GR), except that m and m' denote the elliptic moduli, instead of k and k'.

Identity 1

$$\mathrm{sn}(u+2K) = -\mathrm{sn}(u) = \mathrm{sn}(-u), \qquad \mathrm{sn}(u+\mathrm{i}K') = (m\,\mathrm{sn}\,u)^{-1}. \tag{A1a, b}$$

These relations are given in Section 8.151 of GR.

Identity 2

$$\mathrm{sn}(\tfrac{1}{2}\mathrm{i}K') = \mathrm{i}m^{-\frac{1}{2}}. \tag{A2}$$

This is given in equation 8.151.3 of GR.

Identity 3

$$(m^{\frac{1}{2}}\,\mathrm{sn}\,u + m^{-\frac{1}{2}}/\mathrm{sn}\,u)\{m^{\frac{1}{2}}\,\mathrm{sn}(u+\tfrac{1}{2}\mathrm{i}K') + m^{-\frac{1}{2}}/\mathrm{sn}(u+\tfrac{1}{2}\mathrm{i}K')\} = 2(m^{\frac{1}{2}}+m^{-\frac{1}{2}}), \tag{A3}$$

for all complex numbers u.

Proof. From equations (A1) we can verify that the left-hand side of (A3) is a doubly periodic function of u, with periods $2K, \frac{1}{2}\mathrm{i}K'$. Also, $\mathrm{sn}\,u$ is a meromorphic function with only simple zeros and poles, at $u = 2rK+2sK'\mathrm{i}$ and $2rK+(2s+1)K'\mathrm{i}$ respectively, for all integers r and s (equation 8.151.1 of GR). Thus the left-hand side of (A3) is a meromorphic function of u. The residue of the pole at $u = 0$ is

$$\mathrm{sn}(\tfrac{1}{2}\mathrm{i}K') + m^{-1}/\mathrm{sn}(\tfrac{1}{2}\mathrm{i}K').$$

From the identity (A2) we see that this residue is zero, so the pole is spurious. From periodicity, so are all other possible poles. Thus the left-hand side of (A3) is an entire doubly-periodic function, and hence is bounded. From the Cauchy–Liouville theorem it must be a constant. Setting $u = K$ and using

$$\mathrm{sn}\,K = 1, \qquad \mathrm{sn}(K+\tfrac{1}{2}\mathrm{i}K') = m^{-\frac{1}{2}}$$

(equations 8.151.2, 3 of GR), we find that the constant is $2(m^{\frac{1}{2}}+m^{-\frac{1}{2}})$. This proves the identity.

Identity 4

$$\frac{\mathrm{sn}(u)\,\mathrm{sn}(v+w) - \mathrm{sn}(v)\,\mathrm{sn}(u+w)}{1 - m^2\,\mathrm{sn}(u)\,\mathrm{sn}(v)\,\mathrm{sn}(u+w)\,\mathrm{sn}(v+w)} = \mathrm{sn}(u-v)\,\mathrm{sn}(w), \tag{A4}$$

for all complex numbers u, v and w.

Proof. Regard the ratio of the left-hand side to the right-hand side of (A4) as a function of v. From equations (A1) it has periods $2K$ and $2\mathrm{i}K'$. For these moduli its only possible poles are at $v = u$, $u+\mathrm{i}K'$ and $\mathrm{i}K'-u-w$, but all these have zero residue. Thus the ratio is an entire function of v, and by the Cauchy–Liouville theorem is a constant. Setting $v = 0$ we find that the ratio is unity. This proves the identity.

Identity 5

$$\ln(m^{\frac{1}{4}}\operatorname{sn} u) = \ln\left\{2q^{\frac{1}{4}}\sin\left(\frac{\pi u}{2K}\right)\right\} + 2\sum_{r=1}^{\infty}\frac{q^r}{r(1+q^r)}\cos\left(\frac{\pi r u}{K}\right), \tag{A5a}$$

provided $|\operatorname{Im}(u)| < K'$. This follows from either equation 8.146.20 or 8.146.23 of GR. (The latter equation contains an error: the square root of q should be $q^{\frac{1}{4}}$.)

A simple corollary, obtained by writing the sine function in terms of imaginary exponentials and using the Taylor expansion of $\ln(1+x)$, is

$$\begin{aligned}\ln\{m^{\frac{1}{2}}\operatorname{sn}(u)\} &= \ln\{-m^{\frac{1}{2}}\operatorname{sn}(-u)\}\\ &= -\frac{\mathrm{i}\pi}{2K}\left(u-K-\tfrac{1}{2}\mathrm{i}K'\right) - 2\mathrm{i}\sum_{r=1}^{\infty}\frac{q^{\frac{1}{2}r}}{r(1+q^r)}\sin\left(\frac{\pi r(u-\frac{1}{2}\mathrm{i}K')}{K}\right),\end{aligned} \tag{A5b}$$

provided $0 < \operatorname{Im}(u) < K'$.

Identity 6

If $y = \{m^{\frac{1}{2}}\operatorname{sn}(\frac{2}{3}K)\}^{-2}$ then y is the real root of the equation

$$(y-1)^3(1+3y)/4y^3 = m+m^{-1}-2, \tag{A6}$$

lying in the interval $1 \leqslant y < \infty$.

Proof. Set $u = v = \frac{2}{3}K$ in equation 8.156.1 of GR to obtain

$$\operatorname{sn}(\tfrac{4}{3}K) = 2\operatorname{sn}(\tfrac{2}{3}K)\operatorname{cn}(\tfrac{2}{3}K)\operatorname{dn}(\tfrac{2}{3}K)/\{1-m^2\operatorname{sn}^4(\tfrac{2}{3}K)\}.$$

Also, using $\operatorname{sn} u = \operatorname{sn}(2K-u)$, we have

$$\operatorname{sn}(\tfrac{4}{3}K) = \operatorname{sn}(\tfrac{2}{3}K).$$

Eliminating $\operatorname{sn}(\frac{4}{3}K)$ between these equations, cancelling $\operatorname{sn}(\frac{2}{3}K)$, squaring and using equations 8.154.4, 5 of GR and the above definition of y, we find

$$1 = 4\{1-(my)^{-1}\}(1-my^{-1})/(1-y^{-2})^2.$$

Re-arranging, we obtain the equation (A6) for y. The left-hand side of (A6) increases monotonically from 0 to ∞ as y increases from 1 to ∞, so there is one and only one solution of (A6) in the interval $1 \leqslant y < \infty$. Since we have $0 < m \leqslant 1$ and $0 < \operatorname{sn}(\frac{2}{3}K) \leqslant 1$, y must lie in this interval. Thus this is the desired root of equation (A6).

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