Aust. J. Phys., 1974, 27, 481–93

Nonlinear Thermal Convection in a Layer with Imposed Energy Flux

R. Van der Borght

Department of Mathematics, Monash University, Clayton, Vic. 3168.

Abstract

Results are reported of an investigation into the effect of the chosen boundary conditions on the steady finite-amplitude convective motions in a layer in which the average energy flux is imposed. The boundary conditions are chosen with a view to the application of the results to solar granulation and supergranulation. It is shown that, at high Rayleigh numbers, solutions do in fact exist for which there is no modulation in the energy flux and little fluctuation in the temperature across the boundaries.

Introduction

In studies of finite-amplitude convection, one usually assumes that the average temperature at the convective layer boundaries is given and that the fluctuation of the temperature about this average value vanishes at these boundaries. These assumptions correspond to the usual laboratory situation but are unlikely to be appropriate if one wants to extend the results of such calculations to astrophysical problems. In a star the average energy flux across the convective layer is given and has to be conserved. This requirement, in the Boussinesq approximation, corresponds to the assumption that the temperature gradient on the two boundaries is given. Alternatively, this corresponds to the assumption that the Nusselt number is given.

In order to approximate to an astrophysical situation a little more closely, we assume that the average temperature T_{00} at the bottom of the convective layer is given, whereas the average temperature at the top of the layer is not imposed. Besides the usual assumptions of no overshooting and free boundaries, one has to decide on the appropriate boundary conditions to be applied to the temperature fluctuations. In this paper two cases are considered: (A) in which the temperature fluctuation is assumed to vanish at the top and bottom of the convective layer; (B) in which the gradient of the temperature fluctuation vanishes at the top and bottom of the convective layer. Since in practice there must be some interaction between the convective zone in a star and its surroundings, it is likely that the true picture will be somewhere in between the above two extreme cases.

It is shown that the boundary conditions have a marked influence on the results. The assumption of no temperature fluctuation leads to a marked modulation of the flux, even at low Rayleigh number. On the other hand, the assumption of no modulation in the flux leads to the result that the temperature fluctuation can be made very small for large Rayleigh numbers, i.e. for very deep convective zones. The latter model would be more likely to approximate the situation existing in supergranulation which is only observable as a velocity pattern and not as an intensity pattern (Bray and Loughhead 1967).

Basic Equations

The required basic equations, in the mean-field approximation (Roberts 1966), can be written as

$$(\mathrm{D}^2-a^2)^2 W = Ra^2 F, \qquad (\mathrm{D}^2-a^2)F = W\mathrm{D}T_0, \qquad \mathrm{D}T_0 = FW-N, \qquad (1\mathrm{a,b,c})$$

where $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$, and a is the horizontal wave number, R the Rayleigh number and N the Nusselt number. The Rayleigh number is given by

$$R = g\alpha d^3 T_{00}/\nu\kappa, \qquad (2)$$

where g is the gravitational acceleration, α the coefficient of volume expansion, d the thickness of the layer, T_{00} the temperature at the lower boundary, ν the kinematic viscosity and κ the thermal diffusivity. The vertical velocity W, the temperature fluctuation F and the average temperature T are functions of z to be determined subject to the boundary conditions:

(A) $$W = \mathrm{D}^2 W = F = 0 \quad \text{at} \quad z = 0 \text{ and } 1, \qquad (3\mathrm{a})$$

(B) $$W = \mathrm{D}^2 W = \mathrm{D}F = 0 \quad \text{at} \quad z = 0 \text{ and } 1, \qquad (3\mathrm{b})$$

where $z = 0$ specifies the lower boundary, and the layer thickness d has been taken to be the unit of length. The equations (3) are valid only if the perturbations are two-dimensional rolls or convection cells with square or rectangular planform. The value of a determines the shape of the convective cells, with large values corresponding to elongated cells.

In the present problem the Nusselt number N is assumed to be given. It then follows from the boundary conditions (3) and equation (1c) that $\mathrm{D}T_0 = -N$ on the two boundaries. The elimination of $\mathrm{D}T_0$ and F between equations (1a)–(1c) yields the following sixth-order differential equation in W

$$(\mathrm{D}^2-a^2)^3 W = W^2(\mathrm{D}^2-a^2)^2 W - RNa^2 W, \qquad (4)$$

which has to be solved subject to the following boundary conditions at $z = 0$ and 1:

(A) $$W = \mathrm{D}^2 W = \mathrm{D}^4 W = 0, \qquad (5\mathrm{a})$$

(B) $$W = \mathrm{D}^2 W = \mathrm{D}^5 W - 2a^2\mathrm{D}^3 W + a^4 \mathrm{D}W = 0. \qquad (5\mathrm{b})$$

Linear Problem

Before attempting to find the solution to the nonlinear problem, it is necessary to solve the linear equations. Neglecting the nonlinear term in equation (4) gives

$$(\mathrm{D}^2-a^2)^3 W = -R_c Na^2 W, \qquad (6)$$

where R_c is the so-called critical Rayleigh number; convection only occurs at Rayleigh numbers larger than this critical value. Equation (6) has to be solved, for a given value of N, subject to the boundary conditions (5a) or (5b).

The actual temperature gradient at the boundaries is given from equation (1c) by the expression

$$(\mathrm{d}T_0/\mathrm{d}z)_{\text{actual}} = -NT_{00}/d. \tag{7}$$

It follows that, once a value has been selected for the Nusselt and Rayleigh numbers, the thickness of the layer and the temperature at the lower boundary will be uniquely determined by the assumed value of the temperature gradient at the boundaries, i.e. by the assumed value of the energy flux across the layer. It should be noted that the numerical results given in this paper correspond to the case when the Nusselt number is equal to one, unless otherwise specified.

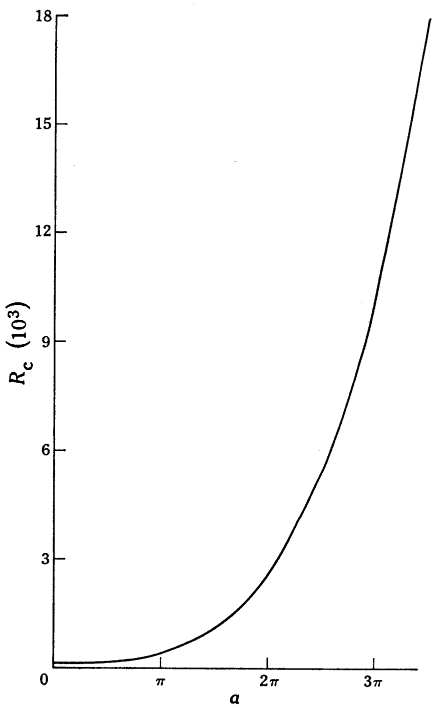


Fig. 1. Critical Rayleigh number R_c as a function of the horizontal wave number a when there is no modulation of the energy flux at the boundaries.

The dependence of the critical Rayleigh number R_c on the horizontal wave number a is now examined. In the absence of temperature fluctuations at the boundaries, i.e. for $F = 0$ (Case A), this dependence is given by (Chandrasekhar 1961)

$$R_c = (\pi^2 + a^2)^3/a^2. \tag{8}$$

In the absence of fluctuations in the energy flux on the boundaries, i.e. $\mathrm{D}F = 0$ (Case B), we have, on solving the differential equation (6) by the initial-value technique, the results as summarized in Fig. 1 for $N = 1$. It can be seen that in this case R_c increases with a but does not tend to infinity as a tends to zero. The minimum of the $R_c(a)$ curve occurs at $a = 0$, which is unusual. In most linear calculations (e.g. Chandrasekhar 1961) $R_c(a)$ has a minimum for some finite value of a, called the critical horizontal wave number a_c, and tends to infinity as a tends to zero.

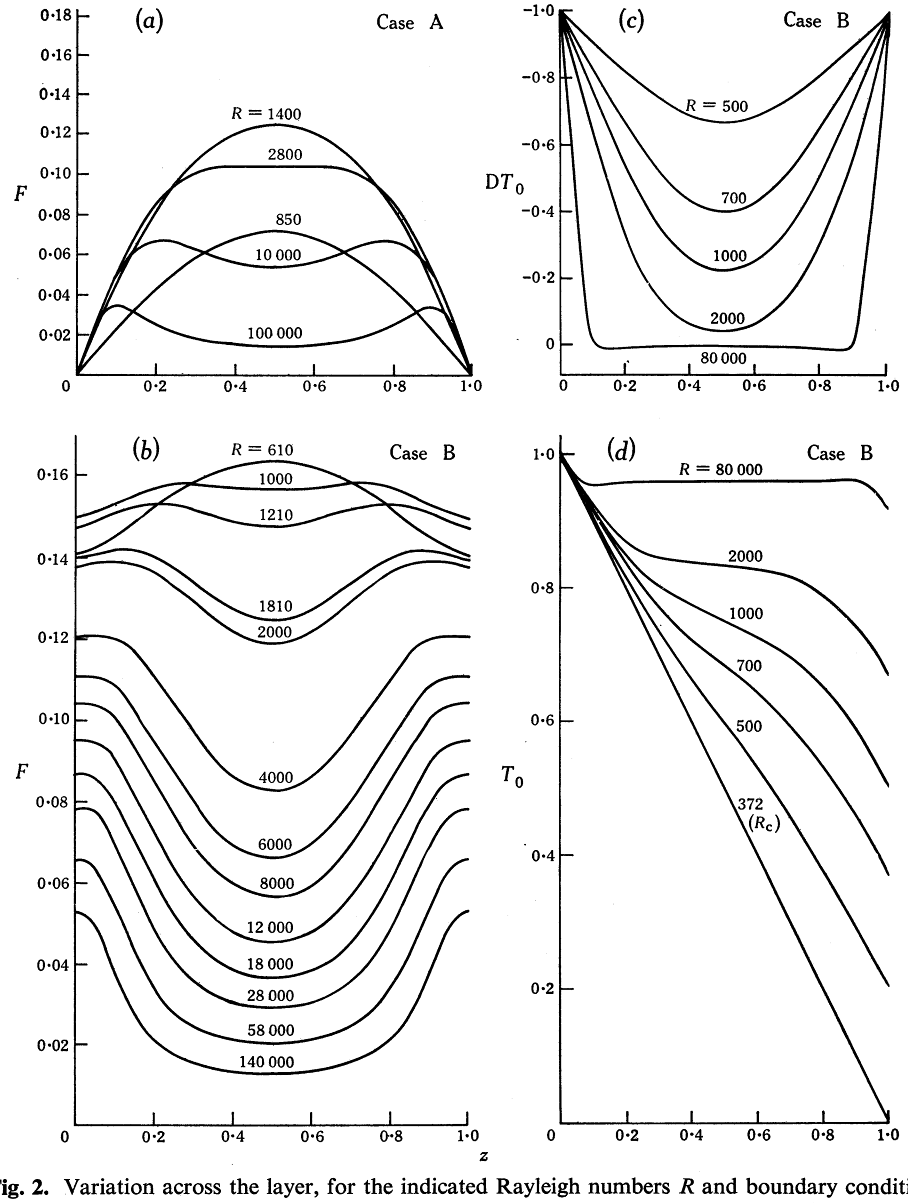


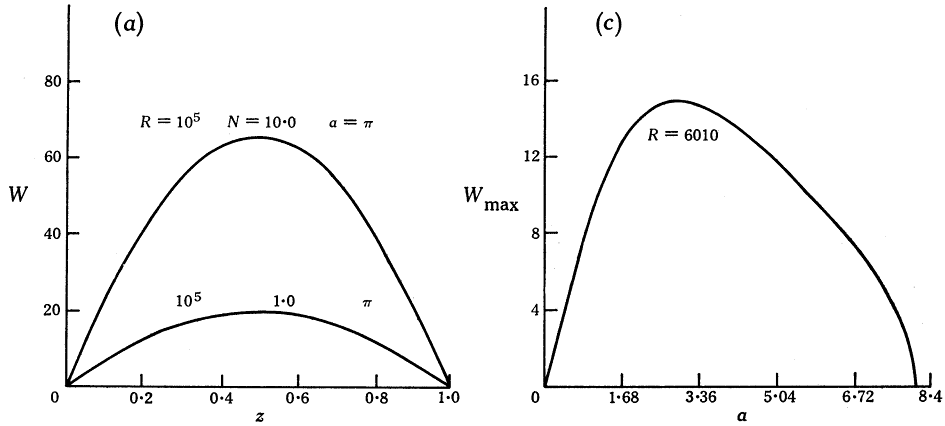
Fig. 2. Variation across the layer, for the indicated Rayleigh numbers R and boundary conditions (Case A or B), of: (*a*) and (*b*) the temperature fluctuation F, (*c*) the temperature gradient DT_0 and (*d*) the average temperature T_0.

Numerical Solutions of Nonlinear Equations

The numerical integration of the nonlinear equation (4), with boundary conditions (5a) or (5b) was carried out by initial-value and quasi-linearization techniques, and the results of the integrations are summarized in Figs 2 and 3.

Case A

When no temperature fluctuations are allowed on the boundaries, the shape of the $F(z)$ curve follows the usual pattern except that the maxima decrease with increasing



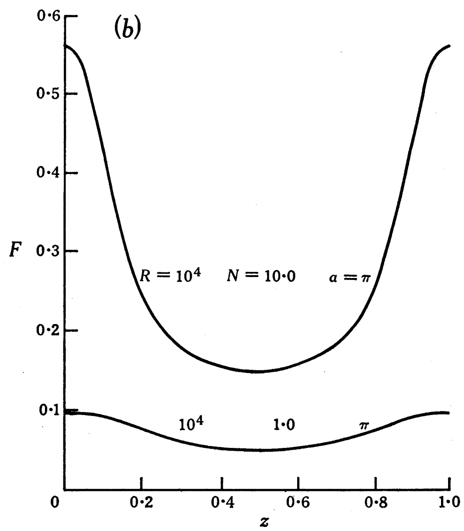


Fig. 3. Variation, for Case B and the indicated parameters, of

(*a*) the vertical velocity W across the layer,

(*b*) the temperature fluctuation F across the layer and

(*c*) the maximum vertical velocity W_{max} as a function of the horizontal wave number a.

Rayleigh number for $R > 1400$, whereas the gradient of F at the boundaries tends to a constant value. This behaviour is illustrated in Fig. 2*a*.

Case B

Figs 2*b*–2*d* illustrate the behaviour of the temperature fluctuation $F(z)$, temperature gradient $DT_0(z)$ and average temperature $T_0(z)$ for various values of the Rayleigh number when no modulation of the energy flux is allowed on the boundaries. It can be seen from Fig. 2*b* that the shape of the F curve changes drastically as the Rayleigh number changes from the low to the large values, and that an isothermal layer develops in the central region of the layer while the temperature at the upper boundary increases gradually. In the compressible case, the isothermal layer corresponds to a region where the temperature gradient equals the adiabatic temperature gradient.

Figs 3*a* and 3*b* show the variation of the W and F curves with changes in the assumed value of N from $N = 1$. Fig. 3*c* show the variation of the maximum vertical velocity W_{max} with wave number for all values of a for which convection exists at $R = 6010$ (for $N = 1$).

Asymptotic Solutions

When the Rayleigh number is large, the numerical integrations become much more difficult and time consuming. Since the Rayleigh number is probably large in astrophysical applications (Spiegel 1971) we now derive analytical expressions, that are valid at high Rayleigh numbers, for the limiting amplitude and the temperature fluctuations at the boundaries.

As explained in detail in previous papers (Van der Borght *et al.* 1972; Van der Borght and Murphy 1973*a*, 1973*b*) the following equation should be satisfied in the main stream

$$\Psi(\mathrm{D}^2-a^2)^2\Psi = 1, \qquad \text{where} \qquad W = (NRa^2)^{\frac{1}{2}}\Psi. \tag{9a, b}$$

Following a method first introduced by Howard (1965) we use the following truncated Fourier expansion for Ψ

$$\Psi = A_1 \sin \pi z + A_3 \sin 3\pi z. \tag{10}$$

The coefficients A_1 and A_3 are then given by (see Appendix 1)

$$A_1 \approx \frac{2^{\frac{1}{4}}}{\pi^2+a^2}\left(1-\tfrac{1}{2}\frac{(\pi^2+a^2)^2}{(9\pi^2+a^2)^2}\right) \qquad \text{and} \qquad A_3 \approx \frac{(\pi^2+a^2)^2}{(9\pi^2+a^2)^2}A_1. \tag{11a, b}$$

It is then evident that the maximum amplitude $W_{\max}$ is given by

$$W_{\max} = (NRa^2)^{\frac{1}{4}}(A_1-A_3). \tag{12}$$

Values of $W_{\max}$ for $N = 1$ and $a = \pi$, as derived by numerical integrations, are compared in Fig. 4*a* with the values predicted from the formula (12) at large Rayleigh numbers. It is seen that the computed values do in fact tend asymptotically to the theoretical values predicted by the expression (12). Since, in the main stream, we have

$$FW = N \tag{13}$$

at high Rayleigh numbers, it is also possible to predict the values of $F(\frac{1}{2})$ from the formula (12).

In order to derive asymptotic expressions for the modulation of the flux (Case A) and temperature modulation at the boundaries (Case B) we have to proceed with the integration of equations (1b) and (1c). From equation (10) it can be seen that, near the origin, we have

$$\Psi = Az, \qquad \text{where} \qquad A = \pi(A_1+3A_3). \tag{14a, b}$$

On eliminating $\mathrm{D}T_0$ between equations (1b) and (1c), we obtain the following equation in F

$$(\mathrm{D}^2-a^2)F = W^2F-NW. \tag{15}$$

Using the following scalings in the boundary layer

$$\zeta = P^{\frac{1}{4}}z, \qquad W = P^{\frac{1}{4}}\psi, \qquad F = P^{-\frac{1}{4}}fN, \qquad \text{where} \qquad P = NRa^2, \tag{16a, b, c, d}$$

we can write the differential equation (15) as

$$\mathrm{d}^2f/\mathrm{d}\zeta^2 = \psi^2f-\psi, \qquad \text{where} \qquad \psi = A\zeta. \tag{17a, b}$$

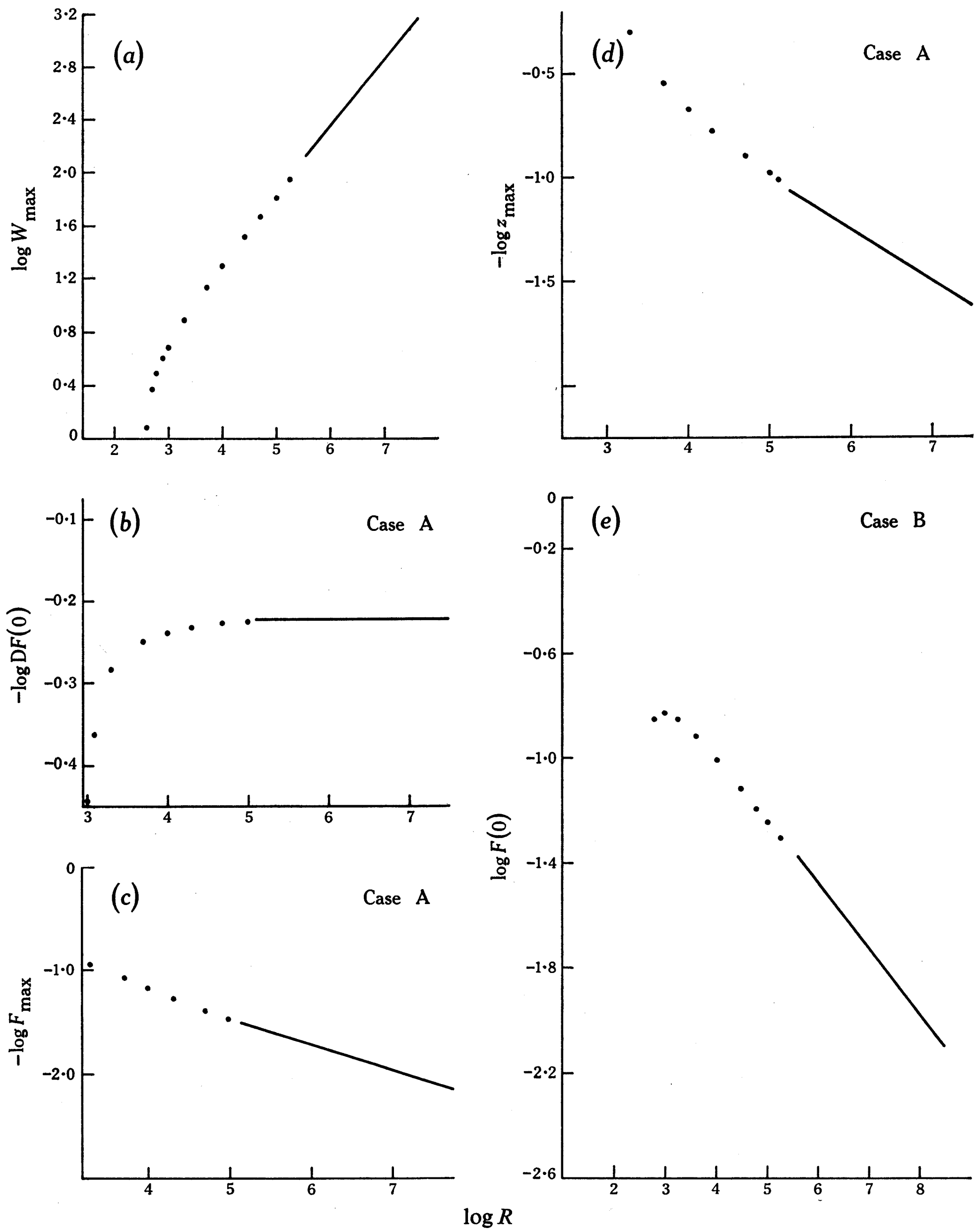


Fig. 4. Comparison as a function of the Rayleigh number R of values predicted by the asymptotic theory (curves) with those derived by numerical integration (points) for:

(*a*) the vertical velocity W_{max},

(*b*) the modulation of the flux $DF(0)$ at the boundaries,

(*c*) the maximum of the temperature fluctuation F_{max},

(*d*) the position z_{max} of F_{max},

(*e*) the temperature fluctuation $F(0)$ at the boundaries.

Equation (17a) must be solved subject to the boundary conditions:

(A) $$f = 0 \quad \text{at} \quad \zeta = 0, \qquad f = (A\zeta)^{-1} \quad \text{at} \quad \zeta = \infty; \tag{18a$_{1,2}$}$$

(B) $$\mathrm{d}f/\mathrm{d}\zeta = 0 \quad \text{at} \quad \zeta = 0, \qquad f = (A\zeta)^{-1} \quad \text{at} \quad \zeta = \infty. \tag{18b$_{1,2}$}$$

These boundary value problems can be solved by numerical methods, and this has in fact been done in order to check the accuracy of the following analytic solutions.

Eliminating ψ between equations (17a) and (17b) yields

$$\mathrm{d}^2f/\mathrm{d}\zeta^2 - A^2\zeta^2 f = -A\zeta. \tag{19}$$

In order to derive the general solution of this second-order linear inhomogeneous differential equation in f, we have to find the complementary function and a particular integral. The complementary function is the general solution of the homogeneous equation

$$\mathrm{d}^2f/\mathrm{d}\zeta^2 - A^2\zeta^2 f = 0. \tag{20}$$

The general solution of this equation (e.g. Kamke 1959) can be expressed in terms of the modified Bessel functions of order $\frac{1}{4}$ as follows

$$f = C_1\,\zeta^{\frac{1}{2}} I_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) + C_2\,\zeta^{\frac{1}{2}} I_{-\frac{1}{4}}(\tfrac{1}{2}A\zeta^2). \tag{21}$$

We now show that a particular solution of equation (19) can be obtained in terms of the modified Struve functions. It is well known (Abramowitz and Stegun 1965) that the modified Struve function $L_\nu(z)$ is a solution of the differential equation

$$z^2\frac{\mathrm{d}^2w}{\mathrm{d}z^2} + z\frac{\mathrm{d}w}{\mathrm{d}z} - (z^2+\nu^2)w = \frac{4(\frac{1}{2}z)^{\nu+1}}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}. \tag{22}$$

This result can be generalized. It can be verified by successive differentiation and substitution that the expression

$$y = x^\alpha L_\nu(\beta x^\gamma) \tag{23}$$

is a solution of

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{1-2\alpha}{x}\frac{\mathrm{d}y}{\mathrm{d}x} - \left((\beta\gamma x^{\gamma-1})^2 + \frac{\nu^2\gamma^2-\alpha^2}{x^2}\right)y = \frac{4\beta^{\nu+1}(\frac{1}{2})^{\nu+1}}{\pi^{\frac{1}{2}}\Gamma(\nu+\frac{1}{2})}\gamma^2 x^{\gamma(\nu+1)+\alpha-2}. \tag{24}$$

A comparison between equations (24) and (19) shows that the latter admits the following particular integral

$$f = -\{(2\pi)^{\frac{1}{2}}\Gamma(\tfrac{3}{4})/4A^{\frac{1}{2}}\}\zeta^{\frac{1}{2}} L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2). \tag{25}$$

Adding equation (25) to (21) then gives the general solution of (19). It now remains to determine the constants C_1 and C_2 from the boundary conditions (18a) or (18b).

Case A

It follows from the properties of the modified Bessel and Struve functions that (Appendix 2)

$$\lim_{\zeta\to 0}\{\zeta^{\frac{1}{2}} I_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\} = 0, \quad \lim_{\zeta\to 0}\{\zeta^{\frac{1}{2}} I_{-\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\} = A^{-\frac{1}{4}}2^{\frac{1}{2}}/\Gamma(\tfrac{3}{4}), \quad \lim_{\zeta\to 0}\{\zeta^{\frac{1}{2}} L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\} = 0. \tag{26a, b, c}$$

The boundary condition ($18a_1$) is then satisfied if

$$C_2 = 0. \tag{27}$$

With the help of the appropriate asymptotic expansions for I_ν and L_ν it follows that, for z large (Appendix 3),

$$L_{\frac{1}{4}}(z) - I_{\pm\frac{1}{4}}(z) \approx -\Gamma(\tfrac{1}{2})/\pi\,\Gamma(\tfrac{3}{4})\,(\tfrac{1}{2}z)^{\frac{3}{4}}. \tag{28}$$

If we select for C_1 the value

$$C_1 = (2\pi)^{\frac{1}{2}}\Gamma(\tfrac{3}{4})/4A^{\frac{1}{4}}, \tag{29}$$

it follows (Appendix 3) that

$$f = \{(2\pi)^{\frac{1}{2}}\,\Gamma(\tfrac{3}{4})/4A^{\frac{1}{4}}\}\zeta^{\frac{1}{2}}\{I_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) - L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\} \tag{30}$$

is a solution of the differential equation (19) and that it satisfies the boundary condition ($18a_2$). From this expression for f we have

$$(\mathrm{d}f/\mathrm{d}\zeta)_{\zeta=0} = \pi^{\frac{1}{2}}\,\Gamma(\tfrac{3}{4})/4\Gamma(\tfrac{5}{4}), \tag{31}$$

and therefore

$$(\mathrm{d}F/\mathrm{d}z)_{z=0} = N\pi^{\frac{1}{2}}\,\Gamma(\tfrac{3}{4})/4\Gamma(\tfrac{5}{4}). \tag{32}$$

We see that the modulation of the flux at the boundaries is appreciable and of the order of 59%. In Fig. 4*b* a comparison is given between the computed values of this quantity and those predicted by the asymptotic theory.

It is also possible to derive the position and value of the maximum temperature fluctuation either by analytic means, making use of equation (30), or by numerical integration. We have chosen the numerical method for the sake of expediency. For $N = 1$ and $a = \pi$, we find that

$$f_{\max} = 1\cdot 08345 \qquad \text{and} \qquad \zeta_{\max} = 3\cdot 15. \tag{33}$$

The values of $F_{\max}$ and $z_{\max}$ are then given by

$$F_{\max} = Nf_{\max}/(NRa^2)^{\frac{1}{4}}, \qquad \text{and} \qquad z_{\max} = \zeta_{\max}/(NRa^2)^{\frac{1}{4}}. \tag{34a, b}$$

These values, for $a = \pi$ and $N = 1$, are compared in Figs 4*c* and 4*d* with those derived from numerical integrations.

Case B

We use a similar procedure to that adopted in Case A. It can be shown (Appendix 2) that

$$\lim_{\zeta\to 0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\zeta^{\frac{1}{2}}I_{-\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\right) = 0, \qquad \lim_{\zeta\to 0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\zeta^{\frac{1}{2}}L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\right) = 0, \tag{35a, b}$$

$$\lim_{\zeta\to 0}\frac{\mathrm{d}}{\mathrm{d}\zeta}\left(\zeta^{\frac{1}{2}}I_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\right) = A^{\frac{1}{4}}/2^{\frac{1}{2}}\,\Gamma(\tfrac{5}{4}). \tag{35c}$$

In order to satisfy the boundary condition ($18b_1$) we have to adopt the value

$$C_1 = 0. \tag{36}$$

It then follows, as for Case A, that

$$f = \{(2\pi)^{\frac{1}{2}}\,\Gamma(\tfrac{3}{4})/4A^{\frac{1}{4}}\}\zeta^{\frac{1}{2}}\{I_{-\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) - L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\} \tag{37}$$

satisfies the boundary condition ($18b_2$) (Appendix 3). A similar expression was obtained by J. Latour (personal communication) in an investigation of the asymptotic behaviour of the Nusselt number at large Rayleigh numbers when the average temperature is given on the two boundaries. In the problem under consideration in the present paper, the heat flux at the boundaries, i.e. the Nusselt number, is given and no further integrations are required.

From equation (37) we obtain

$$f(0) = \tfrac{1}{2}(\pi/A)^{\frac{1}{2}}, \tag{38}$$

and therefore the temperature fluctuation at the lower boundary ($z = 0$) is given by

$$F(0) = \tfrac{1}{2}(\pi/A)^{\frac{1}{2}}\,N^{\frac{3}{4}}\,(Ra^2)^{-\frac{1}{4}}. \tag{39}$$

Due to the symmetry of the problem, the temperature fluctuation $F(1)$ at the upper boundary is given by the same expression. In Fig. 4*e* are given the values of $F(0)$ as evaluated by numerical integration and those predicted by equation (39) for large values of the Rayleigh number, for $a = \pi$ and $N = 1$. It is seen that the computed values do in fact tend asymptotically to those predicted by the theory.

Conclusions

The present results show that some interesting characteristics are possessed by the steady finite-amplitude solutions of the convection problem for non-self-interacting planforms within the Boussinesq approximation in the case of an imposed average energy flux across the convective layer. Two families of solutions have been considered:

(A) In the case of no fluctuations in the average temperature on the boundaries, the solutions have the usual form and show a strong modulation of the convective flux on the boundaries. This could provide a picture of ordinary solar granulation.

(B) On the other hand, the supergranulation is observed as a velocity pattern and not an intensity pattern. If one imposes zero fluctuation of the energy flux at the boundaries, the present results show that finite-amplitude solutions do in fact exist and that for deep layers the resulting fluctuation in the temperatures at the boundaries is also quite small. For such deep layers the Boussinesq approximation is unlikely to be valid and compressibility would have to be taken into account.

Acknowledgment

The author wishes to thank Mr J. R. Crawford for help with the numerical computations.

References

Abramowitz, M., and Stegun, I. A. (1965). 'Handbook of Mathematical Functions', p. 498 (Dover: New York).

Bray, R. J., and Loughhead, R. E. (1967). 'The Solar Granulation', p. 103 (Chapman & Hall: London).

Chandrasekhar, S. (1961). 'Hydrodynamic and Hydromagnetic Stability', p. 35 (Oxford Univ. Press).
Howard, L. N. (1965). Notes from Summer Study Program in Geophysical Fluid Dynamics, Woods Hole Oceanographic Inst., 65–51, Vol. 1, p. 125.
Kamke, E. (1959). 'Differentialgleichungen, Lösungsmethoden und Lösungen', p. 440 (Chelsea: New York).
Roberts, P. H. (1966). *In* 'Non-equilibrium Thermodynamics, Variational Techniques and Stability' (Eds. R. J. Donnelly, R. Herman and I. Prigogine), p. 125 (Chicago Univ. Press).
Spiegel, E. A. (1971). *A. Rev. Astr. Astrophys.* **9**, 323.
Van der Borght, R., and Murphy, J. O. (1973*a*). *Aust. J. Phys.* **26**, 341.
Van der Borght, R., and Murphy, J. O. (1973*b*). *Aust. J. Phys.* **26**, 617.
Van der Borght, R., Murphy, J. O., and Spiegel, E. A. (1972). *Aust. J. Phys.* **25**, 703.

Appendix 1

Equation (9a) can be written

$$(\mathrm{D}^2-a^2)^2\Psi = \Psi^{-1}. \tag{A1}$$

Multiplying this equation by $2\sin k\pi z$ and integrating between the limits 0 and 1, we obtain

$$2\int_0^1 (\mathrm{D}^2-a^2)^2\,\Psi\sin k\pi z\,\mathrm{d}z = 2\int_0^1 (\sin k\pi z)/\Psi\,\mathrm{d}z. \tag{A2}$$

Keeping in mind that a Fourier expansion of the form

$$\Psi = \sum_1^\infty A_n \sin n\pi z \tag{A3}$$

requires the boundary conditions

$$\Psi = \mathrm{D}^2\Psi = 0 \qquad \text{at} \qquad z = 0 \text{ and } 1 \tag{A4}$$

to be satisfied, we obtain by means of a series of integrations by parts:

$$\int_0^1 \mathrm{D}^4\,\Psi\sin k\pi z\,\mathrm{d}z = (k\pi)^4\int_0^1 \Psi\sin k\pi z\,\mathrm{d}z \tag{A5}$$

and

$$\int_0^1 \mathrm{D}^2\,\Psi\sin k\pi z\,\mathrm{d}z = -(k\pi)^2\int_0^1 \Psi\sin k\pi z\,\mathrm{d}z. \tag{A6}$$

It then follows that the left hand side of equation (A2) can be written

$$2\int_0^1 (\mathrm{D}^2-a^2)^2\,\Psi\sin k\pi z\,\mathrm{d}z = 2(k^2\pi^2+a^2)^2\int_0^1 \Psi\sin k\pi z\,\mathrm{d}z. \tag{A7}$$

Substituting equation (A3) in this equation, we find that

$$2\int_0^1 (\mathrm{D}^2-a^2)^2\,\Psi\sin k\pi z\,\mathrm{d}z = (k^2\pi^2+a^2)^2 A_k. \tag{A8}$$

Restricting ourselves to a two-term Fourier sine expansion and neglecting higher order terms, we have

$$\frac{1}{\Psi} = \frac{1}{A_1 \sin \pi z + A_3 \sin 3\pi z} \approx \frac{1 - (A_3/A_1)(1 + 2\cos 2\pi z)}{A_1 \sin \pi z}. \tag{A9}$$

Substituting equations (A8) and (A9) in (A2) gives

$$(k^2\pi^2 + a^2)^2 A_k = 2\int_0^1 (\sin k\pi z / A_1 \sin \pi z)\{1 - (A_3/A_1)(1 + 2\cos 2\pi z)\}\, dz. \tag{A10}$$

Thus, when $k = 1$, we have

$$(\pi^2 + a^2)^2 A_1 = (2/A_1)(1 - A_3/A_1), \tag{A11}$$

and, when $k = 3$, we have

$$\begin{aligned}(9\pi^2 + a^2)^2 A_3 &= (2/A_1)\int_0^1 (1 + 2\cos 2\pi z)\{1 - (A_3/A_1)(1 + 2\cos 2\pi z)\}\, dz \\ &= (2/A_1)(1 - 3A_3/A_1).\end{aligned} \tag{A12}$$

From equations (A11) and (A12) we get, in the first approximation, the approximate expression (11b) for A_3/A_1. On using this expression in equation (A11), we obtain the approximation (11a) for A_1.

Appendix 2

The series expansions for the modified Bessel and Struve functions can be written, to the leading order term, as

$$I_\nu(z) \approx (\tfrac{1}{2}z)^\nu/\Gamma(\nu+1) \quad \text{and} \quad L_\nu(z) \approx (\tfrac{1}{2}z)^{\nu+1}/\{\Gamma(\tfrac{3}{2})\Gamma(\nu+\tfrac{3}{2})\}. \tag{A13}$$

It then follows from these expressions that

$$\zeta^{\frac{1}{2}} I_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) = \frac{A^{\frac{1}{4}}}{2^{\frac{1}{4}}\Gamma(\frac{5}{4})}\zeta, \quad \zeta^{\frac{1}{2}} I_{-\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) = \frac{A^{-\frac{1}{4}}2^{\frac{1}{4}}}{\Gamma(\frac{3}{4})}, \quad \zeta^{\frac{1}{2}} L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) = \frac{A^{5/4}}{2^{5/2}\Gamma(\frac{3}{2})\Gamma(\frac{7}{4})}\zeta^3. \tag{A14a, b, c}$$

The general solution to equation (19) can be written (see equations (21) and (25))

$$f = C_1\zeta^{\frac{1}{2}} I_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) + C_2\zeta^{\frac{1}{2}} I_{-\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) - \{(2\pi)^{\frac{1}{2}}\Gamma(\tfrac{3}{4})/4A^{\frac{1}{4}}\}\zeta^{\frac{1}{2}} L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2). \tag{A15}$$

In Case A the boundary condition ($18a_1$) must be satisfied. It then follows from equations (A14) and (A15) that $C_2 = 0$. Similarly in Case B the boundary condition ($18b_1$) must be satisfied. It then follows from equations (A14) and (A15) that $C_1 = 0$.

Appendix 3

Asymptotic expansions for the modified Bessel and Struve functions I_ν and L_ν can be written (e.g. Abramowitz and Stegun 1965)

$$I_\nu(z) \to \frac{\exp z}{(2\pi z)^{\frac{1}{2}}}\left(1 - \frac{\mu-1}{8z} + \frac{(\mu-1)(\mu-9)}{2!\,(8z)^2}\dots\right), \qquad \text{where} \qquad \mu = 4\nu^2, \tag{A16}$$

and

$$L_\nu(z) \to I_{-\nu}(z) + \pi^{-1}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}\,\Gamma(k+\frac{1}{2})}{\Gamma(\nu+\frac{1}{2}-k)\,(\frac{1}{2}z)^{2k-\nu+1}}. \tag{A17}$$

Equation (28) then follows from equations (A16) and (A17) for large values of z. Thus for large values of ζ, we have

$$\zeta^{\frac{1}{2}}\{I_{\pm\frac{1}{4}}(\tfrac{1}{2}A\zeta^2) - L_{\frac{1}{4}}(\tfrac{1}{2}A\zeta^2)\} \approx \{\pi^{-\frac{1}{2}}\,2^{3/2}/\Gamma(\tfrac{3}{4})\,A^{\frac{3}{4}}\}\zeta^{-1}. \tag{A18}$$

In Case A, with C_1 and C_2 given by equations (29) and (27), equation (A15) of Appendix 2 then yields the expression (30) for f. Alternatively, using equation (A18), we have for large ζ

$$f \approx (A\zeta)^{-1},$$

and the solution (30) therefore satisfies the boundary condition ($18a_2$). Similarly, in Case B, with C_1 given by equation (36) and letting

$$C_2 = (2\pi)^{\frac{1}{2}}\,\Gamma(\tfrac{3}{4})/4A^{\frac{1}{4}}, \tag{A19}$$

equation (A15) of Appendix 2 then yields the expression (37) for f. This expression satisfies the boundary condition ($18b_2$).

Manuscript received 5 December 1973