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Convection in the Lower Atmosphere

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Abstract

From the observations of Warner and Telford (1963, 1967), it would appear that fair-weather daytime conditions can produce a field of isotropically packed convection cells in the lower atmosphere. It is argued here that turbulence plays an insignificant role in the mechanics of one such cell, whose horizontal extent is small compared with its height. Thus a simple model of a convection cell is developed, and the predictions of this model are apparently consistent with the observations in the atmosphere.

1. Introduction

Warner and Telford (1963, 1967) present observations of an apparently regular convection pattern which extends from a uniform horizontal surface to a well-defined inversion about a kilometre above the ground during fair-weather daytime conditions. The pattern is most obvious in the temperature structure which contains pulses with an excess of up to 1 K above the surrounding air temperature. Each pulse, as observed from an aircraft flying horizontally, extends about 200 m and the spacing between pulses is of the same order of magnitude. This geometry is essentially independent of height and shows no preferred horizontal orientation. (We note that other observations (e.g. Grant 1965) imply that the geometry does change with height, but Warner and Telford (1967) suggest that such observations probably correspond to situations in which the convection pattern is not fully developed.) Although the turbulence level is comparable with the mean vertical velocity, it is apparent that there is an updraft of about 1 $m s^{-1}$ associated with each temperature pulse. Continuity requires a corresponding downdraft in the surrounding air.

Simultaneous measurements of the temperature at four levels between $1 \cdot 5$ and 32 m above the ground imply that the physical structure associated with a temperature pulse is an essentially steady plume which is passively advected by the mean wind (Priestley 1959). Such a structure is confirmed by Coulman (1970), who records temperature and velocity fluctuations up to 225 m above the ground. On the other hand, it has been suggested (e.g. James 1953) that the relevant structures are isolated thermals or 'bubbles'. Indeed, small-scale laboratory experiments by Sparrow *et al.* (1970) involving the uniform heating of a flat plate produce sequences of individual thermals regularly separated horizontally in space. The source of each sequence (an eruption of the thermal boundary layer) is fixed in space, and the thermals within each sequence are generated at regular intervals of time. James (1953) also states that the source of his observed atmospheric thermals are probably fixed. Thus, averaged over a time that is large compared with the generation period of the thermals, a mean updraft and a mean temperature excess can be defined for each sequence. That is, the average structure is equivalent to that of a plume.

It therefore appears that a fair-weather convection pattern exists essentially in the form of isotropically packed cells extending from the ground to an inversion about 1 km above the surface. The horizontal dimension of each cell is somewhat less than its height and the cross section of the updraft or plume is independent of altitude. The purpose of the present work is to study the restriction placed by this geometry on any possible fluid motions and to develop a simple model of a convection cell, i.e. a plume and its associated downdraft, which predicts some of the observed features.

Numerical simulations of an unstable atmospheric boundary layer by Deardorff (1972) display plume-like features which contribute to the vertical heat transfer of the system. However, the stochastic nature of that work makes it difficult to isolate the precise mechanics of these features. A deterministic model of a convection plume has been developed by Telford (1970, 1972). Although his model is more sophisticated than the one studied here, and produces more extensive results, the present work shows that a complicated model is not required to explain some of the basic features of the convection process.

2. The Role of Turbulence

At heights less than the Obukhov length scale, which is typically some tens of metres, there generally exists a mechanically mixed, thermally unstable layer owing to solar heating of the ground during fair-weather daytime conditions (Webb 1964). In this region the background turbulence tends to mask any organized cell-like structure, although it is clear that the plumes observed at higher levels do originate here.

The convection plumes of Warner and Telford (1963, 1967) are observed above the well-mixed layer. The behaviour of the turbulent fluctuations of velocity and temperature in this region appears to be somewhat paradoxical. The temperature pulses associated with the plumes are well defined because, whereas there are random fluctuations within each plume, the surrounding air is devoid of temperature fluctuations. On the other hand, the turbulence intensity of the vertical velocity is essentially uniform horizontally, inside and outside the plumes. It is also found that, although the temperature fluctuations decrease with height, the turbulence intensity does not vary greatly with altitude. It is proposed that these features are consistent with the concept of a regular system of convection cells whose horizontal extent R is small compared with their vertical dimension H and in which turbulence plays a minor role.

In the following discussion it is assumed that the mean wind shear above the well-mixed layer, which is observed to be small compared with that near the ground, is unimportant to the convection process. We also take the upper boundary of the convection layer to be fixed, stress free and nonconducting. In practice, the inversion rises at a rate that is slow compared with the mean updraft, and there is often a small downward heat flux from the warmer air above the inversion. We expect the magnitude of this heat flux and the associated shear stress to depend upon the mean wind shear across the inversion.

The basic physical purpose of the fluid motion within each cell is to transport uniformly over the whole cell the heat produced in the well-mixed layer near the ground. (This may be compared with an isolated plume for which the environment acts as an infinite sink of heat.) The well-mixed layer is the only source or sink of heat in the system. Therefore, if all the air in the region is to be heated then either each parcel of air must be advected through the heated region (the well-mixed layer) or the heat must be transferred by turbulent diffusion. (Molecular diffusion is clearly unimportant in a system with a scale of 1 km.) There is also the possibility that each of these mechanisms is of equal importance.

Because the mean wind shear is neglected, the source of any turbulence is associated with the heat transfer itself. If there were no organized motion in the convection layer then buoyancy forces within the well-mixed layer would be the only turbulence source. This turbulence would have a maximum length scale of the order of the Obukhov scale (tens of metres) and hence it would be unlikely to be sustained by diffusion over the whole region to be heated (about 1 km in scale). It would seem therefore that some sort of organized mean motion is required to transport the heat vertically. As there is no preferred horizontal direction, such motions ought to be within isotropically packed cells—ideally hexagonal cells. Now, the shearing motion associated with the mean convection itself acts as a source of turbulence and so some turbulence is always present with the mean flow. However, the basic vertical transport mechanism must be convective rather than diffusive.

To consider the relative importance of the diffusive and convective terms in the equations of motion, we introduce a mean vertical velocity scale W, a vertical length scale of a cell H and a horizontal cell scale R. Thus any turbulent motion has a velocity scale of W, at most, and a length scale of R, and this implies that a turbulent diffusivity coefficient has a magnitude no greater than WR. The ratio of the rate of vertical advection to the rate of vertical diffusion of heat or momentum is of the order

$$(W/H)\div(WR/H^2) = H/R.$$

Therefore the condition that convection must be the dominant vertical transport mechanism implies that

$$R/H \ll 1. \tag{1}$$

The ratio of the rate of vertical advection to the rate of horizontal diffusion is from the condition (1)

$$(W/H)\div(WR/R^2) = R/H \ll 1,$$

that is, the magnitude of the horizontal diffusion terms in the equations of motion is large compared with that of the advection terms. On the other hand, the physical action of the turbulence is essentially unimportant to the basic convection process within a cell. We have assumed that the upper boundary of the convection layer is not a momentum sink. Also, the symmetry of the isotropically packed cells requires that there is no momentum or heat transfer across the vertical side boundaries of each cell. Thus the top and side boundaries of a cell do not support any shear stresses. Symmetry imposes the further restriction of no shear stress at the vertical axis of a cell. The geometry of the convection cell severely restricts the effects of lateral diffusion. Unlike turbulence in an unbounded plume, diffusion cannot attenuate the updraft by lateral spreading because the entire cell must be involved in the mean convection, and so there must be a mean updraft and a corresponding downdraft over the whole cell.

The source of turbulence leading to lateral diffusion is expected to be the mean velocity field rather than the mean temperature field. This is because the temperature difference between the updraft and the downdraft (about 1 K) is of order 1/300 compared with the mean temperature, while the vertical velocity changes sign which implies a normalized velocity difference of order 2. That is, the velocity discontinuity is much greater than the temperature discontinuity. Thus the shear flow at the interface of the updraft and downdraft ought to generate turbulence over the whole vertical extent of the cell. However, the lateral constraints imposed by the geometry, namely no shear stresses at either the side boundaries or the central axis, and also the condition (1), imply that the turbulence is anisotropic with more energy in the vertical direction than the horizontal direction. This means that there is little diffusive transport of momentum or heat between the updraft and the downdraft. Therefore, although vertical velocity fluctuations occur over the whole cell because of the shear flow generation process, significant temperature fluctuations do not arise from the lateral exchange of air between the updraft and downdraft.

Because the mean vertical potential temperature gradient is small, the vertical velocity fluctuations ought not to act as a source of temperature fluctuations. Indeed any temperature fluctuations in a parcel of air in the convection layer are simply residual from the original source in the well-mixed layer, and so they ought to decrease with time after the parcel leaves the well-mixed layer. Therefore we expect the temperature fluctuations to decrease with altitude in the updraft and consequently to be negligible in the return flow of the downdraft.

In summary, we have found that the vertical transport of heat from a relatively thin well-mixed layer throughout a deep layer of air cannot be supported by diffusion alone: a system of convection cells is required whose geometry satisfies the condition (1). This geometric constraint implies that the turbulence within each cell is anisotropic and behaves in the manner observed by Warner and Telford (1963, 1967). We have seen also that the only significant mechanical action of the turbulence is to produce a smooth profile between the updraft and downdraft within a cell; turbulent diffusion does not tend to attenuate the updraft, as occurs in an isolated plume.

The turbulence has also an unimportant thermodynamic effect on the convection process. Firstly, it does not affect the overall energy balance because, while the dissipation of turbulent energy acts to decrease the mechanical energy of the flow, the dissipated energy gives rise to a corresponding increase in heat energy. Since W and R are representative velocity and length scales for the turbulence, the rate of dissipation of turbulent energy per unit mass is of order W^3/R, that is, of order $10^2\ \mathrm{cm^2\,s^{-3}}$ for $W \sim 1\ \mathrm{m\,s^{-1}}$ and $R \sim 100$ m. Now the heat flux at the ground into the air is typically of order 300 $\mathrm{W\,m^{-2}}$. Thus, taking the height of the convection layer $H \sim 1$ km and the air density $\rho_0 \sim 1{\cdot}2\times10^{-3}\ \mathrm{g\,cm^{-3}}$, we find that the ratio of the total rate of dissipation of turbulent energy in the layer to the rate of input of heat energy from the ground is of order 0·04. Therefore the turbulence acts as an insignificant heat source compared with the external heat source. We further note that, because there is turbulence over the whole convection layer, it acts essentially as a uniform heat source and so drives no mean circulation.

Hence it is seen that the turbulence, which exists throughout the convection layer, is unimportant to both the dynamics and thermodynamics of the heat transport processes within the layer. This contention is supported by the observation of Warner and Telford (1967) that the high frequency components of their measured vertical velocity are uncorrelated with the temperature fluctuations associated with the mean heat transport. They therefore 'conclude that there is little difference between the turbulence in the thermals and in the surrounding, thermally quiescent, descending air'.

3. Formulation of Model

We consider a model of a typical convection cell involved in the vertical transport of heat from a thin well-mixed layer near the ground to the ambient air up to a strong inversion about 1 km above the ground. The model is simplified by the inclusion of only those physical processes which the above discussion suggests play a dominant role.

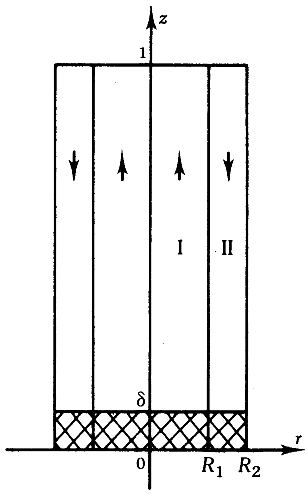

Fig. 1. Axisymmetric convection cell with $R_1 \ll 1$. (r, z) are cylindrical coordinates with z increasing vertically upwards. The well-mixed layer $0 < z < \delta \ll 1$ is modelled as a region in which there is a uniform heat source.

Isotropically packed cells ideally have a hexagonal planform. However, as a good first approximation, we consider an axisymmetric cell of constant radius. The observations of Warner and Telford (1963, 1967) imply that the cross section of the updraft within a cell is independent of height. Thus the model cell is divided into two regions, as shown in Fig. 1, where (r, z) are cylindrical polar coordinates normalized with respect to the height H of the inversion above the ground; z increases vertically upwards from the smooth ground at $z = 0$. Region I, $0 < r < R_1 = R/H$, is a cylindrical core in which the primary mean motion is upwards, while region II, $R_1 < r < R_2$, is an annular domain involving the remainder of the cell in which the mean motion is essentially downwards. The condition (1) now becomes

$$R_1 \ll 1. \tag{2}$$

Because it has been shown in Section 2 that turbulence, although present throughout the whole cell, is physically insignificant with regard to the transport of heat within the cell, viscous and turbulent diffusion terms in the governing equations are neglected. To account for the heat input to the system from the ground, which is clearly a diffusive process, the well-mixed layer $0 < z < \delta \ll 1$ is modelled as a region in which there is a uniform internal heat source. Thus, if F_0 is the heat flux from the ground into the air, then a model heat source is required which produces an average heating rate per unit volume of F_0/H. The thermodynamics of the problem are further simplified by assuming that there is a single mass of a perfect gas within a cell, and so changes due to mixing with the external air are neglected. Also we consider the case where there is no cloud formation in the cell and where radiation heat losses are negligible, which implies that any water vapour acts simply as a passive scalar.

Representative scales are now introduced in order to normalize the variables in the problem: the mean updraft in region I is W, the density and temperature at the ground are initially ρ_0 and T_0 respectively, the average rate of heating per unit volume of fluid within the cell is F_0/H, and the gas constant for the fluid is R_0. Thus the normalized equations for the conservation of momentum, mass and thermodynamic energy and the equation of state may be written as

$$M^2\,\mathrm{d}\boldsymbol{u}/\mathrm{d}t + \gamma^{-1}\rho^{-1}\,\nabla p + G\boldsymbol{3} = 0, \tag{3a}$$

$$\mathrm{d}\rho/\mathrm{d}t + \rho\nabla\,.\,\boldsymbol{u} = 0, \tag{3b}$$

$$\varepsilon\,\mathrm{d}Q = \mathrm{d}T + (\gamma-1)p\,\mathrm{d}(\rho^{-1}), \tag{3c}$$

$$p = \rho T, \tag{3d}$$

where Ht/W is the time; $W\boldsymbol{u} = (u, w)W$ is the fluid velocity at the point $(r, z)H$; $\rho_0\,\rho$, $T_0\,T$ and $\rho_0\,R_0\,T_0\,p$ are the density, temperature and pressure of the fluid; $F_0\,Q/\rho_0\,W$ is the heat input per unit mass to the fluid; and $\boldsymbol{3}$ is a unit vector in the upward vertical direction. The constant parameters in equations (3) are

$$M = W/(\gamma R_0\,T_0)^{\frac{1}{2}}, \quad G = gH/\gamma R_0\,T_0, \quad \varepsilon = (\gamma-1)F_0/\rho_0\,R_0\,T_0\,W, \tag{4a, b, c}$$

and γ is the ratio of the specific heats of the fluid.

Introducing a normalized entropy S defined by

$$\mathrm{d}S = \varepsilon\,\mathrm{d}Q/(\gamma-1)T, \tag{5}$$

we find that equations (3c) and (3d) may be integrated because there is a single air mass. Thus p and ρ may be eliminated from (3), and the equations of motion become

$$M^2\,\mathrm{d}\boldsymbol{u}/\mathrm{d}t + (\gamma-1)^{-1}\,\nabla T - \gamma^{-1}T\,\nabla S + G\boldsymbol{3} = 0, \tag{6a}$$

$$(\gamma-1)^{-1}T^{-1}\,\mathrm{d}T/\mathrm{d}t - \mathrm{d}S/\mathrm{d}t + \nabla\,.\,\boldsymbol{u} = 0, \tag{6b}$$

$$(\gamma-1)T\,\mathrm{d}S/\mathrm{d}t = \varepsilon\,\mathrm{d}Q/\mathrm{d}t, \tag{6c}$$

where $\mathrm{d}Q/\mathrm{d}t$ is the constant and uniform rate of heating within the well-mixed layer, normalized such that

$$\int_0^1 \{\mathrm{d}Q(z)/\mathrm{d}t\}\,\mathrm{d}z = 1. \tag{7}$$

We wish to solve equations (6) in regions I and II subject to certain boundary conditions which are discussed below. However, the equations are clearly nonlinear and so we consider a corresponding set of linear equations which approximately model the physical processes described by (6). It is noted first that the driving force in the system is the heat source term on the right-hand side of equation (6c), which is of order ε. Because the normalized variables are of order unity, it follows that the change in the thermodynamic variables due to the heating is of order ε. The heat flux at the ground F_0 is typically of order $300\,\mathrm{W\,m^{-2}}$, $H \sim 1$ km, $T_0 \sim 288$ K and $\rho_0 \sim 1{\cdot}2\times10^{-3}\,\mathrm{g\,cm^{-3}}$. Taking $\gamma \sim 1{\cdot}4$, $R_0 \sim 2{\cdot}9\times10^{6}\,\mathrm{cm^2\,s^{-2}\,K^{-1}}$ and $W \sim 1\,\mathrm{m\,s^{-1}}$, we see from equation (4c) that ε is of order $1{\cdot}2\times10^{-3}$. Thus the change in the thermodynamic variables induced by the heating is small. The change due to adiabatic movement of a parcel of fluid over the cell is of order G, which for $g \sim 980\,\mathrm{cm\,s^{-2}}$ and $H \sim 1$ km is equal to $0{\cdot}09$. Provided that the vertical extent of the cell is not much greater than 1 km, it is seen therefore that the thermodynamic variables do not vary markedly within the cell. Thus, wherever the temperature T occurs as a coefficient of a differential term in equations (6), we replace the variable by its reference value, that is, by unity.

The second approximation applied to equations (6) concerns the total derivative $\mathrm{d}/\mathrm{d}t$. Within region I the motion is essentially upwards. Although continuity requires a corresponding radial flow, the geometric condition (2) implies that this is small compared with W. Also, if the dependent variables do not change markedly in the radial direction within region I, then it would seem that the radial advection terms can be neglected. This is consistent with the notion that vertical advection is the primary function of the mean flow. Because the vertical velocity is not expected to change sign within region I, the behaviour of the nonlinear vertical advection term should be modelled well by replacing the variable vertical velocity with its average value over the region. Applying the corresponding arguments to region II, we therefore make the approximation

$$\mathrm{d}/\mathrm{d}t \approx \partial/\partial t + U\partial/\partial z, \tag{8}$$

where

$$\begin{aligned} U &= 1 \quad \text{in region I,} \\ &= -\alpha \quad \text{in region II,} \end{aligned}$$

α being the magnitude of the normalized mean downdraft.

By introducing the approximation (8) and the linearization of the thermodynamic terms, equations (6) reduce to the linear system

$$M^2\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial z}\right)\boldsymbol{u}+(\gamma-1)^{-1}\nabla T-\gamma^{-1}\nabla S+G\boldsymbol{3}=0, \tag{9a}$$

$$\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial z}\right)\left((\gamma-1)^{-1}T-S\right)+\nabla.\boldsymbol{u}=0, \tag{9b}$$

$$(\gamma-1)\left(\frac{\partial}{\partial t}+U\frac{\partial}{\partial z}\right)S=\varepsilon\frac{\mathrm{d}Q}{\mathrm{d}t}. \tag{9c}$$

Thus the equations (9) for the conservation of momentum, mass and entropy within the convection cell can be expressed in terms of the velocity vector $\boldsymbol{u}$ and the thermodynamic variables T and S.

We note that equation (9a) does not contain the common Boussinesq approximation in which the coefficient T of the entropy gradient is replaced by its adiabatic value, i.e. by $1-(\gamma-1)Gz$. Such an approximation permits of the generation of vorticity owing to the presence of horizontal entropy (or density) gradients. However, we have assumed already that radial gradients within regions I and II are small, and therefore the neglect of the Boussinesq term is consistent with this. Moreover, the Boussinesq term produces a formally second-order effect because it is of magnitude εG, where both ε and G are small compared with unity. Thus any such hydrostatic imbalance at the interface between regions I and II is small compared with the heat-induced imbalance of order ε.

Because the well-mixed layer provides a steady heat source for the system, the overall entropy and temperature of the air must increase with time and so equations (9) do not admit of a steady solution. On the other hand, the assumption that the geometry of the cell is fixed implies that the mean air density within the cell is fixed. We therefore seek a quasi-steady solution within regions I and II such that the velocity and density fields are independent of time, that is, all the input heat energy appears as internal energy and not as kinetic energy or external work. Thus the 'initial' conditions on the system (9) may be written as

$$\partial \boldsymbol{u}/\partial t = 0 \qquad \text{and} \qquad \partial\{(\gamma-1)^{-1}T - S\}/\partial t = 0, \tag{10a}$$

$$S = 1 \quad \text{and} \quad T = 1 \qquad \text{at} \qquad t = 0,\ z = 0. \tag{10b}$$

We further introduce the energy integral for the overall system:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\int_0^1 \mathrm{d}z \int_0^{R_2} \mathrm{d}r\, 2\pi r\, \rho T\right) = \varepsilon \int_0^1 \mathrm{d}z \int_0^{R_2} \mathrm{d}r\, 2\pi r\, \rho \frac{\mathrm{d}Q}{\mathrm{d}t}.$$

This may be linearized and simplified by using the conditions (7) and (10) to yield

$$\int_0^1 \mathrm{d}z \int_0^{R_2} \mathrm{d}r\, 2\pi r \frac{\partial S}{\partial t} = \frac{\pi R_2^2 \varepsilon}{\gamma-1}. \tag{11}$$

If the top and bottom of the cell are rigid then the boundary conditions on the vertical velocity component are

$$w = 0 \qquad \text{at} \qquad z = 0, 1. \tag{12}$$

These conditions are not satisfied in the physical situation where the convection cell is capped by an inversion. In this case the interface rises as the temperature of the air within the cell increases, and so the top boundary condition ought to ensure continuity of the normal stress across the interface. However, because the rate of heating is of order ε which is much less than unity, it follows that the rate of rise of the interface is also of order ε provided that the inversion is sufficiently strong. It would seem therefore that (12) is valid because we are interested primarily in the internal mechanics of the convection cell.

The remaining boundary conditions on equations (9) are taken to be

$$u = 0 \qquad \text{on} \qquad r = R_2, \tag{13a}$$

$$u = 0 \qquad \text{on} \qquad r = 0, \tag{13b}$$

$$u \qquad \text{is continuous at} \qquad r = R_1, \tag{13c}$$

$$(\gamma-1)^{-1}T - \gamma^{-1}S \qquad \text{is continuous at} \qquad r = R_1, \tag{13d}$$

$$\partial T/\partial r = 0 = \partial S/\partial r \qquad \text{at} \qquad z = 0. \tag{13e}$$

The symmetry of the cell pattern imposes the condition (13a), while (13b) is a consequence of continuity at the axis of symmetry. The condition (13c) is found from integration of equation (9b) across the interface $r = R_1$, and it corresponds to the conservation of mass flux across the interface. Similarly (13d) is obtained from equation (9a), and it ensures the continuity of force at the interface between regions I and II. The conditions (13e) assert that the thermodynamic conditions at the ground are homogeneous so that each plume is not associated with a localized 'hot spot'.

Applying the boundary conditions (10), (11) and (13e), we can integrate equation (9c) to obtain

$$S = 1+(\gamma-1)^{-1}\varepsilon t+(\gamma-1)^{-1}\varepsilon q/U, \tag{14}$$

where

$$q(z) = \int_0^z \frac{dQ}{dt}\,dz - z.$$

Also, by using the conditions (10), (12) and (13e), the vertical component of (9a) can be integrated to yield

$$(\gamma-1)^{-1}T-\gamma^{-1}S = \{\gamma(\gamma-1)\}^{-1}+\gamma^{-1}\varepsilon t-Gz-M^2Uw. \tag{15}$$

Equations (14) and (15) show that within either region I or region II the radial gradients of the thermodynamic variables are of order M^2. However, M^2, which is the square of the Mach number of the flow, is equal to $W^2/\gamma R_0 T_0$ and is generally of order 10^{-5}. Thus the assumption used to obtain the approximation (8), namely that radial gradients are small within each region, is satisfied by the thermodynamic variables. Neglecting the term of order M^2, we find from (14) and (15) that the normalized temperature of the air in the cell is

$$T = 1+\varepsilon t-(\gamma-1)Gz+\gamma^{-1}\varepsilon q/U. \tag{16}$$

Equation (15) shows that the matching condition (13d) on the force at the interface between regions I and II implies that the dynamic pressure must be continuous. That is, any possible imbalance in the force at the interface is of order M^2, and so the condition (13d) is satisfied to that order provided that the velocity is correct to order unity. Therefore, by putting the relations (9a), (10), (14) and (15) into (9b), it is found that the conservation of mass is expressed by the equation

$$\frac{1}{r}\frac{\partial(ru)}{\partial r}+\frac{\partial w}{\partial z} = GU+\varepsilon\gamma^{-1}\frac{dq}{dz}, \tag{17}$$

in which a term of order M^2, corresponding to the last term in equation (15), has been neglected. As expected, the driving force for the velocity field is the heat source in the well-mixed layer which produces a divergence in the field. The first term on the right-hand side of (17) accounts for the adiabatic expansion of a parcel of air as it rises.

By taking the curl of equation (9a) and using the conditions (12) and (13e), it can be shown that the velocity field is irrotational within each region, that is,

$$\partial u/\partial z = \partial w/\partial r. \tag{18}$$

This is consistent with the results (14) and (15), which imply that the density gradient and the pressure gradient are directed essentially in the same direction, that is, vertically. The equation for the vertical velocity is found from equations (17) and (18) to be

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial w}{\partial r}\right)+\frac{\partial^2 w}{\partial z^2} = \varepsilon\gamma^{-1}\frac{\mathrm{d}^2 q}{\mathrm{d}z^2}.$$

This equation is readily solved and the application of the boundary conditions (12) and (13b) yields, in region I,

$$w(r,z) = \varepsilon\gamma^{-1}\left(q(z)+\sum_{n=1}^{\infty} a_n I_0(n\pi r)\sin(n\pi z)\right) \tag{19}$$

and, in region II,

$$w(r,z) = \varepsilon\gamma^{-1}\left(q(z)+\sum_{n=1}^{\infty} \{b_n I_0(n\pi r)+d_n K_0(n\pi r)\}\sin(n\pi z)\right), \tag{20}$$

where I_0 and K_0 are the zeroth order modified Bessel functions and the constants a_n, b_n and d_n are to be determined. The functional form of the radial velocity component u may be found from equations (17), (18), (19) and (20). Application of the boundary conditions (13a) and (13b) then gives, in region I,

$$u = \tfrac{1}{2}Gr - \varepsilon\gamma^{-1}\sum_{n=1}^{\infty} a_n I_1(n\pi r)\cos(n\pi z) \tag{21}$$

and, in region II,

$$u = \tfrac{1}{2}\alpha Gr\{(R_2/r)^2-1\}+\varepsilon\gamma^{-1}\sum_{n=1}^{\infty}\{d_n K_1(n\pi r)-b_n I_1(n\pi r)\}\cos(n\pi z), \tag{22}$$

where

$$d_n = b_n I_1(n\pi R_2)/K_1(n\pi R_2). \tag{23}$$

It is seen from equations (19)–(22) that the heat source in the well-mixed layer gives rise to a uniform vertical flow over the whole cell, while the adiabatic expansion of a rising parcel of air produces a radial flow. The remaining terms in the equations for the velocity components correspond to an incompressible flow induced by the matching conditions at the interface between regions I and II. The normalizing condition that w is of order unity implies that a_n, b_n and d_n are of order ε^{-1}, and so the motion is predominantly incompressible.

From the condition (13c) and equations (21) and (22), the continuity of mass flux at $r = R_1$ requires that

$$\alpha^{-1} = (R_2/R_1)^2-1 \tag{24}$$

and

$$a_n = \{1 - I_1(n\pi R_2)\,K_1(n\pi R_1)/I_1(n\pi R_1)\,K_1(n\pi R_2)\}\,b_n\,. \tag{25}$$

The condition (24) on the normalized mean updraft α implies that the volume flux is equal in regions I and II, that is, it is a further incompressibility condition and is therefore consistent with ε being small. The heat source term $q(z)$, which is zero at $z = 0$ and $z = 1$, may be expanded in the form

$$q(z) = \sum_{n=1}^{\infty} q_n \sin(n\pi z)\,, \tag{26}$$

where

$$q_n = 2\int_0^1 q(z)\sin(n\pi z)\,\mathrm{d}z\,.$$

Hence the condition (13d) on the continuity of force at the interface $r = R_1$ and equations (15), (19) and (20) give

$$(a_n + \alpha b_n)\,I_0(n\pi R_1) + \alpha d_n\,K_0(n\pi R_1) = -(1+\alpha)\,q_n\,. \tag{27}$$

Thus the constants a_n, b_n and d_n can be determined in terms of q_n from equations (23), (25) and (27), and so the flow is specified fully to the zeroth order in M^2.

Now the velocity components have been normalized with respect to the mean updraft W in region I, which implies that

$$\int_0^{R_1} \mathrm{d}r\, 2\pi r \int_0^1 \mathrm{d}z\; w(r,z) = \pi R_1^2\,.$$

Using equation (19), we find that this becomes

$$\gamma\varepsilon^{-1} = \sum_{n=1}^{\infty} (n\pi)^{-1}\{1-\cos(n\pi)\}\{2a_n I_1(n\pi R_1)/n\pi R_1 + q_n\}\,. \tag{28}$$

Equation (28) therefore specifies ε, and hence W, in terms of the rate of heating in the well-mixed layer.

4. Discussion

The velocity field of the present model is internally consistent with the initial assumptions used to obtain the approximation (8). Even for R_2 as large as $0{\cdot}2$, that is, when the cell diameter is $0{\cdot}4$ of the cell height, the magnitude of the radial velocity u is found to be small compared with that of the vertical velocity w. However, because the flow is essentially incompressible, the radial gradient $\partial u/\partial r$ is not small everywhere. This suggests that perhaps the term $u\,\partial u/\partial r$ ought to have been included in the radial momentum equation (9a). On the other hand, away from the top and bottom of the cell, u and so $\partial u/\partial r$ are small, while w and $\partial u/\partial z$ are not small. Thus the vertical advection term dominates over most of the cell. Equation (16) implies that the radial gradients of the thermodynamic variables are negligible within region I and within region II. Similarly, the radial gradients of w within each region are found to be very small. It appears therefore that the approximation (8) is valid, even when the condition (2) is relaxed somewhat.

We now compare the predicted properties of the model convection cell described in Section 3 with the observations of Warner and Telford (1963, 1967). It is consistent with the linearized model to define the normalized potential temperature θ by

$$\theta = T+(\gamma-1)Gz.$$

Hence equation (16) gives

$$\theta = 1+\varepsilon t+\gamma^{-1}\varepsilon q/U. \tag{29}$$

Thus, in agreement with the findings of Warner and Telford, the time rate of change of temperature $\partial\theta/\partial t$ is independent of height, and it is equal to ε. For a heat flux at the ground F_0 of 300 W m^{-2}, a cell height H of 1 km and an updraft W of 1 m s^{-1}, we found in Section 3 that ε is of order $1\cdot2\times10^{-3}$. This therefore corresponds to a potential temperature increase of about 1·2 K per hour. We note that the rigid-end conditions on the model cell imply that the fluid does no external work overall, and so $\partial\theta/\partial t$ is probably less than ε in practice.

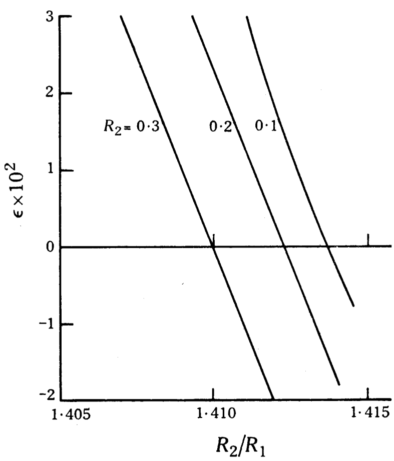

Fig. 2. Behaviour of the heating parameter ε with the radius ratio R_2/R_1 from equation (28) for the indicated values of R_2 when the heat source is described by the conditions (31).

The fundamental property of the model convection cell is that its horizontal extent is small compared with its height, that is, R_1 is small. By consideration of the behaviour of the modified Bessel functions for small arguments, it is straightforward to show from equations (23)-(25), (27) and (28) that the condition that ε be small gives

$$(R_2/R_1)^2 \approx 2. \tag{30}$$

It follows from the condition that the vertical velocity must not change sign in region 1 that the first term in the expansion (19) dominates the behaviour of w. Hence ε and the result (30) are essentially independent of the detailed behaviour of the heat source $\mathrm{d}Q/\mathrm{d}t$, provided that the height δ of the well-mixed layer is small. A plot of ε as a function of R_2/R_1 is shown in Fig. 2 for the case where the heat source is given by

$$\mathrm{d}Q/\mathrm{d}t = 10 \quad \text{for} \quad 0<z<0\cdot1, \tag{31a}$$

$$= 0 \quad \text{for} \quad 0\cdot1<z<1. \tag{31b}$$

It is seen that for a fixed radius ratio R_2/R_1 the heating parameter ε decreases with increasing cell radius R_2. Since the overall rate of heating is determined by the heat flux F_0 and the cell height H, this result implies that the updraft velocity W must increase with increasing cell radius R_2. Indeed equation (4c) defines the actual relationship between ε and W.

Fig. 2 shows that as R_2/R_1 increases for a fixed cell radius R_2, ε decreases and eventually becomes negative. A negative value of ε corresponds to W being negative, that is, the downdraft occurs in region I while the updraft is in the outer region II. It appears that the convection cells in the atmosphere are generated by initially isolated plumes rising from the well-mixed layer, and so solutions with negative values of ε must be considered as unphysical.

The relation (30) implies that the areas of the updraft and downdraft are almost equal, and hence the normalized downdraft α is approximately unity. The latter result can be seen intuitively from the condition (13d) and equation (15), which show that the continuity of force at the interface between regions I and II leads to the continuity of dynamic pressure. Thus the flow in region II ought to be simply equal in magnitude and opposite in sign to the flow in region I.

Warner and Telford (1967) found that the ratio of the average space between plumes to their average horizontal dimension (the space to pulse ratio) was approximately equal to unity. However, this result was obtained from their raw data, that is, from an average along an arbitrary horizontal path through the convection field, and so it must be corrected for this statistical effect before it may be compared with the present model. The probability that any given point on a horizontal plane lies within a plume clearly equals the fraction of the cross sectional area of the field taken up by the plumes. Thus the space to pulse ratio is given by the ratio of the average area of the downdraft to that of the updraft, and so it equals unity when equation (30) holds. In fact, Warner and Telford found that the space to pulse ratio was often slightly larger than unity. This is still consistent with the model prediction if the convection cells in practice are not packed completely regularly, that is, if quiescent gaps occur between some cells.

On the other hand, the model described by Telford (1972) yields values of R_2/R_1 which vary greatly with the heat flux F_0 and with the cell height H (e.g. see Fig. 3). The observations of Warner and Telford (1963, 1967) do not seem to indicate such a vast variation in the value of R_2/R_1. Telford (1970) asserts that the solutions for which R_2/R_1 is less than $2^{\frac{1}{4}}$ are unphysical. He reasons that, because horizontal pressure gradients have been neglected and because there is no energy source at the upper boundary, the kinetic energy of the descending air must not be greater than that of the rising air. All the solutions with $R_2/R_1 \geqslant 2^{\frac{1}{4}}$ are, however, taken to be significant and, in fact, are taken to show that organized convection plumes cannot exist once the cell height reaches a critical value which depends upon the heat flux and surface turbulence intensity.

The main difference between Telford's (1972) model and the present one is that the former includes the effects of turbulence. This suggests that the manner in which Telford models the turbulence (which is deemed to be unimportant in the present case) gives rise to the different predictions. Morton (1968) points out that the entrainment assumption of Telford leads to a gross overestimation of the magnitude of the production term in the turbulent energy equation. Moreover, the rate of dissipation of turbulent energy is surely underestimated by taking the turbulence length scale to be the diameter of the plume, rather than some fraction of the diameter. Thus the net rate of change of turbulence intensity tends to be dominated by the production term which causes the turbulence to increase with height in the updraft and to decrease in the downdraft. Because the turbulence is taken to be horizontally uniform at the base of the cell, the difference between the turbulence intensity of the updraft and that of the downdraft tends to increase with height. The deeper the convection layer, the greater this difference tends to become. But the imposed boundary conditions require the turbulence in the updraft to equal that in the downdraft at the top and at the bottom of the cell, and so the turbulence must be forced not to behave in the above manner if a solution to the system is to be obtained. It can be shown from equations

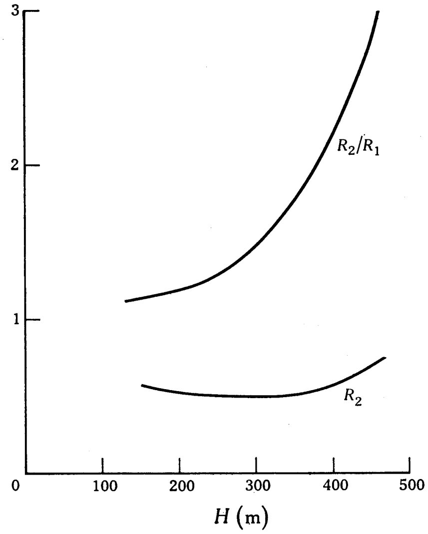

Fig. 3. Variation of the downdraft to updraft radius ratio R_2/R_1 and the normalized cell radius R_2 with the cell height H, as predicted by Telford (1972) for a heating rate of 2 K h^{-1} and a turbulence intensity at the ground of 1 m s^{-1}.

(11), (17) and (18) of Telford (1970) that the ratio of the production term to the dissipation term in the updraft turbulence equation involves the factor $R_2^2/(R_2^2-R_1^2)$ while that in the downdraft equation involves the factor $R_1\,R_2^2/(R_2^2-R_1^2)^{3/2}$. The relative magnitude of the production term in the turbulent energy equations therefore decreases as R_2/R_1 increases. In order to limit the difference between the updraft and downdraft turbulence, a larger value of R_2/R_1 would seem to be required as the cell height increases. Thus it appears that the large values of R_2/R_1 predicted by Telford arise primarily from the overestimation of the rate of production of turbulent energy.

It is seen from equations (8), (14) and (29) that the potential temperature gradient in region II above the well-mixed layer is

$$\partial\theta/\partial z = \varepsilon/\gamma\alpha .$$

For $T_0 \sim 288$ K, $H \sim 1$ km, $\varepsilon \sim 1\cdot2\times10^{-3}$ and $\gamma \sim 1\cdot4$, this corresponds to a stable temperature gradient of about $0\cdot25$ K km^{-1}, which is compatible with the observations that the environment through which the plumes rise is 'neutral to slightly stable'.

Measurements of the average potential temperature at any level show that it is essentially independent of height. Using equations (14) and (29), we find that the potential temperature gradient averaged over the horizontal cross section of a cell is given by

$$(\partial\theta/\partial z)_{av} = -\gamma^{-1}\varepsilon\{2-(R_2/R_1)^2\} \qquad \text{for} \qquad z > \delta .$$

However, the derivation of equation (30) implies that $2-(R_2/R_1)^2$ is of order ε, and so the overall air mass above the well-mixed layer is neutrally stratified to first order in ε, as observed by Warner and Telford (1963, 1967). Their observations also indicate that the temperature excess of each pulse relative to the environment, which corresponds in the model to the difference between θ in regions I and II at any level, is described well by a 'top-hat' profile and it decreases approximately linearly with height. In comparison, equation (29) yields precisely a top-hat profile for θ at any level within the cell with a normalized temperature excess of

$$\Delta\theta = (R_2/R_1)^2\gamma^{-1}\varepsilon(1-z) \qquad \text{for} \qquad z > \delta . \tag{32}$$

Thus, for $\varepsilon \sim 1{\cdot}2\times10^{-3}$ and $T_0 \sim 288$ K, the maximum temperature excess is about $0{\cdot}5$ K, which is the magnitude of the results of Warner and Telford.

The simple method presented here does not predict the actual value of the mean updraft velocity W corresponding to a given heat flux at the ground F_0. However, it can be shown readily from equations (4) and (32) that W and the maximum (dimensional) temperature excess ΔT_0 are related by

$$W\Delta T_0 = (R_2/R_1)^2(1-\delta)F_0/\rho_0\, C_p ,$$

where C_p is the specific heat of the fluid at constant pressure.

It would thus appear from the above discussion that the geometry and temperature structure of the model convection cell are compatible with the observations of Warner and Telford (1963, 1967).

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