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A Cosmological Model of Class One in Lyra's Manifold

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Abstract

A study is presented of a spherically symmetric class-one cosmological model based on Lyra's geometry. The static universe is shown to be physically unrealistic. The nonstatic model parallels Lemaitre's in the Riemannian case but the law of mass–energy conservation does not hold.

Introduction

Lyra's (1951) modification of Riemannian geometry (and of Weyl's (1918) theory of gravitation) by introducing a gauge function into the structureless manifold, and the subsequent investigations by Sen (1957, 1960), Halford (1970, 1972) and Sen and Dunn (1971) of various aspects of the resulting field equations, have interesting consequences. These studies have shown that in a cosmology based on Lyra's geometry: the redshift of spectral lines from extragalactic nebulae arises as a consequence of an inherent geometrical property of the model universe (as in relativistic cosmology) but does so independently of the general expansion; the scalar–tensor theory of gravitation assigns an intrinsic geometrical significance to both scalar and tensor fields, in contrast to the well-known Brans–Dicke (1961) theory, in which the tensor field alone is geometrized and the scalar field remains alien to the geometry; and the principle of mass–energy conservation is violated. The sacrifice of this conservation law is an aspect of the theory requiring further investigation.

In the present paper we examine the scalar–tensor fields of Lyra's manifold for the spherically symmetric class-one cosmological model in the case of a perfect fluid. The solution obtained corresponds to Lemaitre's universe in Einstein's gravitational theory. It also shows the redshift but again at the cost of the mass–energy conservation law.

Field Equations

The flat metric in spherical polar coordinates is given by

$$\mathrm{d}s^2 = -\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}t^2\,. \tag{1}$$

The introduction of a gravitational disturbance function ψ, where ψ is a function of r and t only, converts the metric to the form

$$\mathrm{d}s^2 = -\mathrm{d}r^2 - r^2\,\mathrm{d}\theta^2 - r^2\sin^2\theta\,\mathrm{d}\phi^2 + \mathrm{d}t^2 - \{\mathrm{d}\psi(r,t)\}^2\,.$$

This is a spherically symmetric nonstatic line-element of class one (Singh and Pandey 1960; Tiwari 1971) and can be written as

$$ds^2 = -(1+\psi_1^2)\,dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + (1-\psi_4^2)\,dt^2 - 2\psi_1\psi_4\, dr dt, \tag{2}$$

where

$$\psi_1 = \partial\psi/\partial r, \qquad \psi_4 = \partial\psi/\partial t, \qquad \psi_{14} = \partial^2\psi/\partial r\partial t, \quad \text{etc.}$$

The field equations in normal gauge for Lyra's manifold as obtained by Sen (1957) are

$$R_{\mu\nu} - \tfrac{1}{2}g_{\mu\nu}R + \tfrac{3}{2}\varphi_\mu\varphi_\nu - \tfrac{3}{4}g_{\mu\nu}\varphi_\alpha\varphi^\alpha = -\kappa T_{\mu\nu}, \tag{3}$$

where φ_μ is a displacement field and the other symbols have their usual meanings as in Riemannian geometry. We now assume the vector displacement field φ_μ to be the time-like constant vector

$$\varphi_\mu = (0, \quad 0, \quad 0, \quad \beta = \text{const.}). \tag{4}$$

The nonvanishing components of the energy–momentum tensor in view of equations (3) and (4) are

$$-\kappa T_1^1 = -\psi_1^2/r^2S + \{(1+\psi_1^2)\psi_{44} - \psi_1\psi_4\psi_{14}\}2\psi_1/rS^2 - \tfrac{3}{4}\beta^2(1+\psi_1^2)/S, \tag{5a}$$

$$\begin{aligned} -\kappa T_2^2 &= -\kappa T_3^3 \\ &= -[\{(1-\psi_4^2)\psi_{11} - (1+\psi_1^2)\psi_{44} + 2\psi_1\psi_4\psi_{14}\}\psi_1/r - (\psi_{11}\psi_{44} - \psi_{14}^2)]/S^2 \\ &\quad -\tfrac{3}{4}\beta^2(1+\psi_1^2)/S, \end{aligned} \tag{5b}$$

$$-\kappa T_4^4 = -\psi_1^2/r^2S - \{(1-\psi_4^2)\psi_{11} + \psi_1\psi_4\psi_{14}\}2\psi_1/rS^2 + \tfrac{3}{4}\beta^2(1+\psi_1^2)/S, \tag{5c}$$

$$-\kappa T_1^4 = -\{(1+\psi_1^2)\psi_{14} - \psi_1\psi_4\psi_{11}\}2\psi_1/rS^2, \tag{5d}$$

$$-\kappa T_4^1 = \{(1-\psi_4^2)\psi_{14} + \psi_1\psi_4\psi_{44}\}2\psi_1/rS^2 - \tfrac{3}{2}\beta^2\psi_1\psi_4/S, \tag{5e}$$

where $S = 1+\psi_1^2-\psi_4^2$. The spherically symmetric class-one cosmological model may be viewed as an ideal fluid whose fundamental particles are identified with galaxies, and hence we have

$$\kappa T_\nu^\mu = (p+\rho)u^\mu u_\nu - \delta_\nu^\mu p, \tag{6}$$

where p is the pressure and ρ is the density.

Solution of Field Equations

Static Universe

In the static case, ψ is a function of r only. We then have from equation (6)

$$T_1^1 = T_2^2 = T_3^3 = -p, \qquad T_4^4 = \rho \qquad \text{and} \qquad T_\nu^\mu = 0 \quad \text{for} \quad \mu \neq \nu, \tag{7}$$

where the Greek indices run from 1 to 4 and the identification $x^1 = r$, $x^2 = \theta$, $x^3 = \phi$, $x^4 = ct$ has been made. From equations (5a)–(5e) and (7), the field equations reduce to

$$\kappa p = -\psi_1^2/r^2 S - \tfrac{3}{4}\beta^2, \tag{8a}$$

$$\kappa p = -\psi_1 \psi_{11}/rS^2 - \tfrac{3}{4}\beta^2, \tag{8b}$$

$$-\kappa \rho = -\psi_1^2/r^2 S - 2\psi_1 \psi_{11}/rS^2 + \tfrac{3}{4}\beta^2, \tag{8c}$$

where in the static case $S = 1+\psi_1^2$. The field equations (8a)–(8c) may be compared with those of normal relativistic cosmology based on Riemannian geometry

$$\kappa p = -\psi_1^2/r^2 S - \Lambda, \tag{9a}$$

$$\kappa p = -\psi_1 \psi_{11}/rS^2 - \Lambda, \tag{9b}$$

$$-\kappa \rho = -\psi_1^2/r^2 S - 2\psi_1 \psi_{11}/rS^2 - \Lambda. \tag{9c}$$

We observe that, apart from a difference of sign between the last terms on the right-hand side of equations (8c) and (9c), the two sets of equations are identical, with the number β^2, and therefore φ_μ, playing the role of the cosmological constant Λ.

The solutions of the field equations (8a)–(8c) are

$$\psi = -(X-r^2)^{\frac{1}{2}}, \qquad \kappa p = -Y - \tfrac{3}{4}\beta^2, \qquad \kappa \rho = 3Y - \tfrac{3}{4}\beta^2, \tag{10a, b, c}$$

where X and Y are arbitrary constants. The expressions (10b) and (10c) for the pressure and density yield

$$\beta^2 = -\kappa(\rho + 3p)/3. \tag{11}$$

Since the density and the pressure are positive quantities, we conclude at once that β^2 must be negative. This shows that, for the static case, the displacement vector φ_μ in Lyra's manifold is imaginary and thus this model is unrealistic physically. However, it may be remarked that Sen (1957) has derived a spectral shift for the static model by considering β to be imaginary.

Nonstatic Universe

In the nonstatic case, ψ is a function of r and t only. The eigenvalues of T^μ_ν are given by the determinantal equation

$$|T^\mu_\nu - \lambda \delta^\mu_\nu| = 0, \tag{12}$$

which in the present case reduces to

$$(T^2_2 - \lambda)(T^3_3 - \lambda)\{(T^1_1 - \lambda)(T^4_4 - \lambda) - T^4_1 T^1_4\} = 0. \tag{13}$$

In view of the relation (5b), two of the eigenvalues, λ_2 and λ_3, which from equation (13) are equal to T^2_2 and T^3_3 respectively, are also equal to each other. The other two eigenvalues are given by the quadratic equation

$$(T^1_1 - \lambda)(T^4_4 - \lambda) - T^4_1 T^1_4 = 0. \tag{14}$$

For a perfect fluid distribution, the three spatial eigenvalues λ_1, λ_2 and λ_3 are equal and hence, from equation (14), we obtain

$$(T_1^1 - T_2^2)(T_4^4 - T_2^2) - T_1^4 T_4^1 = 0. \tag{15}$$

The equation (15) is satisfied if we assume that

$$T_1^1 = T_2^2 \quad \text{and} \quad T_1^4 = 0. \tag{16}$$

The equations (5a)–(5e) and (16) lead to two differential equations in terms of ψ only:

$$(1+\psi_1^2)\psi_{14} - \psi_1 \psi_4 \psi_{11} = 0 \tag{17a}$$

and

$$(\psi_{11}\psi_{44} - \psi_{14}^2) - (\psi_1/r)\{(1-\psi_4^2)\psi_{11} + (1+\psi_1^2)\psi_{44} - (\psi_1/r)(1+\psi_1^2-\psi_4^2)\} = 0. \tag{17b}$$

Both the above differential equations admit the particular solution

$$\psi = (A^2 - r^2)^{\frac{1}{2}}, \tag{18}$$

where $A = A(t)$. In view of this solution, the metric (2) takes the form

$$\mathrm{d}s^2 = -\frac{\mathrm{d}r^2}{1-r^2/A^2} - r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + \frac{2A\dot{A}r}{A^2-r^2}\mathrm{d}r\mathrm{d}t + \frac{A^2-r^2-A^2\dot{A}^2}{A^2-r^2}\mathrm{d}t^2, \tag{19}$$

where an overhead dot indicates differentiation with respect to t. From the relation (6), which hold for a perfect fluid distribution, and the relations $g_{\mu\nu}u^\mu u^\nu = 1$, the flow vectors are

$$u^1 = \frac{r\dot{A}}{A(1-\dot{A}^2)^{\frac{1}{2}}}, \qquad u^2 = u^3 = 0 \qquad \text{and} \qquad u^4 = \frac{1}{(1-\dot{A}^2)^{\frac{1}{2}}}. \tag{20}$$

The pressure and density are given by

$$\kappa p = \frac{\dot{A}^2 - 1 - 2A\ddot{A}}{A^2(1-\dot{A}^2)^2} - \frac{\frac{3}{4}\beta^2}{1-\dot{A}^2} \qquad \text{and} \qquad \kappa\rho = \frac{3}{A^2(1-\dot{A}^2)} - \frac{\frac{3}{4}\beta^2}{1-\dot{A}^2}. \tag{21a, b}$$

We notice that the pressure and density are functions of time and, at a given value of t, are independent of position in the space. (If A is taken to be constant, the universe degenerates to the static case.) Since A is a function of time alone, we have the important consequence that β^2 need not be imaginary in order to obtain a physically viable model. According to Tolman (1966) the total energy density which directly corresponds to the mass of the nebulae is given by

$$\rho_m = \rho - 3p. \tag{22}$$

From equations (21a), (21b) and (22), the density of matter in the universe then has the approximate expression

$$\kappa\rho_m = 6(1-\dot{A}^2 + A\ddot{A})/A^2(1-\dot{A}^2)^2 - \tfrac{3}{2}\beta^2/(1-\dot{A}^2). \tag{23}$$

The line element (19) may be written in the form

$$ds^2 = -dr^2 - r^2(d\theta^2 + \sin^2\theta\, d\phi^2) + dt^2 - \{d(A^2 - r^2)^{\frac{1}{2}}\}^2 . \tag{24}$$

On substituting

$$z^1 = r\cos\phi\sin\theta, \qquad z^2 = r\sin\phi\sin\theta,$$

$$z^3 = r\cos\theta \quad \text{and} \quad z^4 = (A^2 - r^2)^{\frac{1}{2}},$$

in the line element (24), we obtain

$$ds^2 = -(dz^1)^2 - (dz^2)^2 - (dz^3)^2 - (dz^4)^2 + dt^2 ,$$

where

$$(z^1)^2 + (z^2)^2 + (z^3)^2 + (z^4)^2 = A^2 .$$

This gives the three-dimensional cross section of the V_4 at any time t as a sphere of radius A which varies with time. The transformation $r = A\sin\chi$ carries the line element (19) into

$$ds^2 = -A^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\} + (1 - \dot{A}^2)\, dt^2$$

which, by the further transformation $d\tau = (1 - \dot{A}^2)^{\frac{1}{2}}\, dt$, can be reduced to the usual form for Lemaitre's universe:

$$ds^2 = -A^2\{d\chi^2 + \sin^2\chi(d\theta^2 + \sin^2\theta\, d\phi^2)\} + d\tau^2 . \tag{25}$$

On differentiating equation (21b) with respect to t, we obtain

$$\kappa\dot{p} = -6\dot{A}(1 - \dot{A}^2 - A\ddot{A})/A^3(1 - \dot{A}^2)^2 - \tfrac{3}{2}\beta^2 \dot{A}\ddot{A}/(1 - \dot{A}^2)^2 .$$

Making use of both equations (21a) and (21b) in the above relation, we get

$$\frac{d}{dt}(\rho A^3) + p\frac{d}{dt}(A^3) = -h\left(\frac{3A^2\dot{A}}{1 - \dot{A}^2} + \frac{A^3\dot{A}\ddot{A}}{(1 - \dot{A}^2)^2}\right), \tag{26}$$

where $h\ (= 3\beta^2/2\kappa)$ is a constant. Considering an element of volume V, in the three-space defined by t = const., we may write $V = A^3$ and $M = \rho V$, where M is the mass of the volume V. The total energy E in V is then $E = Mc^2 = \rho A^3$, as c is unity, and so equation (26) may be written in the form

$$\frac{dE}{dt} + p\frac{dV}{dt} = -h\left(\frac{3A^2\dot{A}}{1 - \dot{A}^2} + \frac{A^3\dot{A}\ddot{A}}{(1 - \dot{A}^2)^2}\right). \tag{27}$$

Consequently the mass–energy conservation law does not hold in this cosmology (in contrast with the condition $dE + p dV = 0$, which holds in the Riemannian based geometry).

Spectral Shift

The equation for a geodesic in Lyra's geometry is

$$x^0\ddot{x}^\alpha+\Gamma^\alpha_{\mu\nu}x^0\dot{x}^\mu x^0\dot{x}^\nu-\tfrac{1}{2}(\varphi_\mu-\varphi^0_\mu)x^0\dot{x}^\mu x^0\dot{x}^\alpha = 0, \tag{28}$$

where

$$\varphi^0_\alpha = \frac{1}{x^0}\frac{\partial\log(x^0)^2}{\partial x^\alpha}.$$

We may choose the natural gauge $x^0 = 1$, as the gauge is entirely arbitrary, and then obtain (as did Sen 1957)

$$\ddot{x}^\alpha+\Gamma^\alpha_{\mu\nu}\dot{x}^\mu\dot{x}^\nu-\tfrac{1}{2}\varphi^\alpha g_{\mu\nu}\dot{x}^\mu\dot{x}^\nu = 0. \tag{29}$$

Except for the last term, the equations (29) are those of curves of extremal length. On letting $\alpha = 1,2,3,4$ for the line element (25), we obtain

$$\frac{d^2\chi}{ds^2}+\frac{2\dot{A}}{A}\frac{d\chi}{ds}\frac{d\tau}{ds}-\sin\chi\cos\chi\left(\frac{d\theta}{ds}\right)^2-\sin^2\theta\sin\chi\cos\chi\left(\frac{d\phi}{ds}\right)^2 = 0, \tag{30}$$

$$\frac{d^2\theta}{ds^2}+2\cot\chi\frac{d\chi}{ds}\frac{d\theta}{ds}-\sin\theta\cos\theta\left(\frac{d\phi}{ds}\right)^2+\frac{2\dot{A}}{A}\frac{d\theta}{ds}\frac{d\tau}{ds} = 0, \tag{31}$$

$$\frac{d^2\phi}{ds^2}+2\cot\chi\frac{d\chi}{ds}\frac{d\phi}{ds}+\frac{2\dot{A}}{A}\frac{d\phi}{ds}\frac{d\tau}{ds}+2\cot\theta\frac{d\theta}{ds}\frac{d\phi}{ds} = 0, \tag{32}$$

$$\frac{d^2\tau}{ds^2}+A\dot{A}\left(\frac{d\chi}{ds}\right)^2+\frac{2\dot{A}}{A}\frac{d\chi}{ds}\frac{d\tau}{ds}-\sin\chi\cos\chi\left(\frac{d\theta}{ds}\right)^2-\sin^2\theta\sin\chi\cos\chi\left(\frac{d\phi}{ds}\right)^2$$
$$-\tfrac{1}{2}\beta\left\{-A^2\left(\frac{d\chi}{ds}\right)^2-A^2\sin^2\chi\left(\frac{d\theta}{ds}\right)^2-A^2\sin^2\chi\sin^2\theta\left(\frac{d\phi}{ds}\right)^2+\left(\frac{d\tau}{ds}\right)^2\right\} = 0. \tag{33}$$

For a particle at rest, we have

$$\frac{d\chi}{ds} = \frac{d\theta}{ds} = \frac{d\phi}{ds} = 0,$$

and therefore

$$\frac{d^2\chi}{ds^2} = \frac{d^2\theta}{ds^2} = \frac{d^2\phi}{ds^2} = 0 \quad\text{and}\quad \frac{d^2\tau}{ds^2} = \tfrac{1}{2}\beta\left(\frac{d\tau}{ds}\right)^2. \tag{34}$$

Thus on integration we have

$$-q^{-1} = \tfrac{1}{2}\beta s+\text{const.},$$

where $q = d\tau/ds$. The initial point of measurement may be so chosen that $q = 1$ when $s = 0$, so that we have

$$(d\tau/ds)^2 = (1-\tfrac{1}{2}\beta s)^{-1}. \tag{35}$$

Equation (35) then shows that along a geodesic the proper-time interval $\delta\tau^0_1$ for an observer situated at the particle is related to the coordinate time interval $\delta\tau^0_2$, as measured by an observer at the origin, by

$$\delta\tau_2^0 = (1-\tfrac{1}{2}\beta s)^{-1}\delta\tau_1^0, \tag{36}$$

where s is the metric interval between the observer and the particle. Therefore the spectral shift in wavelength, as measured at the origin, would be

$$(\lambda+\delta\lambda)/\lambda = \delta\tau_2^0/\delta\tau_1^0 = (1-\tfrac{1}{2}\beta s)^{-1}. \tag{37}$$

The results obtained here may be contrasted with those of Sen (1957). In the present case, β in equation (37) is real whereas in the case considered by Sen (his equation (2.30)) it is imaginary.

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