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Unitarity and Regge Branch Points

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Abstract

We show how t-channel unitarity imposes constraints on the discontinuity associated with the leading Regge branch point. If the discontinuity across the leading Regge cut does not vanish at the end point it leads to the 'Gribov paradox'.

1. Introduction

A few years ago it was shown by Gribov (1961) that certain asymptotic forms for the total scattering amplitude $A(s,t)$ are inconsistent with the Mandelstam representation. The 'Gribov paradox' arises if $A(s,t)$, for large positive s and for t above the elastic threshold t_e but below the first inelastic threshold t_i, is bounded by s^N, where N is a real number (positive or negative). Alternatively the form obtained from a single Regge pole

$$A(s,t) \sim \beta(t)\, s^{\alpha(t)} \tag{1}$$

shows a very natural asymptotic behaviour which avoids the Gribov paradox. This is because, in the region of positive momentum transfer and above the elastic threshold, $\alpha(t)$ develops a branch point. In this region $\alpha(t)$ becomes complex and the imaginary part introduces oscillations in the amplitude

$$A(s,t) \sim s^{\alpha(t)} = \exp\{\alpha(t)\log s\} \tag{2}$$

which are described by the resulting phase $\Phi(s,t) = \operatorname{Im}\alpha(t)\log s$. This is just what is needed to avoid the paradox.

With the introduction of moving branch points (Mandelstam 1963) in the l plane, however, the situation changes considerably. The branch points in the l plane are generated by multireggeon exchanges. If $\alpha_c^n(t)$ is a branch point trajectory generated by n-reggeon exchange, we find that $\alpha_c^n(t)$ is real for $t_e < t < (2nm)^2$, while the pole trajectory $\alpha(t)$ is complex already in the elastic region. Therefore, when the moving branch point is the rightmost singularity in the l plane, the Regge amplitude for large s and $t_e < t < t_i$ is bounded by a real power of s.

Now, in the region of positive momentum transfer a branch point is in general not the leading singularity. This is because a branch point appears in the l plane as a consequence of multireggeon exchange, and the single-particle pole will be to the right of such a branch point (Gribov 1967). However, there are important physical situations in which the exchange of a single pole is forbidden by some conservation law and thus a cut becomes the rightmost singularity. For example, in double-charge or double-strangeness exchange scattering the branch points dominate the asymptotic behaviour of the scattering amplitude (Gribov 1967). Another interesting example is the process $\pi^{\pm}\,\mathrm{p} \rightarrow \mathrm{s}^{\pm}\,\mathrm{p}$ (where $\mathrm{s}^{\pm}$ denotes a scalar particle with $I^G J^P = 1^- 0^+$) in which the exchange of a single pomeranchon is forbidden by parity selection rules. Here the branch points generated by multivacuum exchange lie to the right of Regge poles even in that region of positive momentum transfer where elastic unitarity is valid.

In this paper we study the t-channel unitarity in the presence of moving branch points in the l plane. We show that unitarity as applied to the discontinuity across the elastic cut imposes constraints on the discontinuity across the cut associated with the rightmost branch point in the l plane. In Section 2 we discuss branch points with nonvanishing discontinuities at the end point and show that such branch points are inconsistent with t-channel unitarity. In Section 3 we study branch points whose discontinuities vanish at the end point and find that these branch points do not lead to contradiction with unitarity. Finally, in Section 4 we discuss some implications of our result.

2. Branch Points in the l Plane

Here we study a branch point as the leading singularity and examine its contribution to the asymptotic behaviour. We wish to find what type of branch point is compatible with t-channel unitarity. Our investigation is confined to equal mass pseudoscalar kinematics.

Let us first consider a logarithmic branch which has a constant discontinuity. Such branch points are of special interest because it has recently been shown by Gribov *et al.* (1965) that the sum of certain perturbation graphs leads to logarithmic branch points in the complex l plane. In particular, if n reggeons are exchanged in the t channel, the partial-wave amplitude $a(l,t)$ has a logarithmic branch point at $\alpha_n(t)$ and near it $a(l,t)$ can be written (Gribov *et al.*)

$$a(l,t) = A_n + B_n\{l-\alpha_n(t)\}^{n-2}\log\{l-\alpha_n(t)\}\,. \tag{3}$$

Here $\alpha_n(t)$, the trajectory for n-reggeon exchange, is given in terms of the single-pole trajectory $\alpha(t)$ by (Gribov 1961)

$$\alpha_n(t) = n\,\alpha(t/n^2) - n + 1\,. \tag{4}$$

We are interested in the contribution of these branch points to the asymptotic s-behaviour of the full amplitude $A(s,t)$ in a situation when the branch point is the rightmost singularity. In this case the exchange of a single pole must be forbidden by some conservation law. Then equation (4) shows that the rightmost singularity in the region of positive t is the branch point with $n = 2$. By carrying out the Sommerfeld–Watson transformation, one finds that this branch point will contribute to $A(s,t)$ a term of the form (Oheme 1965; Collins and Squires 1968)

$$A(s,t) = 8\pi \int^{\alpha_2(t)} (2l+1)g\,\frac{P_l(-z_t)\pm P_l(z_t)}{\sin \pi l}\,\mathrm{d}l\,, \tag{5}$$

which is dominated by the upper limit. Here g is the discontinuity across the logarithmic cut and, depending on the signature of the branch point (Mandelstam 1958), it will contribute to one or the other of the signatured amplitudes. Furthermore, if we define $A_s(s,t)$ as the absorptive part of $A(s,t)$ for fixed s then from equation (5) this discontinuity can be written in the form

$$A_s(s,t) = \{\phi(t) + \mathrm{i}\,\psi(t)\}\, z_t^{\alpha_2(t)}/\log z_t, \qquad \text{for } z_t \to \infty, \tag{6}$$

which shows the large dependence on $z_t = 1 + 2s/(t - 4m^2)$. Here $\phi(t)$, $\psi(t)$ and $\alpha_2(t)$ are real for a given $t < 16m^2$. The reality of α_2 follows from equation (4). The particular z-dependence of (6) has been obtained under the assumption $g(\alpha_2(t)) \neq 0$.

We now examine whether the asymptotic behaviour (6) is compatible with t-channel unitarity. For $4m^2 < t < 16m^2$ we have for the discontinuity across the elastic two-particle cut (Mandelstam 1958)

$$\operatorname{Im} A(z,t) = \frac{1}{4\pi}\frac{t-4}{t}\int \mathrm{d}z'\,\mathrm{d}z''\,\frac{A_s(z',t)\,A_s^*(z'',t)}{(z^2 - 2zz'z'' + z'^2 + z''^2 - 1)^{\frac{1}{2}}}, \tag{7}$$

with an integration performed over the domain

$$z > z'z'' + \{(z'^2+1)(z''^2+1)\}^{\frac{1}{2}}.$$

It should be noted at this stage that in the interval $4m^2 < t < 16m^2$ there may exist an anomalous branch point, for example, one generated by a three-particle intermediate state with any two particles forming a bound state (Mandelstam 1963). In these situations the right-hand side of equation (7) should be interpreted as the discontinuity across the elastic two-particle cut only. A similar procedure has been adopted elsewhere (Jones and Teplitz 1967; Oheme 1967; Joshi and Ramachandran 1968) in connection with the partial-wave elastic unitarity in the presence of moving branch points in the l plane. Following Gribov's (1961) method, we study the Mellin transform. Multiplying both sides of equation (7) by $z^{-(p+1)}$ and integrating over z (see Gribov (1961) for details) we obtain

$$\operatorname{Im}\psi(p) = \text{const.}\,|\psi(p)|^2, \tag{8a}$$

where

$$\psi(p) = \int z^{-(p+1)} A_s(z,t)\,\mathrm{d}z. \tag{8b}$$

Equations (7) and (8) imply, in the limit $p \to \alpha_2(t)$, that the integral

$$\int^{\infty} z^{-\alpha_2(t)} A_s(z,t)\, z^{-1}\,\mathrm{d}z < \infty, \tag{9}$$

i.e. it is finite at the upper limit. If we use $A_s(z,t)$ from equation (6), this requires

$$\int^{\infty} z^{-\alpha_2(t)}\{z^{\alpha_2(t)}/\log z\} z^{-1}\,\mathrm{d}z < \infty, \tag{10}$$

which is inconsistent since

$$\int^{\infty} \{z \log z\}^{-1}\,\mathrm{d}z$$

is logarithmically divergent.

We have therefore shown that a branch point with a constant discontinuity is inconsistent with t-channel unitarity. Our result is also valid, however, for all branch points whose discontinuities do not vanish at the end point, as this was the only essential assumption that was made in deriving equation (6).

3. Branch Points with Vanishing End-point Discontinuities

We now investigate branch points whose discontinuities vanish at the end point and show that such branch points allow the unitarity condition to be satisfied. Note that equation (3) implies that branch points with $n = 2$ belong in the class studied in Section 2. We therefore study next branch points arising from multireggeon exchange with $n > 2$. The partial-wave amplitude near such a branch point will be

$$a(l,t) = \{l-\alpha_n(t)\}^{n-2}\log\{l-\alpha_n(t)\}, \qquad n > 2. \tag{11}$$

After a Sommerfeld–Watson transformation this cut will contribute to the full amplitude as

$$A(s,t) = 8\pi \int^{\alpha_n(t)} (2l+1)\{l-\alpha_n(t)\}^{n-2} \frac{P_l(-z_t)\pm P_l(z_t)}{\sin \pi l}\, dl. \tag{12}$$

For large z_t we then have

$$A(s,t) = z^{\alpha_n(t)}/(\log z)^{n-1}, \qquad n > 2, \tag{13}$$

which is consistent with the condition (9).

A slightly more general example of a branch point with a vanishing end-point discontinuity is found if the amplitude behaves as

$$a(l,t) = c\{l-\alpha_c(t)\}^{\lambda} + \psi(l,t), \tag{14}$$

where c is a complex constant and $\psi(l,t)$ is locally analytic in l near $\alpha_c(t)$ and $\lambda > 0$. This branch point will enter the asymptotic form of the amplitude as

$$A(s,t) = 8\pi c \sin \pi\lambda \int^{\alpha_c(t)} (2l+1)\{l-\alpha_c(t)\}^{\lambda} \frac{P_l(z_t)\pm P_l(-z_t)}{\sin \pi l}\, dl. \tag{15}$$

For large z_t we have

$$A(s,t) = z_t^{\alpha_c(t)}/(\log z_t)^{\lambda+1}, \qquad \lambda > 0, \tag{16}$$

which again is compatible with the unitarity condition (9). This last example would cover a square-root type branch point in the l plane.

4. Discussion

We have shown above that if moving branch points dominate the asymptotic form of the scattering amplitude, in the region of positive momentum transfer but below the inelastic threshold, t-channel unitarity imposes a severe constraint on the discontinuity of the branch points. In particular it requires the discontinuity to vanish at the end point. Our results are in agreement with the finding of Bronzan and Jones (1967), who arrived at their conclusion by studying the partial-wave unitarity condition in the t-channel.

The present result can also be understood in terms of Froissart's (1962) argument. If the asymptotic form

$$A(x,t) \sim (\cos\theta_t)^{\alpha_c}(\log\cos\theta_t)^{\beta}$$

is valid for α_c real and $4m^2 < t < t_i$, then this behaviour leads to a singularity in $a(l,t)$ in the l plane of the form $|l-\alpha_c(t)|^{-(\beta+1)}$. Now, unitarity requires $a(l,t)$ to be bounded and therefore $\beta < -1$, which leads to a zero at the end point. Clearly these arguments hold only when α_c is real.

Finally, our results are compatible with the work of Creutz *et al.* (1973), who show that hard cuts are also consistent with the partial-wave unitarity. These authors demonstrate that under certain conditions hard cuts can exist when the momentum transfer is below the crossed-channel elastic threshold. However, for $4m^2 < t < 16m^2$ where our analysis is valid, these hard cuts become soft, i.e. they have vanishing discontinuities at the tip.

References

Bronzan, J. B., and Jones, C. E. (1967). *Phys. Rev.* **160,** 1494.
Collins, P. D. B., and Squires, E. J. (1968). 'Regge Poles in Particle Physics' (Springer–Verlag: Heidelberg).
Creutz, M., Page, F., and Wang, L. L. (1973). *Phys. Rev. Lett.* **30,** 343.
Froissart, M. (1962). 'Lectures in Theoretical Physics', Trieste, Italy (I.A.E.A.: Vienna).
Gribov, V. N. (1961). *Nucl. Phys.* **22,** 249.
Gribov, V. N. (1967). *Sov. J. Nucl. Phys.* **5,** 138.
Gribov, V. N., Pomeranchuk, I. Ya., and Ter-Martirosyan, K. A. (1965). *Phys. Rev.* **139,** 1354.
Jones, C. E., and Teplitz, V. L. (1967). *Phys. Rev.* **159,** 1271.
Joshi, G. C., and Ramachandran, R. (1968). *Phys. Rev.* **166,** 1832.
Mandelstam, S. (1958). *Phys. Rev.* **112,** 1344.
Mandelstam, S. (1963). *Nuovo Cimento* **30,** 113, 1127, 1148.
Oheme, R. (1965). *In* 'Strong Interaction and High Energy Physics' (Ed. R. G. Moorhouse), p. 129 (Oliver & Boyd: Edinburgh).
Oheme, R. (1967). *Phys. Rev. Lett.* **18,** 1222.

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