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Multiple Scattering Effects in a Three-particle System

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Abstract

We show that the phenomenological target wavefunction cutoff prescription arises naturally in the fixed scatterer approximation to particle scattering from a two-particle bound state. The wave-function modification is found to describe the effects of higher order multiple scattering.

Introduction

We study here the effects of multiple scattering in a simple model for three-particle scattering known as the fixed scatterer approximation (FSA; Brueckner 1953). The intention is to explore a possible general scheme which permits a simple description of strong multiple scattering effects, in three-particle systems. The description is by way of a smooth cutoff factor for the two-particle target bound-state wave-function, the cutoff being effective for small separation of the target particles. In this scheme the exact scattering amplitude is approximated by a few low order modified multiple scattering amplitudes. The modification is to use a cutoff target wavefunction to incorporate the effects of all higher order multiple scattering.

The general formulation of three-particle scattering theory by Faddeev (1961) has been followed, in the last decade, by many intricate numerical solutions for various systems. However, the analytical study of the Faddeev integral equation, in order to understand general three-particle dynamics, has been slow to develop, owing to the coupled and singular nature of the Faddeev integral equation. Yet, despite the dynamical complexity of the three-particle system, some reaction processes are capable of a simple though phenomenological description; for example, the quasi-free scattering region of the reaction $\mathrm{n+d \rightarrow n+n+p}$. The positions of the quasi-free scattering peaks kinematically correspond to one target nucleon undergoing no change in momentum. This suggests a single nucleon–nucleon scattering approximation, which then describes the peak shape by the deuteron momentum distribution $|\phi(q)|^2$, where $\phi(q)$ is the deuteron wavefunction; its Fourier transform $\psi(r)$ is the coordinate wavefunction. However, in order to successfully describe the experimental data this model has been phenomenologically modified, for example by Paic *et al.* (1970), by introducing a cutoff factor $D(r)$. Then the effective wavefunction is $D(r)\psi(r)$. Typically the very simple form used is $D(r) = 0$ for $0 < r < R$, and $D(r)$ set equal to some (normalization) constant for $r > R$, where R is some cutoff radius. This phenomenological procedure assumes that quasi-free scattering is a peripheral single scattering process, and that the effect of higher order multiple scattering processes is to suppress scattering to the quasi-free scattering region when the target particles have a separation smaller than R. A similar cutoff description is perhaps applicable in n–d elastic scattering, as suggested by the diffraction-type differential cross sections. Cahill (1974*a*), in analysing quasi-free scattering, has shown by summing parts of all multiple scattering amplitudes from the Faddeev equations that one can obtain a smooth cutoff factor $D(r)$. This cutoff effect also plays a part in the description of final-state interactions (Cahill 1974*b*).

We investigate the above cutoff description by considering a very simple model, in which the target particles are fixed during the scattering (Brueckner 1953; Foldy and Walecka 1969). The complete amplitude is then found by averaging over the target particle positions, with $|\psi(r)|^2$ as a weighting factor. An ideal application of the FSA is to π–d scattering (Brueckner 1953; Kudryavstev 1972), since it appears to be appropriate to high energy and near forward scattering. However, Kowalski and Peiper (1971) have suggested that the approximation may even be applicable to quite low energy n–d scattering. By using separable interactions, an exact solution may be obtained to the FSA (Foldy and Walecka 1969), and we find that the cutoff description occurs naturally.

Faddeev-type Formulation of FSA

The scattering of a single particle by two fixed target particles is properly described by the Lippmann–Schwinger equation

$$T(E) = (V_1+V_2)+(V_1+V_2)\,G_0(E)\,T(E), \tag{1}$$

where V_i is the interaction between the projectile and the ith target particle fixed at position $\boldsymbol{r}_i$, and

$$G_0(E) = (E-H_0)^{-1} \tag{2}$$

is the usual single-particle propagator. Hence we ignore recoil of the target particles and their mutual interaction. Let us recast the equation into the Faddeev type by defining

$$T_i = V_i + V_i\,G_0\,T, \tag{3}$$

so that

$$T = \sum T_i.$$

Introduce the scattering operator t_i, where

$$t_i = V_i + V_i\,G_0\,t_i, \tag{4}$$

so that t_i describes scattering from the ith target particle only. Then from equations (3) and (4) we obtain the coupled integral equation for T_1 and T_2

$$T_i = t_i + t_i\,G_0 \sum_{j\neq i} T_j. \tag{5}$$

This has the same form as the Faddeev equations when two of the particles do not interact. Of course, a fundamental mathematical difference is that the operators here act in single-particle space, whereas the Faddeev operators act in three-particle space. Nevertheless the physical content is very similar.

Let us assume, for simplicity only, that the target particles are identical. If we define t as the solution of equation (4) when the target particle is fixed at the origin, we have

$$t = V + VG_0 t. \tag{6}$$

Then (Messiah 1964)

$$\langle \boldsymbol{k}' \,|\, t_j(E) \,|\, \boldsymbol{k} \rangle = \exp\{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}') \cdot \boldsymbol{r}_j\} \langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k} \rangle, \tag{7}$$

since the jth target particle is located at $\boldsymbol{r}_j$ relative to the origin. We use $\boldsymbol{r}_1 = -\boldsymbol{r}_2 = \frac{1}{2}\boldsymbol{r}$, where $\boldsymbol{r}$ is the separation of the target particles. The complete scattering amplitude is then, in the FSA,

$$\mathscr{T}(\boldsymbol{k}', \boldsymbol{k}; E) = \int \mathrm{d}^3 r \, \langle \boldsymbol{k}' \,|\, T(E, \boldsymbol{r}) \,|\, \boldsymbol{k} \rangle \, |\, \psi(\boldsymbol{r}) \,|^2 . \tag{8}$$

Here $|\, \psi(\boldsymbol{r}) \,|^2$ is the probability density for finding the target particles with separation $\boldsymbol{r}$.

In matrix form, equation (5) is

$$\begin{aligned}\langle \boldsymbol{k}' \,|\, T_i(E, \boldsymbol{r}) \,|\, \boldsymbol{k} \rangle &= \exp\{\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}') \cdot \boldsymbol{r}_i\} \langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k} \rangle \\ &\quad + \sum_{j \neq i} \int \mathrm{d}^3 k'' \, \frac{\exp\{\mathrm{i}(\boldsymbol{k}''-\boldsymbol{k}') \cdot \boldsymbol{r}_i\} \langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}'' \rangle}{E + \mathrm{i}\varepsilon - E(k'')} \langle \boldsymbol{k}'' \,|\, T_j(E, \boldsymbol{r}) \,|\, \boldsymbol{k} \rangle, \end{aligned} \tag{9}$$

with $E(k'') = \hbar^2 k''^2/2m$ and $\varepsilon = 0^+$. With identical target particles, equation (9) gives the obvious result

$$\langle \boldsymbol{k}' \,|\, T_1(E, \boldsymbol{r}) \,|\, \boldsymbol{k} \rangle = \langle \boldsymbol{k}' \,|\, T_2(E, -\boldsymbol{r}) \,|\, \boldsymbol{k} \rangle, \tag{10}$$

which allows equation (9) to be written as an uncoupled integral equation. To exhibit further the similarity between the FSA and the more general Faddeev formalism, we generalize equation (8) by defining the amplitude

$$\mathscr{T}(\boldsymbol{k}', \boldsymbol{k}, \boldsymbol{K}; E) = \int \mathrm{d}^3 r \, \langle \boldsymbol{k}' \,|\, T(E, \boldsymbol{r}) \,|\, \boldsymbol{k} \rangle \exp(\mathrm{i}\boldsymbol{K} \cdot \boldsymbol{r}) \, |\, \psi(r) \,|^2 . \tag{11}$$

Equation (9) then gives an integral equation, similar to the Faddeev equations, for the complete FSA amplitude:

$$\begin{aligned}\mathscr{T}(\boldsymbol{k}', \boldsymbol{k}, \boldsymbol{K}; E) &= 2S(\boldsymbol{K} - \tfrac{1}{2}\boldsymbol{k}' + \tfrac{1}{2}\boldsymbol{k}) \langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k} \rangle \\ &\quad + \int \mathrm{d}^3 k'' \, \frac{\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}'' \rangle}{E + \mathrm{i}\varepsilon - E(k'')} \, \mathscr{T}(\boldsymbol{k}'', \boldsymbol{k}, \tfrac{1}{2}\boldsymbol{k}' - \tfrac{1}{2}\boldsymbol{k} - \boldsymbol{K}; E). \end{aligned} \tag{12}$$

The physical amplitude is $\mathscr{T}(\boldsymbol{k}', \boldsymbol{k}, 0; E)$, with $E(k') = E(k) = E$, and

$$S(q) = \int \mathrm{d}^3 r \exp(\mathrm{i}\boldsymbol{q} \cdot \boldsymbol{r}) \, |\, \psi(r) \,|^2 = \int \mathrm{d}^3 q \, \phi(\boldsymbol{q}') \, \phi(\boldsymbol{q}' - \boldsymbol{q}) \tag{13}$$

is the target form factor.

We note that the amplitude $\mathscr{T}(\boldsymbol{k}', \boldsymbol{k}; E)$ is non-unitary, which is a defect of the FSA. This is seen by noting that $\langle \boldsymbol{k}' \,|\, T(E, \boldsymbol{r}) \,|\, \boldsymbol{k} \rangle$ is a unitary amplitude, but that the averaging process in equation (8) does not preserve this unitarity. However this deficiency can be overcome to some extent by defining the 'channel' amplitudes

$$\mathscr{T}_{mn}(\boldsymbol{k}',\boldsymbol{k};E) = \int \mathrm{d}^3r\, \psi_m^*(\boldsymbol{r})\langle \boldsymbol{k}' \,|\, T(E,\boldsymbol{r}) \,|\, \boldsymbol{k}\rangle\, \psi_n(\boldsymbol{r}).$$

Then $\mathscr{T}_{mn}$ satisfies multichannel unitarity if the set $\{\psi_m(\boldsymbol{r})\}$ is complete (Kowalski and Peiper 1971). Hence $\mathscr{T}$ as defined in equation (8) satisfies the inelastic unitarity inequality

$$\operatorname{Im}\mathscr{T}(\boldsymbol{k},\boldsymbol{k}) > \rho(k) \int_{k'=k} \mathrm{d}\hat{k}' \,|\, \mathscr{T}(\boldsymbol{k}',\boldsymbol{k}) \,|^2. \tag{14}$$

For comparison, with the exact solution, we write the physical single plus double scattering amplitude, obtained from equation (12) by iteration and with $K = 0$, as

$$\begin{aligned}\mathscr{T}^{(2)}(\boldsymbol{k}',\boldsymbol{k};E) &= 2S\big(\tfrac{1}{2}(\boldsymbol{k}'-\boldsymbol{k})\big)\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}\rangle \\ &\quad +2\int \mathrm{d}^3k''\, \frac{S\big(\boldsymbol{k}''-\tfrac{1}{2}(\boldsymbol{k}'+\boldsymbol{k})\big)\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}''\rangle\langle \boldsymbol{k}'' \,|\, t(E) \,|\, \boldsymbol{k}\rangle}{E+\mathrm{i}\varepsilon-E(k'')}.\end{aligned} \tag{15}$$

We solve equations (9) or (12) analytically for the case when the two-particle t-matrix is approximated by the rank-one separable and S-wave form

$$\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}\rangle = g(k')\,\tau(E)\,g(k). \tag{16}$$

Equation (9) then reduces to an algebraic equation, the solution of which may be written in the form, valid only when equation (16) holds,

$$\begin{aligned}\langle \boldsymbol{k}' \,|\, T(E,\boldsymbol{r}) \,|\, \boldsymbol{k}\rangle &= \exp\{\tfrac{1}{2}\mathrm{i}(\boldsymbol{k}-\boldsymbol{k}')\,.\,\boldsymbol{r}\}\, D^2(E,r)\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}\rangle \\ &\quad +D^2(E,r)\int \mathrm{d}^3k''\, \frac{\exp\{\mathrm{i}(\boldsymbol{k}''-\tfrac{1}{2}\boldsymbol{k}'-\tfrac{1}{2}\boldsymbol{k})\,.\,\boldsymbol{r}\}\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}''\rangle\langle \boldsymbol{k}'' \,|\, t(E) \,|\, \boldsymbol{k}\rangle}{E+\mathrm{i}\varepsilon-E(k'')},\end{aligned} \tag{17}$$

where

$$D^2(E,r) = \{1-B^2(E,r)\}^{-1} \tag{18}$$

and

$$B(E,r) = \int \mathrm{d}^3k''\, \frac{\exp(\mathrm{i}\boldsymbol{k}''\,.\,\boldsymbol{r})\langle \boldsymbol{k}'' \,|\, t(E) \,|\, \boldsymbol{k}''\rangle}{E+\mathrm{i}\varepsilon-E(k'')}. \tag{19}$$

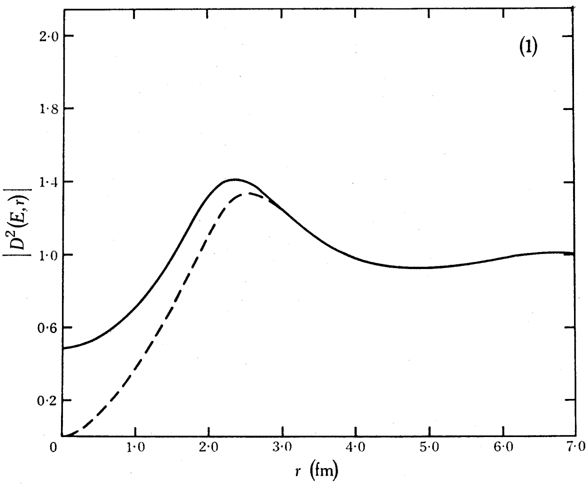
The physical solution of equation (12) then follows from equations (8) or (11) and (17),

$$\begin{aligned}\mathscr{T}(\boldsymbol{k}',\boldsymbol{k};E) &= 2S_{\mathrm{m}}\big(\tfrac{1}{2}(\boldsymbol{k}'-\boldsymbol{k})\big)\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}\rangle \\ &\quad +2\int \mathrm{d}^3k''\, S_{\mathrm{m}}(\boldsymbol{k}''-\tfrac{1}{2}\boldsymbol{k}'-\tfrac{1}{2}\boldsymbol{k})\frac{\langle \boldsymbol{k}' \,|\, t(E) \,|\, \boldsymbol{k}''\rangle\langle \boldsymbol{k}'' \,|\, t(E) \,|\, \boldsymbol{k}\rangle}{E+\mathrm{i}\varepsilon-E(k'')}.\end{aligned} \tag{20}$$

The exact solution (20) differs from the single plus double scattering amplitude (15) only by the presence of the modified target form factor

$$S_{\mathrm{m}}(q) = \int \mathrm{d}^3r\, \exp(\mathrm{i}\boldsymbol{q}\,.\,\boldsymbol{r})\, D^2(E,r) \,|\, \psi(r) \,|^2. \tag{21}$$

Thus $D^2(E,r)$ acts like a modification factor for the target probability density, albeit complex. Here, within the context of the FSA and separable interaction approximation, $D^2(E,r)$ describes exactly the effect of all the higher order multiple scatterings.



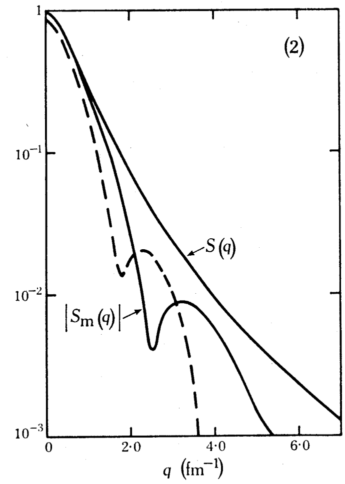


Fig. 1 (*left*). Form of the function $|D^2(E,r)|$ showing the cutoff effect for small r. The dashed curve is $|D^2(E,r)|$ calculated with only on-shell contributions included.

Fig. 2 (*right*). Normal target form factor $S(q)$ and the modified form factor $|S_m(q)|$. The dashed curve is $|S_m(q)|$ calculated with only on-shell contributions included.

Cutoff Effect

We now determine if $D^2(E,r)$ produces a cutoff effect for small r. From equation (19), $B(E,r)$ depends on the off-shell as well as the on-shell behaviour of the two-particle amplitude. To evaluate $B(E,r)$, we choose the Yamaguchi (1954) form for the two-particle t-matrix, with

$$g(q) = (q^2+\beta^2)^{-1} \qquad \text{and} \qquad \tau(E) = -\frac{\lambda\hbar^2}{2m}\left(1+\frac{\beta+\alpha}{k+\mathrm{i}\beta}\right)^{-2}.$$

Here $E = \hbar^2k^2/2m$, $\lambda = \beta(\beta+\alpha)^2/\pi^2$, and k is the on-shell wave number. Equation (19) then gives

$$B(E,r) = -\frac{4\pi^2 m}{\hbar^2}\langle k\,|\,t(E)\,|\,k\rangle\frac{\exp(\mathrm{i}kr)}{r} + \frac{4\pi^2 m}{\hbar^2}\tau(E)\frac{\exp(-\beta r)}{r}\left(g(k)+\frac{r}{2\beta}\right)g(k). \quad (22)$$

The first term comes solely from the on-shell part of $\langle k''\,|\,t(E)\,|\,k''\rangle$ in equation (19), while the second term arises solely from the off-shell behaviour. We see immediately that the off-shell behaviour is important only when the separation r of the target particles is comparable with the range of the potential $\sim\beta^{-1}$. Also, in general $|B(E,r)|$ increases with decreasing r, and in particular $B(E,r)$ is finite at the origin. However, if we neglect the off-shell behaviour, by neglecting the second term in equation (22), then $|B(E,r)|$ is unbounded as $r \to 0$, and $D(E,r) \to 0$ as $r \to 0$; that is, there is a complete cutoff as $r \to 0$. But with retention of off-shell effects the cutoff effect is weakened.

Fig. 1 shows the function $|D^2(E,r)|$ both with and without retention of the off-shell contributions, when $E = 10$ MeV and with α and β values chosen to fit low energy nucleon–nucleon $^3S_1$ data ($\alpha = 0\cdot232\ \mathrm{fm}^{-1}$, $\beta = 1\cdot406\ \mathrm{fm}^{-1}$, m = nucleon mass). We see from this figure that $|D^2(E,r)|$ produces, as well as the cutoff effect, an enhancement effect for $r \approx 2\cdot3$ fm. An enhancement occurs in the FSA when $B(E,r) \approx \pm 1$ and is a resonance phenomenon. Its occurrence depends on the detailed nature of the two-particle interaction. For large r, $|D^2(E,r)| \to 1$.

Fig. 2 shows the form factor $S(q)$ and also the modified form factor $|S_m(q)|$ calculated both with and without off-shell contributions to $B(E,r)$. For simplicity we have used the Hulthen S-wave deuteron wavefunction for $\psi(r)$,

$$\psi(r) = Nr^{-1}\{\exp(-\alpha r) - \exp(-\beta r)\}.$$

We see that $S_m(q)$ acquires structure due to $D^2(E,r)$.

There is an interesting connection between the strength of the cutoff effect and the convergence of the multiple scattering series. The latter series expansion of the amplitudes in equations (17) or (20) is obtained by expanding $D^2(E,r)$ in powers of $B^2(E,r)$ as

$$D^2(E,r) = 1 + B^2 + B^4 + \dots.$$

This series diverges for $|B^2| \geqslant 1$. The cutoff effect is very pronounced for a given r value when $|B^2(E,r)|$ is large compared with unity. Then the multiple scattering series must diverge for that separation r. A similar effect is seen in a proper three-particle treatment of the n–d system. There the Cahill (1974*a*) analysis of the cutoff effect shows that the strong cutoff, for small target separation, is determined by the first few angular momentum amplitudes. But in this system, at moderate energies, the multiple scattering series for these angular momentum states is divergent (Sloan 1969).

Conclusions

The much studied fixed scatterer approximation has again provided valuable insight into dynamical aspects of few-particle scattering. We have found in this model that the cutoff target wavefunction description is indeed a valid and useful interpretation. It also provides an understanding of the basis of the phenomenological cutoff prescription.

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