Aust. J. Phys., 1974, 27, 859–63

A Radiating Charged Particle in an Expanding Universe

L. K. Patel[A] and H. C. Shukla[B]

[A] Department of Mathematics, University School of Sciences, Gujarat University, Ahmedabad 380009, India.
[B] Department of Mathematics, M.G. Science Institute, Ahmedabad 380009, India.

Abstract

The field of a radiating charged particle embedded in an expanding universe is discussed and approximate solutions of the field equations are obtained.

Introduction

McVittie (1933) obtained a generalization of Schwarzschild's exterior solution in isotropic coordinates which represented the field of a mass particle in an expanding universe. The geometry of McVittie's solution is described by the line element

$$\mathrm{d}s^2 = \left(\frac{1-\mu(t)/2r}{1+\mu(t)/2r}\right)^2 \mathrm{d}t^2 - \{1+\mu(t)/2r\}^4 \exp(\beta(t))\{\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2)\}, \tag{1}$$

with $\dot{\beta} = -2\dot{\mu}/\mu$, where both here and subsequently an overhead dot indicates differentiation with respect to t. The space around this mass particle is not empty but is occupied by a distribution of matter of nonzero density and isotropic pressure. Vaidya and Shah (1957) obtained a generalization of this solution which corresponded to the field of a radiating mass particle embedded in an expanding universe. The object of the present investigation is to obtain a solution representing the field of a radiating charged particle in an expanding universe. The solution obtained here is shown to reduce, when the electromagnetic field is switched off, to the field obtained by Vaidya and Shah for a radiating mass particle in an expanding universe.

Field Equations

Bonnor and Vaidya (1970) discussed the metric form

$$\mathrm{d}s^2 = -r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + (1+V)\,\mathrm{d}t^2 - 2\,\mathrm{d}r\,\mathrm{d}t \tag{2}$$

in connection with spherically symmetric radiation by a charged particle on the Einstein–Maxwell theory. In this metric, V is a function of r and t given by

$$V = -2mr^{-1} + 4\pi q^2 r^{-2}, \tag{3}$$

where m and q are arbitrary functions of t.

In order to consider the field of a radiating charged particle in an expanding universe, we take the space–time conformal to that described by the metric (2) and choose a line element of the form

$$\mathrm{d}s^2 = \exp(\omega)\{-r^2(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + (1+V)\,\mathrm{d}t^2 - 2\,\mathrm{d}r\,\mathrm{d}t\}, \tag{4}$$

where here V is an as-yet-undetermined function of r and t, and ω is a function of $x = r-t$. The nonvanishing components of the Einstein mixed tensor G_k^i for the metric (4) are

$$G_1^1 \exp\omega = (1+V)\left(\tfrac{3}{4}\omega^{*2} + \frac{2\omega^*}{r} + \frac{\omega^* V'}{2(1+V)} + \frac{V'}{r(1+V)} + \frac{1}{r^2}\right) - \omega^{**} - \omega^{*2} - 2\omega^* r^{-1} - r^{-2}, \tag{5a}$$

$$G_2^2 \exp\omega = G_3^3 \exp\omega = (1+V)\left(\omega^{**} + \tfrac{1}{4}\omega^{*2} + \frac{\omega^*}{r} + \frac{\omega^* V'}{1+V} + \frac{V''}{2(1+V)} + \frac{V'}{r(1+V)}\right) - 2\omega^{**} - \tfrac{1}{2}\omega^{*2} - \omega^* r^{-1}, \tag{5b}$$

$$G_4^4 \exp\omega = (1+V)\left(\omega^{**} + \tfrac{1}{4}\omega^{*2} + \frac{2\omega^*}{r} + \frac{\omega^* V'}{2(1+V)} + \frac{V'}{r(1+V)} + \frac{1}{r^2}\right) - \omega^{**} - \omega^{*2} - 2\omega^* r^{-1} - r^{-2}, \tag{5c}$$

$$-G_4^1 \exp\omega = (1+V)\left(-\omega^{**} + \tfrac{1}{2}\omega^{*2} + \frac{\omega^* V'}{2(1+V)} + \frac{\omega^* \dot V}{2(1+V)} + \frac{\dot V}{r(1+V)}\right) + \omega^{**} - \tfrac{1}{2}\omega^{*2}, \tag{5d}$$

where the indices correspond to the coordinates

$$x^1 = r, \qquad x^2 = \theta, \qquad x^3 = \phi, \qquad x^4 = t. \tag{6}$$

In equations (5) and subsequently, an asterisk and a prime are used to denote differentiation with respect to x and r respectively.

We take the space surrounding the charged particle to be occupied by radially flowing radiation of density σ together with a spherically symmetric matter distribution of nonzero density ρ and pressure p. The energy–momentum tensor for such a distribution can be taken to be

$$T_k^i = (p+\rho)\, v^i v_k - p\delta_k^i + \sigma\xi^i \xi_k + E_k^i, \tag{7}$$

with

$$v^i v_i = 1 \qquad \text{and} \qquad \xi^i \xi_i = 0, \tag{8a, b}$$

and with

$$E_k^i = -F^{ia} F_{ka} + \tfrac{1}{4}\delta_k^i F_{ab} F^{ab}, \tag{9}$$

where F_{ik} is the antisymmetric electromagnetic field tensor satisfying

$$F_{ik} = A_{i,k} - A_{k,i} \qquad \text{and} \qquad (-g)^{-\frac{1}{2}}\{(-g)^{\frac{1}{2}} F^{ik}\}_{,k} = J^i. \tag{10a, b}$$

In equation (10a), A_i is the four-potential, J^i is the four-current vector and a comma preceding an index indicates partial differentiation with respect to the coordinate of that index in the equations (6). For the present problem we take the four-potential to be

$$A_i = r^{-1} q\,\delta_i^4, \tag{11}$$

where q is an arbitrary function of t. The resulting nonzero components of E^i_j and J^i are

$$E^1_1 = -E^2_2 = -E^3_3 = E^4_4 = \tfrac{1}{2}r^{-4}q^2\exp(-2\omega) \tag{12}$$

and

$$J^i = -r^{-2}\dot{q}\exp(-2\omega)\,\delta^i_1\,. \tag{13}$$

It is clear from equation (13) that the current is radial and null, that is, $g_{ik}J^iJ^k = 0$.

For the line element (4) we can take

$$v^1 = v^2 = v^3 = 0 \qquad \text{and} \qquad \xi^2 = \xi^3 = \xi^4 = 0\,. \tag{14a, b}$$

From equations (7), (8), (12) and (14), we have

$$\left.\begin{aligned} T^1_1 &= -p + \tfrac{1}{2}r^{-4}q^2\exp(-2\omega)\,, &\quad T^2_2 = T^3_3 &= -p - \tfrac{1}{2}r^{-4}q^2\exp(-2\omega)\,, \\ T^4_4 &= \rho + \tfrac{1}{2}r^{-4}q^2\exp(-2\omega)\,, &\quad T^1_4 &= -\sigma(\xi^1)^2\exp\omega\,. \end{aligned}\right\} \tag{15}$$

The field equations $G^i_k = -8\pi T^i_k$ then lead to

$$-\tfrac{1}{2}V'' + r^{-2}V = \tfrac{1}{2}V'\omega^* + V(\omega^{**} - \tfrac{1}{2}\omega^{*2} - r^{-1}\omega^*) - 8\pi q^2 r^{-4}\exp(-\omega)\,, \tag{16a}$$

$$8\pi p = G^1_1 + 4\pi q^2 r^{-4}\exp(-2\omega)\,, \tag{16b}$$

$$8\pi\rho = -G^4_4 - 4\pi q^2 r^{-4}\exp(-2\omega)\,, \tag{16c}$$

$$8\pi\sigma = (\xi^1)^{-2}G^1_4\exp(-\omega)\,. \tag{16d}$$

It should be noted that, when $q = 0$, this field reduces to that discussed by Vaidya and Shah (1957). In the following section we obtain approximate solutions to the field equations (16) by the method developed by Vaidya and Shah.

Approximate Solutions of Field Equations

Zeroth Approximation

In the zeroth approximation we put $\omega = 0$. Equation (16a) then becomes

$$-\tfrac{1}{2}V'' + r^{-2}V = -8\pi q^2 r^{-4}\,, \tag{17}$$

which has the solution

$$V = -2mr^{-1} + Ar^2 + 4\pi q^2 r^{-2}\,, \tag{18}$$

where m and A are arbitrary functions of t. We avoid a singularity at $r = \infty$, by taking $A = 0$. Equation (18) then reduces to the same form as equation (3) and the remaining field equations (16b)–(16d) yield

$$p = \rho = 0\,, \qquad 4\pi\sigma = (\xi^1)^{-2}(\dot{m} - 4\pi q\dot{q}r^{-1})r^{-2}\,, \qquad J^i = -r^{-2}\dot{q}\,\delta^i_1\,, \tag{19a, b, c}$$

which is the solution of Bonnor and Vaidya (1970). Moreover, if q = const. we obtain the field of a radiating particle discussed by Sheth (1970), and if $q = 0$ we obtain the isolated shining star solution of Vaidya (1953).

First Approximation

In the first approximation, we neglect the terms in ω^2, ω^{*2}, $\omega\omega^*$ and ω^{**} on the right-hand side of equation (16a). We note, of course, that the assumption that the above quantities are of the same order of smallness is a purely mathematical device. After substitution from equation (3), the field equation (16a) then becomes

$$-\tfrac{1}{2}V'' + r^{-2}V = 3m\omega^* r^{-2} - 8\pi q^2 r^{-3}\{\omega^* + (1-\omega)r^{-1}\}, \tag{20}$$

which has the solution

$$V = -2mr^{-1} + 3m\omega^* + 4\pi q^2(1-\omega)r^{-2}. \tag{21}$$

The remaining field equations (16b)–(16d) now give

$$p = \rho = 0, \qquad J^i = -r^{-2}\dot{q}\exp(-2\omega)\,\delta^i_1, \tag{22a, b}$$

$$8\pi\sigma = (\xi^1)^{-2}\exp(-2\omega)\{-2\dot{m}\omega^* r^{-1} + (2\dot{m} - m\omega^*)r^{-2} - 4\pi q\dot{q}(r\omega^* + 2 - 2\omega)r^{-3}\}. \tag{22c}$$

If we now impose the restrictions

$$2\dot{m}/m = \omega^* \qquad \text{and} \qquad q = \text{const.}, \tag{23a, b}$$

equations (22b) and (22c) give $J^i = 0$ and $\sigma = 0$. Hence the first-approximation solution, with the restrictions (23), describes the field of a charged particle in an expanding universe. The electric field of a point charge, as given by this solution, vanishes at large distances from the charge. Moreover, if we let $q = 0$, we recover McVittie's (1933) solution for a mass particle in an expanding universe.

Second Approximation

In the second approximation, we neglect the terms ω^3, ω^{*3}, $\omega\omega^{**}$, $\omega\omega^{*2}$, $\omega^*\omega^{**}$ etc. The field equation (16a) then becomes, upon substitution of the expressions (21) and (3) for V in the coefficients of ω^* and of ω^{*2} and ω^{**} respectively,

$$\begin{aligned}-\tfrac{1}{2}V'' + r^{-2}V = {} & 3m\omega^* r^{-2} - 2m(\omega^{**} + \omega^{*2})r^{-1} \\ & + 4\pi q^2\{\omega^{**} - \omega^{*2} + 2\omega^*(\omega - 1)r^{-1} - (\omega^2 - 2\omega + 2)r^{-2}\}r^{-2}.\end{aligned} \tag{24}$$

This equation has the solution

$$V = -2mr^{-1} + 3m\omega^* - 2mr(\omega^{**} + \omega^{*2}) + 2\pi q^2\{(\omega^2 - 2\omega + 2)r^{-2} + (\omega^{**} - \omega^{*2})\}. \tag{25}$$

The remaining field equations (16b)–(16d) then give

$$8\pi p = \exp(-\omega)\{-\tfrac{1}{4}\omega^{*2}(1 - 2mr^{-1}) - \omega^{**}(1 + mr^{-1}) + \pi q^2(2\omega^{**} - \omega^{*2})r^{-2}\}, \tag{26a}$$

$$8\pi\rho = \exp(-\omega)\{\tfrac{3}{4}\omega^{*2}(1 - 2mr^{-1}) + 3m\omega^{**}r^{-1} - 3\pi q^2(2\omega^{**} - \omega^{*2})r^{-2}\}, \tag{26b}$$

$$\begin{aligned}8\pi\sigma = {} & (\xi^1)^{-2}\exp(-2\omega) \\ & \times [2\dot{m}r^{-2} - \omega^* r^{-1}(2\dot{m} + mr^{-1}) + \omega^{*2}(\tfrac{1}{2}\dot{m} + mr^{-1}) + \omega^{**}(2\dot{m} + mr^{-1}) \\ & + 2\pi q^2(2\omega^{**} - \omega^{*2})r^{-2} + 4\pi q\dot{q}\{\omega\omega^* - \omega^* - (\omega^2 - 2\omega + 2)r^{-1}\}r^{-2}].\end{aligned} \tag{26c}$$

However, if we let q = const., we obtain $J^i = 0$ and

$$8\pi\sigma = (\xi^1)^{-2}\exp(-2\omega)\{2\dot{m}r^{-2} - \omega^*(2\dot{m}+mr^{-1})r^{-1} + \omega^{*2}(\tfrac{1}{2}\dot{m}+mr^{-1}) + \omega^{**}(2\dot{m}+mr^{-1}) + 2\pi q^2(2\omega^{**}-\omega^{*2})r^{-2}\}. \tag{27}$$

Thus, in the second approximation, we obtain the field of a radiating charged particle embedded in an expanding universe in which the density of flowing radiation surrounding the charge is given by equation (27). The method of approximation used here is quite general, and one can use it to obtain the solution to any desired degree of accuracy.

Acknowledgments

The authors are indebted to Dr M. Misra for many helpful discussions. During the course of this investigation, one of us (H.C.S.) was a COSIP participant at the Department of Mathematics, Gujarat University. He expresses his gratitude to the Director of the COSI program for giving him the opportunity to participate in the program.

References

Bonnor, W. B., and Vaidya, P. C. (1970). *Gen. Relativ. Gravitat.* **1**, 127.
McVittie, G. C. (1933). *Mon. Notic. Roy. Astron. Soc.* **193**, 325.
Sheth, Y. I. (1970). Ph.D. Thesis, Gujarat University, India.
Vaidya, P. C. (1953). *Nature (London)* **171**, 260.
Vaidya, P. C., and Shah, K. B. (1957). *Proc. Nat. Inst. Sci. India* **23**, 534.

Manuscript received 27 March 1974