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The Wick Rotation

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Abstract

A justification is presented for the use of the Wick rotation in the construction of solutions to Wick's equation. This is done by demonstrating the equivalence of the solutions found by Wick (1954) and by Green (1957) with and without recourse to the Wick rotation respectively.

Introduction

Wick (1954) made two important contributions to the theory of the bound states of a pair of particles in relativistic quantum mechanics. The first was his suggestion that the relative momentum of the bound particles could be regarded as a Euclidean four-vector, so that the contour of integration over the relative energy, which appears in the Bethe–Salpeter integral equation for the wave function of the bound state, could be rotated from the real to the imaginary axis. This transformation of the Bethe–Salpeter equation has since come to bear the unfortunate name 'Wick rotation'. The validity of this rotation depends upon the analytic properties of the wave function. Wick's second contribution was to propose a set of approximations to the Bethe–Salpeter equation which rendered this equation soluble, and yet not so simple that all the relativistic features were obliterated. These approximations were that the bound particles should be spinless, that the exchanged particles should be both massless and spinless, and that the bound state could be adequately represented by the ladder Feynman diagram.

Wick's equation for the bound state wave function $\phi(p)$ is

$$(p_1^2 - m_1^2)(p_2^2 - m_2^2)\,\phi(p) = \mathrm{i}\lambda\pi^{-2}\int \mathrm{d}^4k\;\phi(k)/\{(p-k)^2+\mathrm{i}\varepsilon\},$$

where $p = (p_0, \boldsymbol{p})$ is the relative momentum of the bound particles, p_1, p_2 and m_1, m_2 are their four-momenta and masses respectively, and λ is the coupling constant whose magnitude determines the strength of the interaction. It is questionable whether Wick's model has any connection with physics, for the assumption that the exchanged particles have neither mass nor spin excludes all real particles. Nevertheless, applications have been found; e.g. Biswas (1958) used Wick's equation in a composite model for K-mesons (see Nakanishi (1969) for many other applications). However, the value of Wick's equation lies not in its applications, but rather in the fact that it is the only soluble example of the Bethe–Salpeter equation. It is hoped that the relativistic aspects of its solutions, namely, the appearance of a new quantum number and the possibility of imaginary values for the relative time and energy, are not peculiar to the model but are general features of the solutions of the Bethe–Salpeter equation. The purpose of the present paper is to justify the use of the Wick rotation in the construction of solutions to Wick's equation.

The advantage gained by Wick through the rotation was the replacement of the indefinite Minkowski metric in the kernel of the integral equation by the definite Euclidean metric, so that the integral equation became amenable to standard mathematical techniques. To accomplish this change of metric, Wick needed to prove three facts:

(1) that the wave function $\phi(p)$ was analytic in the upper and lower halves of the complex plane of the variable p_0, and could be analytically continued from one region to the other;

(2) that the contour of integration for the p_0 variable could be rotated from the real axis to the imaginary axis without encountering any singularities of the integrand;

(3) that the contributions to the integral from the quarter-circles at infinity in the p_0 plane were zero or, equivalently, that $\phi(p)$ approached zero as rapidly as p_0^{-2} when p_0 approached infinity along any ray in the first or third quadrants of the p_0 plane.

Wick formulated stability conditions for the bound particles and found that these were sufficient to establish points (1) and (2) concerning the wave function. However, he was forced to assume the validity of (3). Wick reduced the four-dimensional integral equation to an ordinary differential operator. He deduced that, at a given centre-of-mass energy, bound states could only occur if the coupling constant assumed one of a countably infinite set of values, and that these values were the eigenvalues of the differential operator.

Green (1957) discovered that it was not necessary to rotate the contour of integration in order to find solutions of Wick's equation. He replaced the integral equation by a partial differential equation with associated boundary conditions and demonstrated that an ingenious bipolar coordinate transformation rendered the partial differential equation separable. Green also found that the permissible values of the coupling constant were the eigenvalues of an ordinary differential operator, which differs from the operator derived by Wick.

To reconcile the two studies, and hence to justify Wick's assumption (3), it must be shown that the two ordinary differential operators have the same spectrum. The solution of this problem is here resolved into two stages. Firstly, the two differential equations are reduced to the same form, so that the operators differ only in the boundary conditions. Secondly, solutions which fulfil one set of boundary conditions are shown to fulfil the other set, and vice versa. Only the case of particles of equal mass is considered since Cutkosky (1954) has shown that the equations for unequal masses can be reduced to the former case.

Differential Operators

Wick's differential operator is comprised by the differential equation

$$\{(1-z^2)\,\mathrm{d}^2/\mathrm{d}z^2 + 2(n-1)z\,\mathrm{d}/\mathrm{d}z - n(n-1) + \lambda/(1-a+az^2)\}\,g(z) = 0 \tag{1}$$

and the boundary conditions

$$g(\pm 1) = 0. \tag{2}$$

The parameter λ, the eigenvalue of the equation, is the coupling constant, n is an integer quantum number, and a is the square of the ratio of the centre-of-mass energy to the total rest mass of the interacting particles, and so is less than unity for bound states. The eigenvalues of this operator are simple, so that it follows that each eigensolution $g(z)$ must have definite parity, for otherwise $g(z)$ and $g(-z)$ would be independent eigensolutions with the same eigenvalue. The boundary conditions (2) may be replaced by

$$g(0) = g(1) = 0$$

for odd solutions of equation (1), and by

$$\mathrm{d}g(0)/\mathrm{d}z = g(1) = 0$$

for even solutions. If we now define

$$x = (1-z^2)^{-1} \qquad \text{and} \qquad f(x) = x^{\frac{1}{4}n} g\big((1-1/x)^{\frac{1}{2}}\big)$$

then $f(x)$ satisfies the following differential equation, a particular case of the Heun equation (Erdélyi 1955):

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \tfrac{1}{2}\left(\frac{1}{x} + \frac{1}{x-1}\right)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\lambda - n^2(x-a)}{4x(x-1)(x-a)}\right\} f(x) = 0. \tag{3}$$

The boundary conditions to be satisfied by an 'odd' solution of equation (3), i.e. a solution of (3) derived from an odd solution of (1), are

$$\lim_{x\to\infty} x^{-\frac{1}{4}n} f(x) = 0 \qquad \text{and} \qquad f(1) = 0. \tag{4,5}$$

'Even' solutions of equation (3) must also satisfy (4) but, instead of (5), they must satisfy

$$\lim_{x\to 1} (x-1)^{\frac{1}{2}} \mathrm{d}f/\mathrm{d}x = 0. \tag{6}$$

Green (1957) produced the equation

$$\{\mathrm{d}^2/\mathrm{d}\alpha^2 + \lambda/(a-\cos^2\alpha) + n^2\}\, u(\alpha) = 0, \tag{7}$$

which is to be satisfied by bounded periodic functions with period 2π. If we have

$$x = \cos^2\alpha \qquad \text{and} \qquad f(x) = u(\arccos x^{\frac{1}{2}})$$

then $f(x)$ also satisfies equation (3). The condition that $u(\alpha)$ should be bounded requires that $u(\alpha)$ should vanish as rapidly as $\cos^2\alpha - a$ near $\cos^2\alpha = a$, for otherwise $u(\alpha)$ would contain a factor $\log(\cos^2\alpha - a)$. (Such a term would confound the definition of periodicity!) For the function $f(x)$, this implies that we have

$$\lim_{x\to a} f(x)/(x-a) < \infty. \tag{8}$$

The second boundary condition for $f(x)$ also depends upon whether $f(x)$ is derived from an odd or even function. If $u(\alpha)$ is odd, we have

$$u(0) = 0$$

and thus

$$f(1) = 0 \tag{9}$$

but, if $u(\alpha)$ is even, we have

$$\mathrm{d}u(0)/\mathrm{d}\alpha = 0$$

and thus

$$\lim_{x \to 1} (x-1)^{\frac{1}{2}} \mathrm{d}f/\mathrm{d}x = 0. \tag{10}$$

The various boundary conditions imposed on solutions of equation (3) are summarized in Table 1.

Table 1. Summary of boundary conditions for equation (3)

Solution	Green (1957) $x = a$	Green (1957) $x = 1$	Wick (1954) $x = 1$	Wick (1954) $x = \infty$
'Even'	$\lim_{x \to a} \frac{f(x)}{x-a} < \infty$	$\lim_{x \to 1} (x-1)^{\frac{1}{2}} \frac{\mathrm{d}f}{\mathrm{d}x} = 0$	$\lim_{x \to 1} (x-1)^{\frac{1}{2}} \frac{\mathrm{d}f}{\mathrm{d}x} = 0$	$\lim_{x \to \infty} x^{-\frac{1}{2}n} f(x) = 0$
'Odd'	$\lim_{x \to a} \frac{f(x)}{x-a} < \infty$	$f(1) = 0$	$f(1) = 0$	$\lim_{x \to \infty} x^{-\frac{1}{2}n} f(x) = 0$

It is shown in the following sections that every solution of equation (3) which satisfies the boundary conditions at $x = 1$ and $x = \infty$ has an analytic continuation which satisfies the boundary condition at $x = a$, and conversely. The work of Erdélyi (1944) on the representation of Heun functions as convergent series of hypergeometric functions contains the solution of this problem. There is, however, a simpler solution which uses only elementary properties of Heun functions.

Heun Functions

The Heun equation is the second-order differential equation with just four singular points, all of which are regular. In its general form

$$\left\{\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \sum_{i=1}^{3} \left(\frac{1-\alpha_i-\beta_i}{x-x_i}\frac{\mathrm{d}}{\mathrm{d}x} + \frac{\alpha_i \beta_i}{(x-x_i)^2}\right) - \frac{x(\alpha_4 \beta_4 - \alpha_1 \beta_1 - \alpha_2 \beta_2 - \alpha_3 \beta_3) - \zeta}{(x-x_1)(x-x_2)(x-x_3)}\right\} f(x) = 0$$

with

$$\sum_{i=1}^{4} (\alpha_i + \beta_i) = 2,$$

the singularities lie at x_1, x_2, x_3 and $x_4 = \infty$, and the parameters may be assumed to satisfy

$$\mathrm{Re}(\alpha_i - \beta_i) \geqslant 0 \quad \text{for} \quad i = 1, 2, 3, 4.$$

Solutions of the Heun equation are represented globally by

$$P\begin{Bmatrix} x_1 & x_2 & x_3 & x_4=\infty & \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 & x \\ \beta_1 & \beta_2 & \beta_3 & \beta_4 & \end{Bmatrix}, \tag{11}$$

a simple extension of the notation invented by Riemann for solutions of the hypergeometric equation. The entries α_i and β_i below a given x_i are the exponents of the two independent branches which can be developed in series near x_i. For certain values of the parameter ζ, two branches of the P function may become linearly dependent; such exceptional solutions being called Heun functions. In this section it is shown that for certain cases of the Heun equation, one of which is equation (3), there are Heun functions which adopt definite exponents at not just two singular points but instead at all four.

Suppose that it is possible to choose one exponent at each singular point so that the sum of the four chosen exponents is an integer. If the difference of the exponents at any singular point is an integer, suppose in addition that the exponent with the larger real part has been chosen. For clarity of argument, we assume that exponents α_1, α_2, α_3, α_4 have been chosen and that we have

$$\alpha_1+\alpha_2+\alpha_3+\alpha_4 = m, \tag{12}$$

where m is an integer. Let $f(x)$ denote a Heun function with exponents α_3 and α_4 at x_3 and x_4 respectively. The analytic continuation of $f(x)$ from some initial point x_0 around a simple closed contour $\mathscr{L}$, which encircles in a positive sense the singular points x_1 and x_2, produces the same effect as continuation around a similar contour which encircles x_3 and x_4 in the opposite sense. This is readily seen if the compactified plane is projected onto the Riemann sphere. The latter continuation replaces $f(x)$ by

$$\exp(-2\pi i(\alpha_3+\alpha_4))\,f(x).$$

By the assumption (12) we have

$$\exp(-2\pi i(\alpha_3+\alpha_4)) = \exp(2\pi i(\alpha_1+\alpha_2)).$$

Thus continuation of $f(x)$ from x_0 around $\mathscr{L}$ maps $f(x)$ into

$$\exp(2\pi i(\alpha_1+\alpha_2))\,f(x).$$

A sufficient condition for this to be true is that $f(x)$ should have exponents α_1 and α_2 at x_1 and x_2 respectively. When the difference of the exponents at one of these points, say x_2, is an integer, this condition is also necessary. The details of the proof of this assertion are messy, but the result is easily understood. The branch of the P function (11) with exponent β_2 at x_2 contains a term with a factor $\log(x-x_2)$ whenever $\alpha_2-\beta_2$ is integral. The assumption that $f(x)$ depends upon this branch, and hence that $f(x)$ also contains a term with a logarithmic factor, is impossible to reconcile with the result that continuation of $f(x)$ around $\mathscr{L}$ merely multiplies $f(x)$

by the factor $\exp(2\pi i(\alpha_1+\alpha_2))$. Thus, when the condition (12) holds and either $\alpha_1-\beta_1$ or $\alpha_2-\beta_2$ is an integer, the Heun function with exponents α_3 and α_4 at x_3 and x_4 also adopts exponents α_1 and α_2 at x_1 and x_2.

Application to Wick's Equation

Equation (3) is a particular case of the Heun equation, the solutions of which in Riemann notation are

$$P\begin{Bmatrix} 0 & a & 1 & \infty & \\ \tfrac{1}{2} & 1 & \tfrac{1}{2} & \tfrac{1}{2}n & x \\ 0 & 0 & 0 & -\tfrac{1}{2}n & \end{Bmatrix}.$$

The boundary conditions at $x = 1$, to be satisfied respectively by 'odd' and 'even' solutions of equation (3), are fulfilled if and only if the 'odd' solutions have exponent $\frac{1}{2}$ and the 'even' solutions have exponent 0 at $x = 1$. The condition at infinity imposed by Wick (1954) requires $f(x)$ to have exponent $\frac{1}{2}n$ at infinity. Thus, the 'odd' and 'even' eigensolutions of Wick's boundary value problem are Heun functions with exponent $\frac{1}{2}n$ at infinity and exponents $\frac{1}{2}$ and 0 respectively at $x = 1$. The condition at $x = a$ imposed by Green (1957) requires $f(x)$ to have exponent 1 at $x = a$, so that the eigensolutions of Green's boundary value problem are also Heun functions.

Suppose $f(x)$ is an 'odd' eigensolution of Wick's boundary value problem. If n is even, then

$$\tfrac{1}{2}+1+\tfrac{1}{2}+\tfrac{1}{2}n$$

is an integer and, since the difference of the exponents at $x = a$ is an integer, it follows from the argument of the preceding section that $f(x)$ has exponents $\frac{1}{2}$, 1, $\frac{1}{2}$ and $\frac{1}{2}n$ at 0, a, 1 and ∞ respectively. Thus $f(x)$ satisfies the boundary condition at $x = a$ and so is an eigensolution of Green's boundary value problem. This argument can be reversed because the difference of the exponents at infinity is also an integer. Consequently $f(x)$ is an eigensolution of one operator if and only if it is an eigensolution of the other. The same conclusion applies in the other cases, corresponding to the other possible choices of parity for the eigensolution and n.

Conclusions

It has been shown above that the analytic continuation of an eigensolution of Wick's differential operator satisfies the differential equation and boundary conditions which together comprise Green's differential operator, and vice versa. The discrete spectra of the operators are identical. Consequently Wick's approach, which employs the contour rotation, is equivalent to Green's which does not. Hence the use of the contour rotation is justified in this model.

The proof of the equivalence rests upon a fortuitous combination of exponents in the Heun equation, which permits the boundary conditions at one pair of singular points to be transferred to equivalent boundary conditions at another pair. It is most certainly not a general feature of Heun functions that they adopt a definite exponent at each of the four singular points. In fact, the argument can be reversed to imply that the contour rotation is only permissible because Wick's equation leads to such a special case of the Heun equation. It could be argued that the unusual trick needed to justify the rotation is a reflection of the special nature of Wick's model and that the possibility of the Wick rotation only arises for solutions of Wick's equation. However, an alternative and broader view is that in any fundamental process the internal or relative momenta can be treated as Euclidean four-vectors or, equivalently, that the relative times may be taken to be imaginary. If this view is accepted, as is done e.g. in Euclidean field theory, then it is not at all surprising that the rotation can be justified.

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