Aust. J. Phys., 1975, **28**, 69–75

A Digital Method for Computation of Time-varying Power Spectra applied to Geomagnetic Pi Pulsations

F. F. F. Yuan

Mawson Institute for Antarctic Research, University of Adelaide, S.A. 5001.

Abstract

A digital method of computing time-varying power spectra involving complex demodulation followed by low pass filtering is described. The method is illustrated by an application to records of geomagnetic Pi pulsations from Macquarie Island. These show a steady decrease in power with increasing frequency in the range 0·1–0·5 Hz.

1. Introduction

Since the development of the sound spectrograph (Koenig *et al.* 1946), the production of time-varying power spectra or 'spectrograms' has been of great utility in many experimental fields, including the study of whistling atmospherics (Storey 1953), speech sounds (see e.g. Flanagan 1965), animal and bird vocalizations (see e.g. Koenig *et al.*) and geomagnetic pulsations (Duffus *et al.* 1958), to name a few applications. The term spectrogram is used here to denote a three-dimensional representation of a time-varying power spectrum, showing power as a function of frequency and time in some form of contour map. Spectrograms of geomagnetic pulsations have commonly been referred to as 'sonagrams' because of the usual production method of recording signals on magnetic tape and then replaying them at speeds many times faster than the recording speed into a sound spectrograph. Such spectrograms have also been produced by recording signals in digital form and then using a computer to calculate the time-varying power spectrum (McPherron 1968); these spectrograms were named 'digital sonagrams', which is surely an inappropriate terminology as no frequency multiplication into the audio frequency range was required before or during computer analysis.

In McPherron's (1968) spectrogram production method, band-pass filtering was achieved by convolving the input time series with the impulse response of a symmetrical filter. The impulse response function was derived by multiplying that of an ideal band-pass filter with a Hanning function, so that the resulting function was nonzero only within a finite time interval. The present paper is devoted to a discussion of another digital method which makes use of complex demodulation followed by low pass filtering. Although the discussion is restricted to the problems of analysis of geomagnetic pulsation data, this technique is generally useful as it can be used to produce spectrograms of any time-varying phenomenon after the appropriate analogue–digital conversion has been performed.

2. Method of Computation

Geomagnetic pulsations were recorded in the field as a serial bit-stream on 6 mm magnetic tape. These data were re-recorded in the laboratory in computer-compatible form (Yuan 1970). A computer program was written to select data sections which were then re-written onto magnetic tape in blocked form for convenience of processing. This program also provided for smoothing and resampling of the data and for filtering out the zero-frequency component prior to spectral analysis (see Section 3).

Band-pass filtering was simulated by complex demodulation, which is equivalent to the heterodyning of a signal with a sine wave (see review by Bingham *et al.* 1967). The process of forming the complex demodulate at a frequency ω and then filtering with a low pass filter having a 3 dB cutoff frequency $\Delta\omega$ is equivalent to the process of filtering the signal with a symmetrical band-pass filter having a centre frequency ω and 3 dB cutoff frequencies $\omega \pm \Delta\omega$.

Given the times series $f(t)$ which has a value $f(nT)$ at time $t = nT$, the complex demodulate of $f(t)$ at frequency ω is given by

$$F_{\omega}(nT) = f(nT)\exp(i\omega nT). \tag{1}$$

If a low pass filtering process, denoted by an operator S, is applied to the complex demodulate, we obtain

$$F'_{\omega}(nT) = S[F_{\omega}(nT)] = S[f(nT)\cos\omega nT] + iS[f(nT)\sin\omega nT]. \tag{2}$$

An instantaneous power function $P'(\omega, t)$ is then given by

$$P'(\omega, t) = |F'_{\omega}(nT)|^2. \tag{3}$$

All low and high pass filtering operations were effected with recursive digital filters, simulated by use of the bilinear Z-transform technique of Golden and Kaiser (1964). Cosine and sine values were efficiently computed using the equation

$$\begin{bmatrix} \sin\omega(n+1)T \\ \cos\omega(n+1)T \end{bmatrix} = \begin{bmatrix} \cos\omega T & \sin\omega T \\ -\sin\omega T & \cos\omega T \end{bmatrix} \begin{bmatrix} \sin\omega nT \\ \cos\omega nT \end{bmatrix}. \tag{4}$$

The time-varying power spectrum $P(\omega, t)$ is estimated by smoothing the power function $P'(\omega, t)$ (Bendat and Piersol 1966). This smoothing is necessary because of a basic uncertainty principle in spectral analysis, namely, it is impossible to increase frequency resolution without decreasing the maximum possible time resolution and vice versa. For an ideal band-pass filter, with passband from frequency ω_0 to $\omega_0 + \Delta\omega$, the maximum time resolution possible can be estimated using a sampling theorem in the time domain (Kohlenberg 1953). Provided ω_0 is an exact multiple of $\Delta\omega$, the output $f(t)$ from the filter is completely determined by samples taken with a sampling interval $\Delta T = \pi/\Delta\omega$. If ΔT is decreased below $\pi/\Delta\omega$, the increased time resolution gives no further information about the input signal; details of the filter impulse response function are revealed instead.

For a gaussian white noise input time series which is stationary, the effect of smoothing the power function $P'(\omega, t)$ can be calculated (see e.g. Blackman and Tukey 1958). If this smoothing is performed by taking a simple moving average of duration ΔT, estimates of the power spectral density can be made by sampling the resulting function. The statistical character of these estimates, as shown by the mean and variance, will be similar to that of a chi-square variate, with the number of degrees of freedom being proportional to the duration ΔT. The variance will be inversely proportional to ΔT. When $\Delta T = \pi/\Delta\omega$, the spectral density estimate has one equivalent degree of freedom.

For the general case, i.e. a nonstationary input time series, the effect of smoothing $P'(\omega, t)$ with a moving average cannot be calculated without comprehensive *a priori* knowledge of the input time series. Also, in trying to improve statistical reliability by increasing ΔT, errors may be introduced because the spectrum may change significantly within the time interval ΔT. From these considerations, it can be seen that a duration $\Delta T = \pi/\Delta\omega$ for the moving average would be close to optimum.

Average frequency characteristics, over some interval of time $t = jmT$ to kmT, can be examined by computing a time-averaged power spectrum

$$P_{\mathrm{av}}(\omega) = K_0 \sum_{r=j}^{k-1} P(\omega, rmT). \tag{5}$$

The normalization constant K_0 was chosen so that $P_{\mathrm{av}}(\omega_0)$ was equal to unity, ω_0 being the lowest centre frequency in the array of filters used to estimate the time-varying power spectrum.

3. Digital Spectrograms of Geomagnetic Pi Pulsations

Fig. 1 shows two spectrograms: (*b*) is a spectrogram of geomagnetic Pi pulsations (H component) recorded at Macquarie Island and was produced by the conventional analogue method (see Section 1); (*a*) is a digital spectrogram produced by simulating an array of band-pass filters with 0·02 Hz spacing between the centre frequencies of adjacent filter channels (the filters were contiguous, i.e. adjacent filters had common 3 dB points). Low pass filtering of complex demodulates was performed using the following scheme. For input and output time series $X(nT)$ and $Y(nT)$ respectively, the value of the output time series at $t = (n+1)T$ was computed using the equation

$$Y((n+1)T) = A\{X(nT) + X((n+1)T)\} + BY(nT), \tag{6}$$

where

$$A = \Omega/(2T^{-1}+\Omega), \quad B = (2T^{-1}-\Omega)/(2T^{-1}+\Omega) \qquad \text{with} \qquad \Omega = 2T^{-1}\tan\tfrac{1}{2}\omega_c T,$$

ω_c being the 3 dB cutoff point for the low pass filter. The frequency response for this filter is shown in Fig. 2.

Use of recursive digital filtering gives a considerable advantage in computation speed over non-recursive filtering schemes, such as that of McPherron (1968). Consider the computational effort, measured by the number of multiplications and additions, required to produce one point in the instantaneous power function $P'(\omega, t)$. Assume that all time-invariant filter coefficients have been calculated. With the complex demodulation technique, firstly 4 multiplications and 2 additions are required to produce sine and cosine values using equation (4). Then 2 more multiplications are needed to compute the complex demodulate. Low pass filtering of the complex demodulate is effected using equation (6) with another 4 multiplications and 4 additions. Another 2 multiplications and 1 addition are required to calculate one point of $P'(\omega, t)$, giving a total of 12 multiplications and 7 additions.

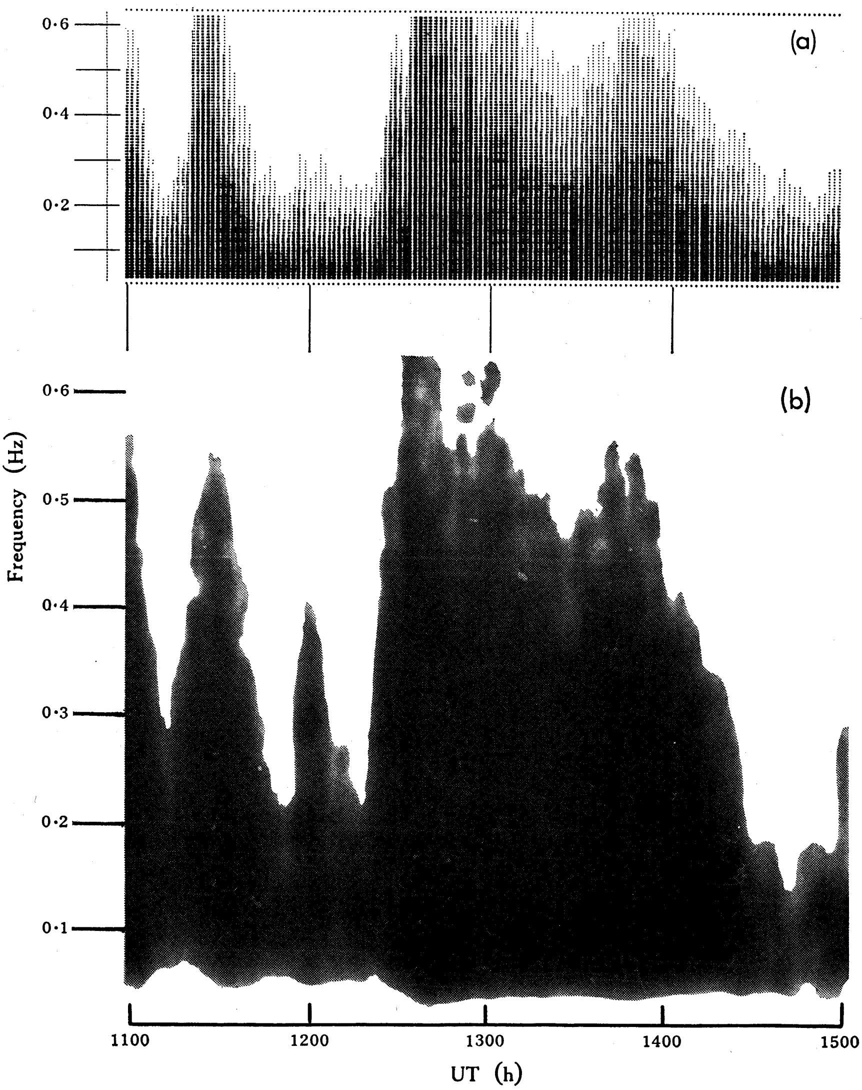

Fig. 1. Spectrograms of geomagnetic Pi pulsations recorded at Macquarie Island over the period 1100–1500 h UT on 15 March 1968: (*a*) digital spectrogram produced by simulating an array of band-pass filters; (*b*) analogue spectrogram recorded by the conventional method. The contour map representation for (*a*) was produced by selective overprinting of line-printer characters; the spacing between contour levels was 3 dB. The feature on (*b*) at 1200 UT is a time mark. The analogue spectrogram was kindly provided by R. R. Heacock.

McPherron's (1968) band-pass filter is implemented by digital convolution over a time interval $T_0 = 2\pi/\Delta\omega$, where $\Delta\omega$ is the filter bandwidth. The number of multiplications, and the number of additions, required to produce one point in the instantaneous power function is approximately $2\pi/T\Delta\omega$, where T is the sampling interval; this is approximately 100 for the value of $T\Delta\omega$ used in producing the digital spectrogram of Fig. 1*a*. Assuming addition to be much faster than multiplication, as is usual, the recursive filter is faster than McPherron's filter when the product $T\Delta\omega$ is less than $\frac{1}{6}\pi$.

The instantaneous power function, derived by squaring the band-pass filter outputs, was smoothed to give the time-varying power spectrum by using a moving average of the form

$$P(\omega, nT) = (2m+1)^{-1} \sum_{j=-m}^{m} P'(\omega, nT - jT). \tag{7}$$

The value of $P(\omega, nT)$ was calculated at every mth datum point.

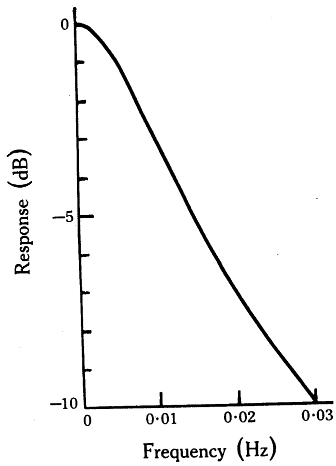

Fig. 2. Frequency response for the low pass filter used in complex demodulation. This is the frequency response of any band-pass filter channel, i.e. the spectral window.

The use of a computer for spectral analysis gives the advantage that data can be analysed repeatedly with ease, varying the frequency and time resolution. However, when analysing any particular type of data, adoption of standardized analysis procedures facilitates the comparison of spectrograms. In order to study Pi pulsations, a frequency resolution of 0·02 Hz was considered to be adequate for routine spectral analysis over the frequency range 0·1–0·5 Hz, as these pulsations resemble band-limited noise (see McPherron *et al.* 1968). After some experimentation, a value of $m = 200$ was chosen for routine analysis, as it provided sufficient time resolution. This gave an integration time of 200 s, which was a factor of eight larger than the optimum time calculated using the sampling theorem (see Section 2). The resulting spectrograms showed similar details of time structuring of Pi pulsation events to those discernible on spectrograms produced at the University of Alaska using analogue methods.

The spectrograms in Fig. 1 show the wideband nature of Pi pulsations. This distinguishes them from the Pc type pulsations, which are characterized by concentration of power in a narrow frequency band (see e.g. Heacock 1970). It is also evident from Fig. 1 that power is concentrated at low frequencies and decreases as the frequency increases. This is more readily seen in the time-averaged power spectra illustrated in Fig. 3. These were obtained from consecutive 1 h time intervals of the digital spectrograms of Fig. 1*a*. (This spectral type is found to be characteristic of Pi pulsations associated with auroral activity at Macquarie Island.)

The digital spectrogram of Fig. 1*a* was computed from data obtained by smoothing and resampling the original data. The latter were obtained by sampling the signal at a rate of $10\ s^{-1}$, each sample being the mean of the signal over an interval of 0·1 s. The smoothing effect of this sampling process may be described by a power transfer function $R(\omega)$ given by

$$R(\omega) = P_s(\omega)/P_u(\omega), \tag{8}$$

where $P_s(\omega)$ is the time-averaged power spectrum of the smoothed data and $P_u(\omega)$ that of the unsmoothed signal. In the present case

$$R(\omega) = \{(\sin\tfrac{1}{2}\omega T)/\tfrac{1}{2}\omega T\}^2, \tag{9}$$

where T is the interval over which the signal mean was taken. In the range 0·1–0·5 Hz,

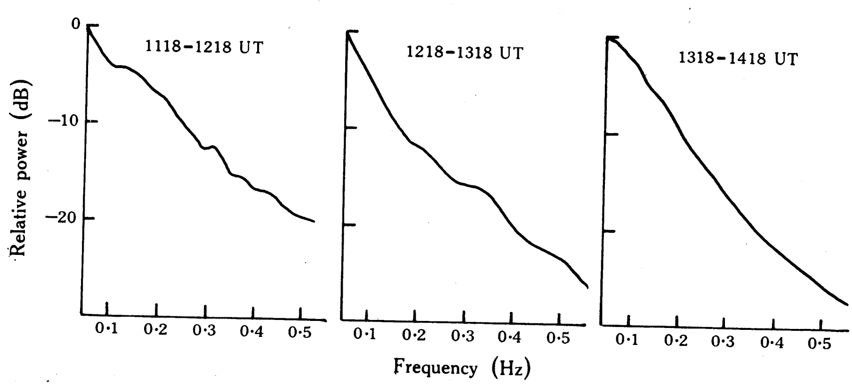

Fig. 3. Time-averaged power spectra for geomagnetic Pi pulsations recorded at Macquarie Island on 15 March 1968.

$R(\omega)$ varied by less than 0·1 dB. Smoothing was then effected by taking an 11-point moving average defined by

$$f'(nT) = (2m+1)^{-1} \sum_{j=-m}^{m} f(nT-jT), \qquad m = 5. \tag{10}$$

The functions $f'(nT)$ and $f(nT)$ respectively represent the values of the smoothed and unsmoothed time series at time $t = nT$, where T is the sampling interval. The smoothed data were then resampled at intervals of 0·5 s. The power transfer function for the moving average is given by

$$R'(\omega) = [\sin\{\tfrac{1}{2}(2m+1)\omega T\}/(2m+1)\sin\tfrac{1}{2}\omega T]^2 \tag{11}$$

(Blackman and Tukey 1958). From 0·1 to 0·5 Hz, $R'(\omega)$ decreased by less than 5 dB. Over this range (and indeed over a much greater range) the response of the pulsations detection system used at Macquarie Island was approximately proportional to frequency. It is thus apparent that the observed decrease in power over the above frequency range in the spectrograms of Pi geomagnetic pulsations is a real effect.

Acknowledgments

I am grateful to Dr F. Jacka and Dr P. W. Seymour for valuable criticisms and discussions, and I would like to thank Mr D. F. Creighton for his supervision of the equipment design. The helpful criticisms of the referee are also gratefully acknowledged. Dr R. R. Heacock of the Geophysical Institute, University of Alaska, provided the analogue spectrogram. This work was supported by a grant from the Radio Research Board, Australia. Logistic support for the observations at Macquarie Island was provided by the Antarctic Division, Department of Science.

References

Bendat, J. S., and Piersol, A. G. (1966). 'Measurement and Analysis of Random Data' (Wiley: New York).

Bingham, C., Godfrey, M. D., and Tukey, J. W. (1967). *IEEE Trans. Audio Electroacoustics* AU **15**, 56–66.

Blackman, R. B., and Tukey, J. W. (1958). 'The Measurement of Power Spectra' (Dover: New York).

Duffus, H. J., Nasmyth, P. W., Shand, J. A., and Wright, C. (1958). *Nature (London)* **181**, 1258–9.

Flanagan, J. (1965). 'Speech Analysis, Synthesis and Perception' (Academic: New York).

Golden, R. M., and Kaiser, J. F. (1964). *Bell Syst. Tech. J.* **43**, 1533–46.

Heacock, R. R. (1970). 'An Atlas of Micropulsations Spectra', Geophys. Inst. Rep. No. UAG R–214 (Univ. Alaska).

Koenig, W., Dunn, H. K., and Lacey, L. Y. (1946). *J. Acoust. Soc. Amer.* **18**, 19–49.

Kohlenberg, A. (1953). *J. Appl. Phys.* **24**, 1432–6.

McPherron, R. L. (1968). A digital technique for obtaining dynamic spectra. Space Sci. Lab. Rep., Univ. California.

McPherron, R. L., Parks, G. K., Coroniti, F. V., and Ward, S. H. (1968). *J. Geophys. Res.* **73**, 1697–1713.

Storey, L. R. O. (1953). *Phil. Trans. Roy. Soc. London* A **246**, 113–41.

Yuan, F. F. F. (1970). Ph.D. Thesis, University of Adelaide.

Manuscript received 21 September 1972, revised 29 January 1974