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Three-wave Interactions Involving One Whistler

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Abstract

Three-wave interactions in which one of the waves is a whistler and the other two are higher frequency waves are examined. The suggestion by Chiu (1970) and Chin (1972) that radio emission near the fundamental plasma frequency might arise in the solar corona from the coalescence of a whistler wave with a Langmuir wave is shown to be unacceptable because the resonance condition for the three-wave interaction cannot be satisfied. Modifications aimed at overcoming this objection are explored. In particular it is pointed out that a spectrum of Langmuir waves evolves into a spectrum of Z-mode waves because of nonlinear interactions in a magnetoactive plasma, but that the coalescence of a Z-mode wave with a whistler can occur only under implausibly restrictive conditions. Explicit expressions which describe the three-wave interactions are derived, and these are used to treat the scattering of transverse waves (and also of Langmuir waves) by whistlers. It is suggested that such scattering may be important in regions of the solar corona where energetic electrons are trapped.

1. Introduction

Three-wave interactions involve either the coalescence of two waves into the third or the decay of one wave into the other two. Such interactions involving whistlers (also known as helicons) have been discussed in connection with the ionosphere and magnetosphere (Harker and Crawford 1969), the solar corona (Chiu 1970; Chin 1972), solid state plasmas (Sudan *et al.* 1967; Bulgakov *et al.* 1970) and computer-simulated plasmas (Wright 1971).

The purposes of this paper are twofold: firstly, to derive specific expressions describing three-wave interactions in which one of the waves is a whistler and the other two are higher frequency waves, the derivation being based on the theory developed by Melrose and Sy (1972; hereinafter referred to as MS); secondly, to discuss the possible importance of such interactions to plasma radiation (i.e. radiation at or near the local plasma frequency) from the solar corona. Plasma radiation is usually attributed to the scattering of stream-excited Langmuir waves by thermal ions in the coronal plasma. In particular, it is shown that a specific three-wave interaction proposed by Chiu (1970) cannot lead to plasma radiation from the solar corona as he suggested, since, in the form proposed by him, the resonance conditions for the three-wave interaction are not satisfied. Because this criticism can be made, and its implications explored, without knowledge of the probabilities which govern three-wave interactions, this particular application precedes the general theory.

Thus in Section 2 the following process is investigated. It was suggested by Chiu (1970), with a further discussion by Chin (1972), that a form of plasma emission in which whistlers coalesce with Langmuir waves (longitudinal electron plasma waves) to produce escaping radiation might operate in the solar corona. The suggestion is attractive, both because whistlers are readily generated by energetic electrons and because there are several types of observed emission (e.g. storm continuum, type I bursts and stationary type IV bursts) for which either there is no well-established theory or the existing theories are unsatisfactory. (Chiu and Chin both applied the suggested mechanism to the specific case of type III bursts.) However, apart from the physical impossibility of the process as originally proposed, the conditions under which this process can occur are found here to be so restrictive that plasma emission involving whistlers is unlikely to be important in the solar corona.

In Section 3 the explicit expressions describing the relevant three-wave processes are derived using the theory of MS. The underlying theory of nonlinear processes in a magnetoactive plasma has been developed in a variety of ways, by the authors cited in the first paragraph of this section and by Yip (1970), Stenflo (1973) and Giles (1974), amongst others. The equivalence of the relevant theory as developed by Bulgakov *et al.* (1970) and by Giles (1974) with that developed by MS is discussed in Appendix 1.

In Section 4 the theory is extended to treat scattering by whistlers of Langmuir waves into Langmuir waves and scattering by whistlers of transverse waves into transverse waves at frequencies higher than the plasma frequency. The possibility is explored briefly that a region of the solar corona where whistlers are excited could become opaque because of this scattering.

2. Plasma Emission involving Whistlers

Plasma Emission

For plasma emission to occur it is usually understood that Langmuir or other waves, such as the Bernstein modes, with frequencies comparable with or greater than the plasma frequency must be excited in the source. Suggestions to the contrary have been made by Zaitsev (1966) and Chiu (1970). These authors suggested that if low frequency waves (ion sound waves in Zaitsev's case and whistlers in Chiu's case) were excited then these could coalesce with thermal plasma waves to produce escaping radiation. Melrose (1970) showed that in such a process the effective temperature T^{σ} of the resulting radiation is restricted by

$$T^{\sigma} \leqslant (\omega^{\sigma'}+\omega^{\sigma''})T^{\sigma'}T^{\sigma''}/(\omega^{\sigma''}T^{\sigma'}+\omega^{\sigma'}T^{\sigma''}), \tag{1}$$

where the coalescing waves are labelled σ' and σ'' and the resulting waves are labelled σ, and ω is the angular wave frequency. By hypothesis Zaitsev and Chiu have (see the conditions, equations (2), below) $\omega^{\sigma''} \ll \omega^{\sigma'} \approx \omega^{\sigma}$, $T^{\sigma''} \gg T_{\mathrm{e}}$ and $T^{\sigma'} = T_{\mathrm{e}}$, where T_{e} is the temperature of the thermal electrons. In this case equation (1) reduces to $T^{\sigma} \leqslant T_{\mathrm{e}}$, that is, the mechanism cannot produce nonthermal radiation. Both the low frequency waves and the Langmuir waves need to be excited well above their respective thermal levels for this coalescence process to produce nonthermal radiation.

The preceding conditions, however, do not rule out the possibility that plasma emission involving whistlers is an important process in the solar corona. Consider a region where energetic electrons are trapped. As argued both by Chiu (1970) and

Chin (1972), relatively mild anisotropies of the distribution of energetic electrons can cause amplification of whistlers, and so a nonthermal level of whistler turbulence is likely to be present. A nonthermal distribution of Langmuir waves should also be present under these conditions because the energetic electrons emit and absorb Langmuir waves. Provided that the emission and absorption of the Langmuir waves are dominated by the energetic electrons, an equilibrium spectrum of these waves will be formed. The effective temperature is then not the thermal temperature but corresponds to the temperature of electrons with random speeds near v_ϕ, the phase speed of the Langmuir waves ($v_\phi = \omega_p/k$, where ω_p is the plasma frequency and k the wave number); this may be shown by, say, inserting a power-law momentum distribution for $f_e(p)$ in equation (26) of Melrose (1970). Thus excited whistlers and excited Langmuir waves should coexist in regions where energetic electrons are trapped.

Resonance Conditions

The resonance conditions

$$\boldsymbol{k} = \boldsymbol{k}' \pm \boldsymbol{k}'', \qquad \omega^\sigma = \omega^{\sigma'} \pm \omega^{\sigma''} \tag{2a, b}$$

prove to be very restrictive when plasma emission involving whistlers is considered. Whistlers with frequencies $\omega'' \ll \Omega|\cos\theta''|$, where Ω is the electron gyrofrequency and θ'' the angle between $\boldsymbol{k}''$ and the background magnetic field, have wave numbers

$$k'' = \frac{\omega_p}{c}\left(\frac{\omega''}{\Omega|\cos\theta''|}\right)^{\frac{1}{2}} \ll \frac{\omega_p}{c}. \tag{3}$$

(Here, and in the remainder of this section, the labels σ are omitted.) The escaping radiation has wave numbers

$$k = (\omega^2 - \omega_p^2)^{\frac{1}{2}}/c \ll \omega_p/c, \tag{4}$$

while Langmuir waves generated directly by particles with speed v have

$$k' = \omega_p/v > \omega_p/c. \tag{5}$$

There is simply no way in which equations (3), (4) and (5) can be made compatible with (2).

Electron Cyclotron Waves

One way in which it could be hoped to satisfy the resonance conditions (2) would be by relaxing the very restrictive condition $\omega'' \ll \Omega|\cos\theta''|$ in equation (3). For ω'' sufficiently close to Ω, when the waves are called electron cyclotron waves, it may be possible to have $k'' > \omega_p/c$. The maximum possible value of k'' is given by (Stix 1962, p. 196)

$$k'' < (\omega_p/c)(\Omega/\omega_p\beta_e)^{\frac{1}{2}}, \tag{6}$$

where β_e is the ratio of the thermal speed of electrons to the speed of light. Under conditions appropriate for the solar corona, e.g. $\beta_e = 10^{-2}$ and $\Omega/\omega_p = 10^{-1}$, this maximum is greater than ω_p/c. However, for k'' close to its maximum value, not only does it require a substantial anisotropy in the distribution of energetic electrons (of speed $\sim(\Omega/\omega_p)c$) to excite nonthermal electron cyclotron waves (Kennel and Petschek 1966), but the waves so excited are in addition strongly damped by thermal electrons. That such waves should exist over a sufficiently large volume to give detectable plasma emission seems implausible. (Note that $k'' \approx k' \gg k$ is not possible for electron cyclotron waves coalescing with longitudinal waves in view of the inequality (6) and the condition (7) below.)

Supraluminous Waves

Alternatively one could satisfy the resonance conditions (2) if the Langmuir waves had phase speeds greater than the speed of light, i.e. for supraluminous waves, in the terminology of Lerche (1968). Furthermore, it would seem that supraluminous waves should be present. This is because coalescence and decay processes involving two Langmuir waves and a whistler should cause Langmuir waves to be pumped continuously from the region $v_\phi < c$ into the region $v_\phi > c$. There is, however, a weakness in this argument: the assumption that the waves are longitudinal must break down at small k values. Langmuir waves in a magnetized plasma are in fact resonant or near-resonant waves in the lower frequency branch of the extraordinary mode. Here this branch will be called the Z-mode. (Its existence is fundamental in the accepted explanation of the Z-trace, or third trace, found in ionosphere sounding experiments. The mode has no widely accepted name.) The properties of the Z-mode are discussed in Appendix 2.

It follows from equation (A15) of Appendix 2 that the above waves cannot be regarded as longitudinal waves, as a first approximation, for

$$k' < (\omega_p/c)(\Omega \sin\theta/\sqrt{3}\,\omega_p \beta_e)^{\frac{1}{2}}. \tag{7}$$

Using again $\beta_e \approx 10^{-2}$ and $\Omega/\omega_p = 10^{-1}$ as appropriate values for the solar corona, one finds from inequality (7) that supraluminous Langmuir waves in the solar corona cannot be treated as even approximately longitudinal. Such waves must be treated as Z-mode waves whose polarization is more nearly transverse than longitudinal.

Coalescence of Z-mode Waves with Whistlers

The only possible plasma emission process involving whistlers is therefore the coalescence of a whistler with a Z-mode wave. Because the frequency of the Z-mode wave is in the range $\omega_p - \frac{1}{2}\Omega \leqslant \omega' < \omega_p$ ($\omega' > \omega_p$ corresponds to $k' > \omega_p/c$), while the frequency of the whistler satisfies $\omega'' \ll \Omega$, it follows that the sum frequency $\omega = \omega' + \omega''$ is less than the cutoff frequency $\omega_p + \frac{1}{2}\Omega$ for the x-mode. Consequently, plasma emission arising from the coalescence of a Z-mode wave and a whistler is 100% polarized in the sense of the o-mode.

Consider the resonance conditions (2). The frequency of the Z-mode wave satisfies $\omega_p - \omega' < \Omega$ and the frequency of the o-mode wave satisfies $\omega - \omega_p \ll \omega_p$. The condition $\omega = \omega' + \omega''$ could be satisfied in three ways:

$$\omega'' \approx \omega_p - \omega' \gg \omega - \omega_p, \tag{8a}$$

$$\omega'' \approx \omega - \omega_p \gg \omega_p - \omega', \tag{8b}$$

$$\omega_p - \omega' \approx \omega - \omega_p \gg \omega''. \tag{8c}$$

The condition $\boldsymbol{k}' + \boldsymbol{k}'' = \boldsymbol{k}$ could also be satisfied in three ways:

$$k'' \approx k' \gg k, \qquad k'' \approx k \gg k', \qquad k' \approx k \gg k''. \tag{9a, b, c}$$

The dispersion relations (equation (3) for the whistlers, equation (A19) for the Z-mode waves and $\omega^2 \approx \omega_p^2 + k^2c^2$ for the o-mode waves) can be approximated by

$$k'' \approx \frac{\omega_p}{c}\left(\frac{\omega''}{\Omega}\right)^{\frac{1}{2}}, \qquad k' \approx \frac{\omega_p}{c}\frac{\Omega}{\omega_p}\left(\frac{\omega_p}{2(\omega_p - \omega')}\right)^{\frac{1}{2}}, \qquad k \approx \frac{\omega_p}{c}\left(\frac{2(\omega - \omega_p)}{\omega_p}\right)^{\frac{1}{2}}. \tag{10a, b, c}$$

When these dispersion relations are substituted in equations (9) one finds that the resonance conditions can be satisfied only with equations (8a) and (9a), i.e. for

$$\omega'' \approx \omega_p - \omega' \gg \omega - \omega_p, \qquad k'' \approx k' \gg k.$$

Furthermore, the resonance conditions can be satisfied only for

$$\omega'' \approx \Omega(\Omega/2\omega_p)^{\frac{1}{2}}. \tag{11}$$

Anisotropic electrons with speed $v \approx (\Omega/\omega_p)^{\frac{3}{4}}c$ would generate such whistlers.

The conclusion that the coalescence is possible only for whistlers with a specific frequency suggests that the process is unlikely to be of physical significance. Of course, the coalescence would occur over a small range of frequencies once allowance is made for the angular dependences which have been ignored in deriving equation (11), but nevertheless the resonance conditions remain very restrictive.

3. Coalescence and Decay Probabilities

The probability of coalescence of two waves σ' and σ'' into a third wave σ was given by MS; apart from minor changes in notation (W_E/W_T is written R_E here) the result is (MS, equation (14))

$$u^{\sigma\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = \frac{8(2\pi)^7 \hbar c^4}{\omega^\sigma \omega^{\sigma'} \omega^{\sigma''}} R_E^\sigma R_E^{\sigma'} R_E^{\sigma''}$$

$$\times |\alpha^{\sigma\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'')|^2 \delta^3(\boldsymbol{k} - \boldsymbol{k}' - \boldsymbol{k}'')\, \delta(\omega^\sigma - \omega^{\sigma'} - \omega^{\sigma''}), \tag{12}$$

with (MS, equation (15))

$$\alpha^{\sigma\sigma'\sigma''}(\boldsymbol{k}, \boldsymbol{k}', \boldsymbol{k}'') = e_i^{\sigma *} e_j^{\sigma'} e_l^{\sigma''} \alpha_{ijl}(\boldsymbol{k}, \omega^\sigma; \boldsymbol{k}', \omega^{\sigma'}; \boldsymbol{k}'', \omega^{\sigma''}), \tag{13}$$

where α_{ijl} is a tensor which characterizes the nonlinear response of the medium and the asterisk denotes complex conjugation. In the applications discussed here the plasma may be regarded as a cold electronic plasma for the purpose of calculating α_{ijl}. The relevant expression for α_{ijl} is (MS, equation (A21))

$$\alpha_{ijl}(\boldsymbol{k},\omega;\boldsymbol{k}',\omega';\boldsymbol{k}'',\omega'')$$

$$= -\frac{e\omega_{\mathrm{p}}^2}{8\pi m_{\mathrm{e}} c^2}\left(\frac{k_r}{\omega'}\tau_{rj}(\omega')\,\tau_{il}(\omega'')+\frac{k_r}{\omega''}\tau_{rl}(\omega'')\,\tau_{ij}(\omega')+\frac{k_r'}{\omega}\tau_{ir}(\omega)\,\tau_{jl}(\omega'')\right.$$

$$\left.+\frac{k_r''}{\omega}\tau_{ir}(\omega)\,\tau_{lj}(\omega')-\frac{k_r'}{\omega''}\tau_{rl}(\omega'')\,\tau_{ij}(\omega)-\frac{k_r''}{\omega'}\tau_{rj}(\omega')\,\tau_{il}(\omega)\right), \tag{14}$$

with (from MS, equation (19))

$$\tau_{ij}(\omega) = (\omega^2\delta_{ij}-\Omega^2 b_i b_j - \mathrm{i}\omega\Omega\varepsilon_{ijl}b_l)/(\omega^2-\Omega^2), \tag{15}$$

where δ_{ij} is the unit tensor, ε_{ijl} is the permutation symbol and $\boldsymbol{b}$ is a unit vector along the ambient magnetic field.

To apply equation (12) to three-wave interactions involving a whistler, one needs to insert the relevant properties of whistlers for one of the modes, σ'' say, and make appropriate approximations to α_{ijl}. The whistler mode is the lower frequency branch of the ordinary mode of magnetoionic theory. For $\omega'' = \omega^{\sigma''} \ll \Omega|\cos\theta''| \ll \omega_{\mathrm{p}}$, the general results written down in Appendix I of MS give

$$\omega^{\sigma''} = \left(\frac{k''c}{\omega_{\mathrm{p}}}\right)^2 \Omega|\cos\theta''|, \qquad R_E^{\sigma''} = \left(\frac{\omega^{\sigma''}}{k''c}\right)^2\frac{1+\cos^2\theta''}{2\cos^2\theta''}$$

and

$$\boldsymbol{e}^{\sigma''} = \frac{\boldsymbol{\kappa}''\sin\theta''+\boldsymbol{\lambda}''\cos\theta''+\mathrm{i}\boldsymbol{a}''|\cos\theta''|}{(1+\cos^2\theta'')^{\frac{1}{2}}}, \tag{16}$$

with

$$\boldsymbol{\kappa}'' = \frac{\boldsymbol{k}''}{k''}, \qquad \boldsymbol{\lambda}'' = \frac{\boldsymbol{\kappa}''\times(\boldsymbol{\kappa}''\times\boldsymbol{b})}{|\boldsymbol{\kappa}''\times\boldsymbol{b}|}, \qquad \boldsymbol{a}'' = \boldsymbol{\lambda}''\times\boldsymbol{\kappa}''. \tag{17}$$

For later purposes we note that the electric vector for the whistler mode is orthogonal to the background magnetic field:

$$\boldsymbol{e}^{\sigma''}.\boldsymbol{b} = 0. \tag{18}$$

In view of the requirement $\omega'' \ll \Omega$ for whistlers, one may use the approximation

$$\tau_{ij}(\omega'') \approx b_i b_j + (\mathrm{i}\omega''/\Omega)\varepsilon_{ijl}b_l \tag{19}$$

in equation (14). If the other two waves have $\omega \approx \omega' \gg \Omega$, one may also use the approximation

$$\tau_{ij}(\omega) \approx \tau_{ij}(\omega') \approx \delta_{ij}. \tag{20}$$

Combining the two approximations (19) and (20), equation (14) reduces to

$$\alpha_{ijl}(\boldsymbol{k},\omega;\boldsymbol{k}',\omega';\boldsymbol{k}'',\omega'')$$
$$\approx -\frac{e\omega_{\mathrm{p}}^2}{8\pi m_{\mathrm{e}}c^2}\left\{\left(\frac{b_i k_j}{\omega}+\frac{k_i'' b_j}{\omega}+\frac{\boldsymbol{k}''.\boldsymbol{b}}{\omega''}\delta_{ij}\right)b_l + \frac{\mathrm{i}}{\Omega}k_r''\varepsilon_{rlm}b_m\delta_{ij}+\frac{k_i''\delta_{jl}-k_j''\delta_{il}}{\omega''}\right\}. \tag{21}$$

However, when equation (21) is substituted into (13) the leading terms, being proportional to b_l, vanish because of equation (18). Thus one finds

$$\alpha^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') \approx -\frac{e\omega_{\mathrm{p}}^2 k''\sin\theta''\,|\cos\theta''\,|}{8\pi m_{\mathrm{e}}c^2\Omega(1+\cos^2\theta'')^{\frac{1}{2}}}\boldsymbol{e}^{\sigma *}.\boldsymbol{e}^{\sigma'}, \tag{22}$$

where

$$\boldsymbol{\kappa}''\times\boldsymbol{e}^{\sigma''}.\boldsymbol{b} = -\mathrm{i}\sin\theta''\,|\cos\theta''\,|(1+\cos^2\theta'')^{-\frac{1}{2}} \tag{23}$$

has been used. The next-order terms in equation (21) would provide corrections of order ω''/Ω in equation (22); these corrections would be significant only for $\sin\theta'' \lesssim \omega''/\Omega$. (It may be noted that the coupling (22) between three waves, one of which is a whistler, vanishes when the whistler propagates along the field lines, i.e. for $\sin\theta'' = 0$, as already pointed out by Harker and Crawford (1969), who gave a physical explanation for it.)

It is possible for the whistler to be involved in either a coalescence process $\sigma'+\sigma''\rightarrow\sigma$ or in a decay process $\sigma'\rightarrow\sigma+\sigma''$. The probabilities for these two processes, and their inverses, are almost identical. They are

$$u^{\sigma\sigma'\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') = \frac{(2\pi)^5\hbar e^2\omega_{\mathrm{p}}^4}{4m_{\mathrm{e}}^2c^2\Omega^2}\frac{\omega^{\sigma''}}{\omega^{\sigma}\omega^{\sigma'}}R_E^{\sigma}R_E^{\sigma'} \times|\boldsymbol{e}^{\sigma *}.\boldsymbol{e}^{\sigma'}|^2\sin^2\theta''\,\delta^3(\boldsymbol{k}-\boldsymbol{k}'\pm\boldsymbol{k}'')\,\delta(\omega^{\sigma}-\omega^{\sigma'}\pm\omega^{\sigma''}), \tag{24}$$

where the plus signs refer to the decay process.

4. Scattering of High Frequency Waves

The three-wave interaction involving whistlers can be effective in scattering Langmuir waves into Langmuir waves and in scattering transverse waves into transverse waves at relatively high frequencies. The equations describing such scattering are formally equivalent to the quasi-linear equations, provided that the wave numbers of the higher frequency waves are much greater than the wave number of the whistlers. These equations are (Tsytovich 1966, Section 3)

$$\frac{\mathrm{d}N^{\sigma}(\boldsymbol{k})}{\mathrm{d}t} = \frac{\partial}{\partial k_i}\left(A_i^{\sigma\sigma'}(\boldsymbol{k})\,N^{\sigma}(\boldsymbol{k})+D_{ij}^{\sigma\sigma'}(\boldsymbol{k})\frac{\partial N^{\sigma}(\boldsymbol{k})}{\partial k_j}\right), \tag{25}$$

$$\frac{\mathrm{d}N^{\sigma'}(\boldsymbol{k}')}{\mathrm{d}t} = \int\frac{\mathrm{d}^3\boldsymbol{k}}{(2\pi)^3}w^{\sigma\sigma'}(\boldsymbol{k},\boldsymbol{k}')\left(N^{\sigma}(\boldsymbol{k})+N^{\sigma'}(\boldsymbol{k}')\,\boldsymbol{k}'.\frac{\partial N^{\sigma}(\boldsymbol{k})}{\partial\boldsymbol{k}}\right), \tag{26}$$

with

$$A_i^{\sigma\sigma'}(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} w^{\sigma\sigma'}(\boldsymbol{k},\boldsymbol{k}')\, k_i', \tag{27}$$

$$D_{ij}^{\sigma\sigma'}(\boldsymbol{k}) = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} w^{\sigma\sigma'}(\boldsymbol{k},\boldsymbol{k}')\, k_i' k_j' N^{\sigma'}(\boldsymbol{k}'). \tag{28}$$

Here the higher frequency waves are denoted by σ and the lower frequency waves by σ'. The quantity $w^{\sigma\sigma'}(\boldsymbol{k},\boldsymbol{k}')$ is the probability per unit time that a photon in the mode σ in the range $d^3\boldsymbol{k}/(2\pi)^3$ at $\boldsymbol{k}$ should emit a photon in the mode σ' in the range $d^3\boldsymbol{k}'/(2\pi)^3$ at $\boldsymbol{k}'$.

The probability $w^{\sigma\sigma''}(\boldsymbol{k},\boldsymbol{k}'')$ for emission and absorption of whistlers may be derived from equation (12) by expanding in k''/k and taking the limit $k''/k \to 0$ in

$$w^{\sigma\sigma''}(\boldsymbol{k},\boldsymbol{k}'') = \int \frac{d^3\boldsymbol{k}'}{(2\pi)^3} u^{\sigma\sigma\sigma''}(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''). \tag{29}$$

One finds

$$w^{\sigma\sigma''}(\boldsymbol{k},\boldsymbol{k}'') = \frac{(2\pi)^2 \hbar e^2 \omega_p^4 \omega^{\sigma''}}{4m_e^2 c^2 \Omega^2 (\omega^\sigma)^2} (R_E^\sigma)^2 \sin^2\theta''\, \delta(\omega^{\sigma''} - \boldsymbol{k}''.\boldsymbol{v}_g^\sigma), \tag{30}$$

where

$$\boldsymbol{v}_g^\sigma = \partial\omega^\sigma/\partial\boldsymbol{k} \tag{31}$$

is the group velocity of the higher frequency waves.

For scattering of, say, o-mode waves at $\omega > \omega_p$, one might think that some x-mode waves would be produced. This is not so because the polarization vectors are orthogonal, to a first approximation, and hence the factor $|\boldsymbol{e}^{\sigma*}.\boldsymbol{e}^{\sigma'}|^2$ in equation (13) is zero to this approximation. However, if the mean free path for scattering were less than the distance over which Faraday rotation would cause a rotation of the plane of polarization through $\sim 90°$, it would be inappropriate to separate into magnetoionic components. If such were the case then it would be necessary to treat the waves as transverse waves with arbitrary transverse polarization; an extension of the formalism, e.g. by introducing polarization tensors, would be required to treat this case.

For transverse waves, or o-mode or x-mode waves, the condition $k'' \ll k$ must break down at frequencies very close to the plasma frequency because $k \to 0$ for $\omega \to \omega_p$. For equations (25)–(28) to apply requires

$$\omega''/\Omega \ll \omega^2/\omega_p^2 - 1. \tag{32}$$

Scattering in Solar Corona

The scattering of high frequency waves by whistlers will not be discussed in detail here. However, it is appropriate to make a rough estimate of the conditions required for effective scattering. According to equation (25) the high frequency waves diffuse in $\boldsymbol{k}$-space with a diffusion rate $\nu \sim k^2 D_{ii}$. The propagation time for escaping radiation passing through a region where whistlers are excited in the solar corona is probably of the order of a second or slightly less. Consequently, the scattering would be important for $\nu \gg 1\ \mathrm{s}^{-1}$. A very rough estimate of the scattering rate gives

$$\nu \approx k^2 D_{ii} \sim W'' \omega_p^4 / n_e m_e c^2 \Omega^3 , \tag{33}$$

where W'' is the energy density in whistlers. The maximum value of W'' for whistlers generated by energetic electrons with number density n_1 and speed $v \sim c$ is $W'' \gtrsim n_1 m_e c^2 (\Omega/\omega_p)^4$. Hence the maximum value of the scattering rate is $\nu_{1,\max} \gtrsim \Omega(n_1/n_e)$. This rate could well be much greater than $1\ \mathrm{s}^{-1}$ in regions where energetic electrons are trapped in the corona.

The possibility of enhanced scattering of radiation by whistlers in the solar corona should be explored in more detail. It appears possible that this scattering could produce broad-band absorption features in radio spectra.

5. Discussion and Conclusions

The conclusions reached in this paper concerning the possibility of plasma radiation resulting from whistlers are essentially negative. The conclusions can be summarized by contrasting them with suggestions made by Chiu (1970). He suggested that whistlers excited by anisotropic electrons could lead to plasma radiation through coalescence of the excited whistlers with thermal Langmuir waves, and that the resulting radiation should be polarized with the same handedness as the incident whistlers. The relevant conclusions reached here are:

(1) The coalescence process involving thermal Langmuir waves cannot produce nonthermal radiation (this has already been pointed out by Chin 1972).

(2) The wave numbers for whistlers are such that they can coalesce only with Langmuir waves with phase speeds greater than the speed of light. (Neither Chiu (1970) nor Chin (1972) noted this.) Such waves cannot be regarded as longitudinal waves. They should be treated using magnetoionic theory, in which case they correspond to the Z-mode (the lower frequency branch of the extraordinary mode).

(3) The resulting radiation should be 100% polarized in the sense of the o-mode. This is the opposite sense to that suggested by Chiu (1970) (see comment below on the polarization).

(4) The resonance conditions for the coalescence of whistlers and Z-mode waves can be satisfied only for whistlers within a very narrow range around the frequency $\omega'' \approx \Omega(\Omega/2\omega_p)^{\frac{1}{2}}$. This is so restrictive that the mechanism is regarded as of no physical significance.

The discussion of plasma emission involving whistlers can be repeated for plasma emission involving hydromagnetic waves. Hydromagnetic waves have k'' less than $(\omega_p/c)(\omega''/43\,\Omega)$, and consequently they too cannot be involved in plasma emission.

A notable feature of the probability (equation (24)) describing three-wave interactions involving one whistler is that the polarization of the scattered wave is independent of the polarization of the whistler. Indeed the factor $|e^{\sigma *} . e^{\sigma'}|^2$ in equation (24) is the same factor that appears in any probability for the scattering of one wave into another by a particle. It might be commented that Chiu's (1970) suggestion that the radiation produced when whistlers coalesce with Langmuir waves should be polarized in the sense of the whistlers is unfounded.

The conclusion reached concerning enhanced scattering of radiation by whistlers is that such scattering may well be important in the solar corona. If this were the case, it would lead to broad-band absorption features in radio spectra. This requires further investigation.

One other point is worthy of comment. An answer has been given in this paper to the question of how a distribution of Langmuir waves in a magnetoactive plasma can evolve through nonlinear processes: it is that the waves evolve into Z-mode waves. This raises the question of what happens to the Z-mode waves, and this problem is of particular interest because a source for the Z-mode waves has been identified but there is no obvious effective sink for them. Collisional damping cannot be the answer, or at least not the direct answer, because the nonlinear processes cause the Z-mode waves to grow exponentially if, and only if, the growth rate exceeds the collision frequency. Two possibilities remain: either the Z-mode waves are converted into escaping radiation (which is the interesting possibility of course), or their growth is balanced by propagation of the Z-mode waves out of the region where growth occurs. This problem also requires further investigation.

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Appendix 1

Bulgakov *et al.* (1970, equation (10)) derived a coupling coefficient which should be proportional to the one from equation (13) here, while Giles (1974, equation (6.7)) derived a coupling coefficient which differs from that obtained by Bulgakov *et al.* only by the inclusion of an additional term associated with a finite sound speed. These coupling coefficients, when Giles's additional term is omitted, are proportional to ($\Omega > 0$ by definition here)

$$v_i^* v_j' v_l'' \left(\frac{k_i}{\omega}\delta_{jl} + \frac{k_j'}{\omega'}\delta_{il} + \frac{k_l''}{\omega''}\delta_{ij} - \frac{\mathrm{i}\Omega(\omega''\boldsymbol{k}' - \omega'\boldsymbol{k}'').\boldsymbol{b}}{\omega\omega'\omega''}\varepsilon_{ijl}\right) = C(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}''). \qquad \text{(A1)}$$

The quantity $\boldsymbol{v}$ (which is Giles's $\boldsymbol{e}$) is proportional to the perturbed fluid velocity of the electrons due to the wave motion; apart from normalization, $\boldsymbol{v}$ and the polarization vectors in equation (13) are related by

$$v_i = \tau_{ij}(\omega)\, e_j, \qquad v_i' = \tau_{ij}(\omega')\, e_j', \qquad v_i'' = \tau_{ij}(\omega'')\, e_j''. \tag{A2}$$

To establish the equivalence of the results of Bulgakov *et al.* and Giles with the result of MS, it must be shown that $C(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')$ is proportional to $\alpha(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'')$ as given by equation (13) (the superscripts σ, σ' and σ'' are omitted for simplicity).

Starting from equations (2), one has

$$\alpha(k,k',k'') = v_a^*\, v_b'\, v_c''\, \tau_{ai}^{-1}(\omega)\, \tau_{jb}^{-1}(\omega')\, \tau_{lc}^{-1}(\omega'')\, \alpha_{ijl}(\boldsymbol{k},\omega;\boldsymbol{k}',\omega';\boldsymbol{k}'',\omega''), \tag{A3}$$

with τ_{ij}^{-1} defined by

$$\tau_{ij}^{-1}\, \tau_{jl} = \delta_{il} \tag{A4}$$

and given explicitly by

$$\tau_{ij}^{-1}(\omega) = \delta_{ij} + \mathrm{i}(\Omega/\omega)\varepsilon_{ijl}\, b_l. \tag{A5}$$

Explicit evaluation of equation (A3), using equations (14) and (A5), gives

$$\alpha(k,k',k'') = v_a^*\, v_b'\, v_c'' \left\{ \frac{k_a}{\omega}\delta_{bc} + \frac{k_b'}{\omega'}\delta_{ac} + \frac{k_c''}{\omega''}\delta_{ab} - \frac{\mathrm{i}\Omega}{\omega\omega'\omega''}\Big(A_a\, \varepsilon_{bcl} + A_b\, \varepsilon_{cal} + A_c\, \varepsilon_{abl} \Big) b_l \right\}, \tag{A6}$$

where $\omega = \omega' + \omega''$ and $\boldsymbol{k} = \boldsymbol{k}' + \boldsymbol{k}''$ have been used to obtain the equalities

$$\boldsymbol{A} = \omega''\boldsymbol{k}' - \omega'\boldsymbol{k}'' = \omega''\boldsymbol{k} - \omega\boldsymbol{k}'' = \omega\boldsymbol{k}' - \omega'\boldsymbol{k}. \tag{A7}$$

The equality

$$\alpha(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') = C(\boldsymbol{k},\boldsymbol{k}',\boldsymbol{k}'') \tag{A8}$$

then follows from the identity

$$\delta_{ir}\, \varepsilon_{stj} + \delta_{is}\, \varepsilon_{trj} + \delta_{it}\, \varepsilon_{rsj} \equiv \delta_{ij}\, \varepsilon_{rst}. \tag{A9}$$

Appendix 2

For $\Omega \ll \omega_\mathrm{p}$ the Z-mode exists in the range

$$\omega_\mathrm{p} - \tfrac{1}{2}\Omega \leqslant \omega \leqslant (\omega_\mathrm{p}^2 + \Omega^2 \sin^2\theta)^{\frac{1}{2}}\,; \tag{A10}$$

it has a cutoff ($\mu = 0$) at the lower frequency limit, a resonance ($\mu = \infty$) at the upper limit, and a refractive index $\mu = 1$ at $\omega = \omega_\mathrm{p}$ (the case $\omega = \omega_\mathrm{p}$, $\sin\theta = 0$ is excluded). Near the resonance, thermal corrections are important and the waves may be identified as longitudinal electron plasma waves (see Appendix I*c* of MS).

Range $\omega_p \lesssim \omega \leqslant (\omega_p^2 + \Omega^2 \sin^2\theta)^{\frac{1}{2}}$

For $\omega \gtrsim \omega_p$ the properties of the Z-mode may be approximated by

$$\mu^2 \approx \frac{Y^2(1 - X\cos^2\theta)}{X-(1 - Y^2\sin^2\theta)}, \quad T \approx \frac{(1-X)\cos\theta}{Y\sin^2\theta}, \quad K \approx -\frac{Y\sin\theta}{X-(1-Y^2\sin^2\theta)}. \tag{A11}$$

The resonance occurs at $X = 1 - Y^2\sin^2\theta$. One can decide whether the waves are to be regarded as Z-mode or longitudinal, as a first approximation, as follows. Suppose the waves are Z-mode and then include finite thermal motions and calculate the correction $\Delta\mu^2$ to μ^2. If this correction is small ($\Delta\mu^2 \ll \mu^2$), the waves can be regarded as Z-mode, but, if the corrections are large ($\Delta\mu^2 \gg \mu^2$), the waves should be regarded as longitudinal.

The correction $\Delta\mu^2$ may be calculated as follows. Ignore thermal motions to find μ^2 and $\boldsymbol{e}$, include thermal motions in a correction $\Delta\varepsilon_{ij}(\boldsymbol{k},\omega)$ to the dielectric tensor, and then solve the equation (cf. MS, equation (A1))

$$e_i^* e_j \Lambda_{ij}(\boldsymbol{k},\omega) = \Delta\mu^2(|\boldsymbol{\kappa}.\boldsymbol{e}|^2 - 1) + e_i^* e_j \Delta\varepsilon_{ij}(\boldsymbol{k},\omega) = 0 \tag{A12}$$

for $\Delta\mu^2$. Because, according to equations (A11), the waves are nearly longitudinal, one has

$$e_i^* e_j \Delta\varepsilon_{ij}(\boldsymbol{k},\omega) \approx \varepsilon^l(\boldsymbol{k},\omega) = -\mu^2 X\beta_e^2 f(Y^2,\theta), \tag{A13}$$

with $\beta_e = V_e/c$, and where equation (A15) of MS has been used. (There is an error in the expression of MS (equation (A16)) for $f(Y^2,\theta)$: the denominator in the penultimate term should be $Y^2(1-Y^2)^3$.) Here one has $X \approx 1$ and $f(Y^2,\theta) \approx 3$. Hence, using the equation for μ^2 in (A11), one finds

$$\Delta\mu^2/\mu^2 \approx -3\mu^4\beta_e^2/Y^2\sin^2\theta. \tag{A14}$$

For $\omega = \omega_p$ one has $\mu = 1$ and equation (A14) implies that for

$$3\beta_e^2\,\omega_p^2/\Omega^2\sin^2\theta \ll 1 \tag{A15}$$

the waves should be treated as Z-mode waves.

Range $\omega_p - \frac{1}{2}\Omega \leqslant \omega \lesssim \omega_p$

To find approximate analytic expressions for μ^2 and K for $\omega \lesssim \omega_p$ the formulae (MS, equations (A12) and (A13))

$$\mu^2 = 1 - \frac{XT}{T - Y\cos\theta}, \qquad K = \frac{XY\sin\theta}{1-X}\,\frac{T}{T-Y\cos\theta} \tag{A16}$$

may be used together with some approximate expression for T. For the Z-mode, one has $T = -\cos\theta$ at the cutoff at $X = 1 + Y$, that is, at $\omega \approx \omega_p - \frac{1}{2}\Omega$, and $T = 0$ at $X = 1$ ($\omega = \omega_p$). The function

$$\tilde{T} = \{(1-X)\cos\theta\}/Y \tag{A17}$$

is equal to T at these two points. A reasonable approximation for μ^2 and K is obtained by equating T to $\tilde{T}$ in equations (A16):

$$\mu^2 \approx \frac{(1-X)^2 - Y^2}{(1-X) - Y^2}, \qquad K \approx \frac{XY\sin\theta}{1-X-Y^2}. \tag{A18}$$

In the text the range of interest is $(1-X)^2 \ll Y^2 \ll X-1$, where equations (A18) reduce to

$$\mu^2 \approx Y^2/(X-1), \qquad K \approx (-Y\sin\theta)/(X-1). \tag{A19}$$

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