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The Nature of *D*-region Scattering of Vertical Incidence Radio Waves. I Generalized Statistical Theory of Diversity Effects between Spaced Receiving Antennas

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Abstract

A theoretical basis for studying the nature of ionospherically reflected radio waves is described. The expected statistical properties of the amplitude and phase variations of the radio wave ground pattern sampled at a pair of spaced receiving antennas are derived for general correlation conditions, assuming different models of the angular spectrum of reflected radio waves. The results provide a basis for experimental measurements of the angular spread and coherence ratio of the angular spectrum of reflected waves.

Introduction

Subject to certain assumptions, the measured statistical properties of the diffraction pattern formed by ionospherically reflected radio waves may be used to derive parameters which describe both the angular spectrum of the reflected waves and the nature of the ionospheric reflecting region. These parameters include:

(i) the angular spread of downcoming reflected waves,

(ii) the ratio of the contributions of specular and random components in the spectrum (sometimes referred to as the 'signal to noise ratio' or 'coherence ratio'),

(iii) the scale or size of the reflecting ionospheric irregularities, and

(iv) the ionospheric drift velocity.

In the present paper attention is confined to parameters (i) and (ii). Two possible definitions of the coherence ratio arise, depending on whether signal amplitudes or signal powers are considered. In the present discussion, the amplitude coherence ratio is used.

Coherence ratios can be determined from amplitude probability distributions alone using the theory of Rice (1945) as a basis. However, in the present theory a more generalized approach based on the use of amplitude and phase information is used. This theory enables the nature of the reflection process to be described in terms of both the angular spread and the coherence ratio of the spectrum of the reflected waves. The theory is based mainly on the work of Bramley (1951) in which the statistical properties of signals received by a spaced pair of antennas were considered. However, the validity of Bramley's work is confined to cases where the two received signals are highly correlated. Since practical situations often arise in which this condition is not met, his work is generalized here to remove this restriction.

Theory

Specular reflection with time-varying direction of arrival

It is convenient to consider first the simple case of a single specular reflection whose direction of arrival varies with time. If we assume that the ray is confined to a vertical plane containing the line joining the centres of two receiving antennas, the phase difference ϕ between the antennas at an instant when the ray is arriving at an angle θ from the vertical is given by

$$\phi = (2\pi d \sin\theta)/\lambda, \tag{1}$$

where d is the antenna spacing and λ is the probing wavelength. Rearrangement of the above expression shows that for small angles the mean absolute deviation $\langle|\theta|\rangle$ of the ray from vertical is given by

$$\langle|\theta|\rangle = \lambda\langle|\phi|\rangle/2\pi d. \tag{2}$$

The quantity $\langle|\theta|\rangle$ may be used as a convenient measure of the angular spread of the received signal over some given observing period.

Model 1

In model 1, a spaced pair of receiving antennas is assumed to be under the influence of a continuous distribution of randomly phased reflected waves which is confined to the vertical plane containing the line joining the centres of the spaced antennas. It is convenient to consider the amplitude and phase statistics for this model separately.

Amplitude Statistics

Let $W(\theta)\,\mathrm{d}\theta$ be the power returned from an angular sector $\theta \pm \frac{1}{2}\,\mathrm{d}\theta$, then the total power received from the entire contributing plane is

$$W_0 = \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} W(\theta)\,\mathrm{d}\theta. \tag{3}$$

Following Bramley (1951) we define a correlation function R for a pair of antennas separated by a distance d by

$$R^2 = (\alpha^2+\beta^2)/W_0^2, \tag{4}$$

where

$$\alpha = \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} W(\theta)\cos\left(\frac{2\pi d\sin\theta}{\lambda}\right)\mathrm{d}\theta \quad \text{and} \quad \beta = \int_{-\frac{1}{2}\pi}^{+\frac{1}{2}\pi} W(\theta)\sin\left(\frac{2\pi d\sin\theta}{\lambda}\right)\mathrm{d}\theta. \tag{5,6}$$

The form of R thus depends on $W(\theta)$. On the assumption that $W(\theta)$ may be represented by the gaussian function

$$W(\theta) = \{W_0/(2\pi)^{\frac{1}{2}}\theta_0\}\exp(-\tfrac{1}{2}(\theta/\theta_0)^2), \tag{7}$$

where θ_0 is the r.m.s. angular half-width of the power distribution (the quantity we wish to determine by experiment), then for small θ_0 it can be shown (Bramley 1951) that

$$R = \exp(-2(\pi d\theta_0/\lambda)^2). \tag{8}$$

The determination of R by experiment would thus allow θ_0 to be calculated. In practice it is more convenient to determine the amplitude correlation coefficient ρ_A which is related to R and is given by

$$\rho_A = \langle(A_1-\bar{A}_1)(A_2-\bar{A}_2)\rangle / \langle(A_1-\bar{A}_1)^2(A_2-\bar{A}_2)^2\rangle^{\frac{1}{2}}, \tag{9}$$

where overhead bars are used to denote averages of A_1 and A_2, and angle brackets are used to denote averages of combinations of them.

The relationship between ρ_A and R may be derived by calculating ρ_A from the joint probability distribution $P(A_1, A_2)$ of the amplitudes A_1 and A_2 received at the two antennas, which is given (Rice 1945) by

$$P(A_1, A_2) = \frac{A_1 A_2}{W_0^2(1-R^2)}\, I_0\left(\frac{RA_1 A_2}{W_0^2(1-R^2)}\right)\exp\left(-\frac{(A_1^2+A_2^2)}{2W_0(1-R^2)}\right). \tag{10}$$

The required relationship between ρ_A and R is derived from this distribution by calculating the appropriate averages of equation (9). This relation involves elliptic integrals of R and is intractable for determining θ_0 from equation (8) unless it is assumed that $R \approx 1$ (i.e. correlation between the two antennas is high), in which case the following result (Bramley 1951) is obtained:

$$(4-\pi)(1-\rho_A) = 1-R^2 \approx (2\pi d\theta_0/\lambda)^2. \tag{11}$$

Rearranging this to give θ_0 in terms of ρ_A we obtain

$$\theta_0 = \{\lambda/2\pi d\}\{(4-\pi)(1-\rho_A)\}^{\frac{1}{2}}. \tag{12}$$

Thus determining ρ_A by experiment allows θ_0 to be calculated provided, however, that the correlation between the spaced receiving antennas is high ($R \approx 1$).

An alternative approach which allows θ_0 to be evaluated for general values of R can be derived from the simple relation which exists between R and ρ_{A^2} (the correlation coefficient between squared amplitudes) and which is valid for general values of R. Equation (10) yields the following quantities needed for evaluating ρ_{A^2}

$$\langle A_1^2 A_2^2\rangle = 4W_0^2(1+R^2) \quad \text{and} \quad \langle A_1^4\rangle = \langle A_2^4\rangle = 8W_0^2. \tag{13a, b}$$

The relation between ρ_{A^2} and R then simply becomes

$$\rho_{A^2} = R^2. \tag{14}$$

Combining equations (8) and (14) then gives

$$\theta_0 = (\lambda/2\pi d)\{-\ln(\rho_{A^2})\}^{\frac{1}{2}}, \tag{15}$$

which enables θ_0 to be obtained by an experimental determination of ρ_{A^2} without restriction on the degree of correlation between the spaced receiving antennas used to determine ρ_{A^2}.

Another estimate of θ_0 from measurements of amplitude differences between the spaced receiving antennas is possible. Bramley (1951) has shown that the joint probability distribution (10) allows the distribution $p(x)$ of amplitude differences $x = A_1 - A_2$ to be derived. For $R \approx 1$ (high correlation) this distribution is normal and may be used to derive an expression for the difference correlation coefficient ρ_D, which can be evaluated by experiment. Bramley showed that

$$\rho_D \equiv \langle |x| \rangle / \bar{A} = 2\pi^{-1}(1-R^2)^{\frac{1}{2}}. \tag{16}$$

Combining equations (16) and (8) gives

$$\theta_0 = \tfrac{1}{4}\lambda \rho_D / d, \tag{17}$$

which allows θ_0 to be evaluated by determining ρ_D from experiment, provided that the correlation between the spaced antennas used to determine ρ_D is high ($R \approx 1$).

An expression for θ_0 in terms of ρ_D which is valid for general values of R may be derived by using the following expression for $p(x)$ (Fürth and MacDonald 1947) which is valid for general values of R:

$$p(x) = \left(\frac{1-Z^2}{8\bar{A}^2}\right)^{\frac{1}{2}} \exp\left(-\frac{\pi x^2}{4\bar{A}^2(1-Z^2)}\right) \int_0^\infty \frac{(1+\xi)\exp(\pi x^2 \xi / 8\bar{A}^2(1-Z^2))}{\xi^{\frac{1}{2}}\{(1+\xi)^2 - Z^2\}^{3/2}}\, \mathrm{d}\xi, \tag{18}$$

where $Z = \exp(-\frac{1}{2}(1-R^2))$. Using this distribution we now obtain

$$\langle |x| \rangle = 2\int_0^\infty x\, p(x)\, \mathrm{d}x = \bar{A}(2-\sqrt{2})\{1-\exp(R^2-1)\}^{\frac{1}{2}}, \tag{19}$$

and thus we have

$$\rho_D \equiv \langle |x| \rangle / \bar{A} = (2-\sqrt{2})\{1-\exp(R^2-1)\}^{\frac{1}{2}}. \tag{20}$$

Combining equations (20) and (8) then gives

$$\theta_0 = (\lambda/2\pi d)\{-\ln(\ln(1-2{\cdot}914\,\rho_D^2)+1)\}^{\frac{1}{2}}. \tag{21}$$

This expression is valid for general values of R, and allows θ_0 to be evaluated from an experimental determination of ρ_D.

Phase Statistics

The angular spread of a continuous distribution of randomly phased reflected rays may also be determined from a knowledge of the difference of the phase variations of the signals received at a pair of spaced antennas. Rice (1945) calculated the form of the joint probability distribution $p(A_1, A_2, \psi_1, \psi_2)$, where A_1, A_2, ψ_1 and ψ_2 are the amplitudes and phases of the signals received at two spaced antennas. The distribution $p(\phi)$ of phase differences $\phi = \psi_1 - \psi_2$ can now be derived from this distribution (Bramley 1951) and is given by

$$p(\phi) = \frac{1-R^2}{2\pi}\left(\frac{1}{1-R^2\cos^2\phi} + \frac{R\cos\phi}{(1-R^2\cos^2\phi)^{3/2}}\{\tfrac{1}{2}\pi + \arcsin(R\cos\phi)\}\right). \tag{22}$$

The mean absolute phase difference $\langle|\phi|\rangle$ between the spaced antennas can now be calculated and leads to the simple result

$$\langle|\phi|\rangle = 2\int_0^\infty \phi\, p(\phi)\, \mathrm{d}\phi = \arccos R. \tag{23}$$

Combining equations (23) and (8) and putting $R \approx 1$ gives

$$\theta_0 = \lambda\langle|\phi|\rangle/2\pi d. \tag{24}$$

The corresponding expression for θ_0 which is valid for general values of R is obtained by directly combining equations (23) and (8), which leads to the expression

$$\theta_0 = (\lambda/\sqrt{2}\,\pi d)\{-\ln(\cos\langle|\phi|\rangle)\}^{\frac{1}{2}}. \tag{25}$$

This expression has also been independently derived by Meeklah *et al.* (1972), and allows θ_0 to be calculated after experimentally determining $\langle|\phi|\rangle$.

At this point it is interesting to note from a comparison of equations (24) and (2) that the same expression involving $\langle|\phi|\rangle$ gives an estimate of the angular spread that is appropriate even for two models of opposite extremes: a single specular reflection whose direction of arrival varies with time, and a fan of randomly phased rays. It therefore seems likely that the same expression may give a reasonable estimate of the angular spread regardless of the detailed nature of the angular spectrum, for any general situation could be regarded as a combination of these two models. It is to be noted, however, that equation (24) was derived on the assumption that $R \approx 1$ so that this generalization might be expected to apply only to situations where the correlation between the spaced receiving antennas is high. This restriction may be expressed in quantitative terms which are of more practical use by examining equation (25) which holds for general values of R. Using power series expansions, equation (25) becomes

$$\theta_0 = \frac{\lambda\langle|\phi|\rangle}{2\pi d}\left(1+\frac{\langle|\phi|\rangle^2}{2^2.2}+\frac{\langle|\phi|\rangle^4}{2^2.3}+\ldots\right)^{\frac{1}{2}} \qquad \text{for} \qquad \langle|\phi|\rangle < 1.$$

From this relation it can readily be deduced that $\theta_0 = \lambda\langle|\phi|\rangle/2\pi d$ gives the angular spread to within 5% of the 'true' value provided that we have $\langle|\phi|\rangle < 0{\cdot}75$. In practice the antenna spacing could be chosen so that this condition holds, and thus should allow θ_0 to be accurately determined regardless of the nature of the reflection process.

Model 2

In model 2, we consider the case of a continuous distribution of randomly phased rays in the presence of a strong signal of fixed amplitude and direction of arrival. This type of situation might arise in practice when a stable strong reflector has weak fine-scale irregularities either superimposed on it or situated below it. The derivation of the theory describing such a situation is rather complicated unless it is assumed that the amplitude coherence ratio b is such that $b > 1$. The following discussion is subject to this assumption. Again we consider the amplitude and phase statistics separately.

Amplitude Statistics

Bramley (1951) showed that, for a continuous distribution of randomly phased rays in the presence of a strong steady signal, the amplitude correlation coefficient between a pair of spaced receiving antennas is given by

$$\rho_A = W_0^{-1} \int_0^{2\pi} W(\theta) \cos((2\pi d/\lambda)(\cos\theta - \cos\theta_s))\, d\theta, \tag{26}$$

where θ_s is the off-vertical angle of arrival of the strong steady signal component and $W(\theta)$ is the angular power distribution of the randomly phased components. If $W(\theta)$ is assumed to have the gaussian form (7) about $\theta = \theta_s$ then the combination of equations (26) and (3) gives

$$\rho_A = R. \tag{27}$$

If we have $R \approx 1$, equations (8) and (27) give

$$\theta_0 = (\lambda/\sqrt{2}\pi d)\{\tfrac{1}{2}(1-\rho_A)\}^{\frac{1}{2}}. \tag{28}$$

Thus a measurement of ρ_A can provide an estimate of the angular spread θ_0 of received radiation around the mean direction θ_s.

The relation between θ_0 and ρ_A for general values of R may be obtained by deducing the analytic expression for θ_0 from equations (8) and (27). The result is

$$\theta_0 = (\lambda/\sqrt{2}\pi d)\{-\ln(\rho_A)\}^{\frac{1}{2}}. \tag{29}$$

This result has also been independently derived by Meeklah *et al.* (1972). An estimate of the coherence ratio b can be made from a measurement of ρ_A as well as the amplitude difference correlation coefficient ρ_D. Bramley (1951) obtained the following result which is valid for general values of R:

$$b = \rho_D^{-1}\{2(1-\rho_A)/\pi\}^{\frac{1}{2}}. \tag{30}$$

Phase Statistics

For the case of a strong coherent signal in the presence of randomly phased incoherent components, Bramley (1951) showed that the resultant phase at a receiving antenna is normally distributed and that the correlation coefficient ρ_ψ between the phases at a spaced pair of receiving antennas becomes equal to the amplitude correlation coefficient ρ_A. Equation (27) thus gives

$$\rho_\psi = R. \tag{31}$$

Combining equations (31) and (8) we find

$$\theta_0 = (\lambda/\sqrt{2}\pi d)\{-\ln(\rho_\psi)\}^{\frac{1}{2}}. \tag{32}$$

This expression is valid for general values of R and provides a means for evaluating θ_0 from an experimental determination of ρ_ψ.

The distribution of the phase difference ϕ between spaced receiving antennas was given by Bramley (1951) as

$$p(\phi) = [b/\{2(1-R)\}^{\frac{1}{2}}]\exp(-\tfrac{1}{2}b^2\phi^2/(1-R)). \tag{33}$$

Assuming $R \approx 1$ and using the relation

$$\langle|\phi|\rangle = 2\int_0^\pi \phi\, p(\phi)\,\mathrm{d}\phi\,, \tag{34}$$

we obtain

$$\langle|\phi|\rangle = b^{-1}\{2(1-R)/\pi\}^{\frac{1}{2}}. \tag{35}$$

Re-arranging this expression and using equation (27), we convert the expression for b into

$$b = \langle|\phi|\rangle^{-1}\{2(1-\rho_A)/\pi\}^{\frac{1}{2}}, \tag{36}$$

showing that b may be evaluated from an experimental determination of $\langle|\phi|\rangle$ and ρ_A. The validity of equations (35) and (36), however, is subject to the restriction that $R \approx 1$. Using equations (27), (33) and (34), we obtain the following general expression

$$\langle|\phi|\rangle = b^{-1}\{2(1-\rho_A)/\pi\}^{\frac{1}{2}}\{1-\exp(-\tfrac{1}{2}b^2\pi/(1-\rho_A)\}. \tag{37}$$

Unfortunately it is not possible to re-arrange this formula to obtain an analytic expression for b, and it becomes necessary to evaluate b either graphically or numerically from an experimental measurement of $\langle|\phi|\rangle$ and ρ_A. In comparing equations (36) and (37), however, calculations show that the values of b determined separately from these equations show good agreement for $b \geqslant 1$. Since the scattering model assumes $b > 1$ it is clear that in practice the generalized equation (37) introduces a negligible correction in the evaluation of b, and thus it is adequate to use equation (36) which allows b to be evaluated more easily and directly.

Conclusions

Models of the ionospheric scattering of vertical incidence radio waves have been described in terms of the diversity effects produced at a spaced pair of receiving antennas. The mean absolute direction of arrival of a single specular reflection whose direction of arrival varies with time is given by equation (2), and this bears a similar relation to the mean absolute phase difference between spaced receiving antennas as does the angular spread for a continuous distribution of randomly phased rays (equation 24). It is expected that the mean absolute phase difference may be used to determine the angular spread independently of the detailed nature of the angular spectrum of reflected waves, provided an antenna spacing is chosen such that the mean absolute phase difference is less than 0·75 radians.

Three independent estimates of the angular spread of a continuous distribution of randomly phased rays may be obtained from equations (15), (21) and (25), which apply for general correlation conditions between spaced receiving antennas. For a model assuming a continuous distribution of randomly phased rays in the presence of a steady specular component, two independent estimates of the angular spread may be obtained using equations (29) and (32), and two independent estimates of the coherence ratio b are possible using equations (30) and (36).

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