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# A Relativistic Treatment of Space–Time Rays and the Doppler Effect in General Media

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*Abstract*

For waves in plasmas and similar media, space–time rays with a 4-vector formulation are used to obtain an approximate expression for the rapidly varying part of the wave, on the assumption that the plasma properties vary slowly in space and time. It is shown that the phase function associated with the rapid variations is invariant with respect to Lorentz transformation and that the ray equations may be written in vector form. These results are used to obtain expressions for the Doppler shift that are consistent with special relativity. The possibility that the medium is moving and time-varying is allowed for, and a variety of possible sources are considered.

## 1. Introduction

Many radio-astronomical observations and space physics experiments involve radio waves propagating over long paths through plasmas. The plasma medium is in general anisotropic and dispersive, and it may change with time as the wave propagates. The transmitter, receiver and plasma may be in relative motion with velocities not negligible in comparison with the free space velocity of light. Supposing that there exists a frame in which the plasma properties vary slowly in space and time, we wish to obtain rays which describe the rapidly varying part of the wave correctly within the framework of special relativity.

For waves propagating in a medium varying slowly in space and time, an approximate solution with a rapidly varying factor $\exp(iS)$ (where $S$ is a function of position both in physical space and time) may be constructed using space–time rays. The approximate solution behaves locally like a plane wave, and the local wave number and frequency are appropriate local derivatives of the phase function $S$ (Synge 1954; Whitham 1961; Poeverlein 1962; Lewis 1965; Rawer and Suchy 1967). Now it is well known that in a homogeneous medium (one in which the medium's properties do not depend upon position or time) there exist plane wave solutions with a phase function that is invariant with respect to Lorentz transformation. In consequence, the 4-gradient of this phase function (which is constructed from the wave number and frequency, see Section 3) transforms like a Minkowski 4-vector (Papas 1965). Furthermore, for the special case of time harmonic plane waves in a homogeneous medium, the space–time ray construction is exact and leads to a function $S$ which is precisely the invariant phase just described.

These remarks taken together suggest that in the general case $S$ may be invariant with respect to Lorentz transformation, and this is shown in Section 4*b* to be the case using a 4-vector approach. The transformation laws for the ray equations are derived in Section 4*c*. From these results, expressions (51) and (52) below are obtained for the Doppler frequency shift experienced when a transmitter and receiver are moving in a medium, which varies slowly in space and time, and which may also be dispersive and anisotropic. These expressions are consistent with the special theory of relativity and generalize results obtained previously for the nonrelativistic case by different treatments (Bennett 1975; Brandstatter and Censor 1974). In Sections 2 and 3, an outline of the necessary background theory is given in 4-vector notation.

## 2. Equation of Eikonal and its Solution by Rays

In the usual way (Lawden 1962) we may introduce the 4-vector $\boldsymbol{X} = (\boldsymbol{x}, \mathrm{i}ct)$ and the operator $\square = (\nabla, -\mathrm{i}c^{-1}\,\partial/\partial t)$ which is sometimes written as $\partial/\partial\boldsymbol{X}$. To avoid further complications, the propagation is assumed to be lossless. Substitution of the proposed approximate solution in the equations governing the propagation leads to the 'equation of the eikonal', a partial differential equation for $S$, that may be written in the form

$$F(\square S, \boldsymbol{X}) = 0. \tag{1}$$

Generally this equation will have multiple sheets so that there is more than one solution for $S$, but in most of the following we concentrate on one of these solutions. Equation (1) may be solved by the method of characteristics, the characteristics being space–time rays. The ray or Hamilton equations then take the compact form (Synge 1954; Poeverlein 1962; Rawer and Suchy 1967)

$$-\boldsymbol{K}' = \theta\,\partial F/\partial\boldsymbol{X} \qquad \text{and} \qquad \boldsymbol{X}' = \theta\,\partial F/\partial\boldsymbol{K}, \tag{2}$$

where $F(\boldsymbol{K}, \boldsymbol{X})$ is obtained by replacing $\square S$ by $\boldsymbol{K}$ in the expression on the left-hand side of equation (1). The operator $\partial/\partial\boldsymbol{K}$ is defined in an analogous way to $\partial/\partial\boldsymbol{X}$, and a prime is used to denote differentiation with respect to $u$, where $u$ is a convenient (real) parameter increasing along the ray. The system (2) is solved subject to

$$F(\boldsymbol{K}, \boldsymbol{X}) = 0. \tag{3}$$

As is discussed in greater detail in the next section, equation (3) is the local dispersion relation for the existence of plane wave solutions. The multiplying function $\theta$ depends upon the choice of $u$ and the particular form in which $F(\boldsymbol{K}, \boldsymbol{X})$ is written. For reasons that will become clearer in the next section we write $\boldsymbol{K} = (\boldsymbol{k}, \mathrm{i}\omega c^{-1})$.

The function $S$ can be determined by integrating along the rays to give

$$S_{\mathrm{B}} - S_{\mathrm{A}} = \int_{u_{\mathrm{A}}}^{u_{\mathrm{B}}} \boldsymbol{K}.\boldsymbol{X}'\,\mathrm{d}u, \tag{4}$$

where $S_{\mathrm{B}}$ and $S_{\mathrm{A}}$ indicate the values of $S$ at space–time points lying on the same ray. The value of $S$ throughout the region of interest is found by integrating along a family of rays emanating from an initial manifold on which data for $S$ are given. The initial values of $\boldsymbol{K}$ needed to integrate the Hamilton equations are determined by the data for $S$ and a consistency requirement. It is usually necessary to resolve a source or initial disturbance into modes associated with the various sheets of the eikonal equation, but we do not need to dwell on this aspect of the problem.

Typically $S = S_0$ is a known function on a given (hyper)surface. The surface may be described parametrically by $\boldsymbol{X} = \boldsymbol{X}_0(\zeta_\alpha)$ for $\alpha = 1,2,3$. Thus $S = S_0(\zeta_\alpha)$ is known on $\boldsymbol{X}_0(\zeta_\alpha)$. Then for a general point $\boldsymbol{X}$, we have

$$S(\boldsymbol{X}) = S_0(\boldsymbol{X}_0) + \int_{u(\boldsymbol{X}_0)}^{u(\boldsymbol{X})} \boldsymbol{K}.\boldsymbol{X}'\,\mathrm{d}u\,, \tag{5}$$

where the integral is taken along the ray from $\boldsymbol{X}_0(\zeta_\alpha)$ that passes through $\boldsymbol{X}$. Here, in order to simplify the notation, the point of intersection with the initial surface is simply denoted $\boldsymbol{X}_0$ (see Fig. 1). As $\boldsymbol{X}$ is varied the point of intersection ranges over the initial surface. The corresponding value of $\boldsymbol{K} = \boldsymbol{K}_0$ is chosen so that

$$\mathrm{d}S_0/\mathrm{d}\zeta_\alpha = \boldsymbol{K}_0\,.\,\mathrm{d}\boldsymbol{X}_0/\mathrm{d}\zeta_\alpha\,. \tag{6}$$

This ensures that the eikonal equation is satisfied in $\boldsymbol{X}_0(\zeta_\alpha)$ (Lewis 1965; Bennett 1974). (In order that, with the given data, the eikonal equation be properly posed it is necessary that the rays leave $\boldsymbol{X}_0(\zeta_\alpha)$; otherwise the solution cannot be carried outside the initial hypersurface, there being no unique solution.)

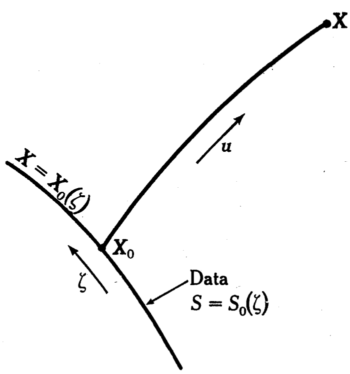



**Fig. 1.** Parametric description of the ray and the initial surface.

As the data play a vital role in determining the rays appropriate to a given problem, we consider equation (6) in more detail. If the initial surface is space like, equation (6) becomes

$$\mathrm{d}S_0/\mathrm{d}\zeta_\alpha = \boldsymbol{k}_0\,.\,\mathrm{d}\boldsymbol{x}_0/\mathrm{d}\zeta_\alpha\,. \tag{7}$$

For example, for the simple case of *plane waves* with $S_0(\zeta_\alpha) = \boldsymbol{\kappa}\,.\,\boldsymbol{x}_0(\zeta_\alpha)$, where $\boldsymbol{\kappa}$ is a constant 3-vector, equation (7) is immediately satisfied by choosing $\boldsymbol{k}_0 = \boldsymbol{\kappa}$ for the space vector corresponding to the first three components of $\boldsymbol{K}_0$. Then substitution in equation (3) determines the fourth component $\mathrm{i}\omega_0 c^{-1}$.

The initial manifold may be of lower dimension, e.g. a *stationary point source of sinusoidal waves* (see Fig. 4*a* below). In this case only one parameter $\zeta$ is needed. The manifold is a time-like line with $S_0(\zeta) = -\omega_0\,t(\zeta)$. Substitution in equation (3) leads in general to a two-parameter family of $\boldsymbol{k}_0$. These $\boldsymbol{k}_0$ may depend upon $\zeta$ if $F(\boldsymbol{K}, \boldsymbol{X})$ is an explicit function of time in the initial manifold. Similarly slow variations of $\omega_0$ with time may be dealt with, as may also the case of moving sources with nearly sinusoidal time dependence (Fig. 4*b* below), but we defer consideration of this case until Section 5*c*. Another interesting and important case is that in which the initial manifold shrinks to a space–time point (Fig. 4*c* below). This represents an *impulsive disturbance* which contains essentially all frequencies. The initial $\boldsymbol{K}_0$ vectors then form a three-parameter family, comprising all the solutions of equation (3) with $\boldsymbol{X} = \boldsymbol{X}_0$ that correspond to rays in the forward light cone.

## 3. Local Plane Wave Behaviour of Approximate Solution

The equation

$$F(\boldsymbol{K}, \boldsymbol{X}_\mathrm{B}) = 0 \tag{8}$$

for fixed $\boldsymbol{X}_\mathrm{B}$ is the dispersion relation for the existence of plane waves proportional to

$$\exp(\mathrm{i}\boldsymbol{K}.\boldsymbol{X}) \tag{9}$$

in a medium having everywhere the properties of the actual medium at $\boldsymbol{X}_\mathrm{B}$ (Synge 1954; Whitham 1961; Poeverlein 1962; Lewis 1965; Rawer and Suchy 1967). This leads to the familiar interpretation $\boldsymbol{K} = (\boldsymbol{k}, \mathrm{i}\omega c^{-1})$ where $\boldsymbol{k}$ is the 3-space wave number and $\omega$ the angular frequency. The approximate solution behaves like a plane wave near $\boldsymbol{X}_\mathrm{B}$ because $\boldsymbol{K}_\mathrm{B}$, as determined from the Hamilton equations, satisfies equation (8). Furthermore, if the wave amplitude is calculated it is found to have a polarization corresponding to the appropriate sheet of equation (8), although this is not in general true if higher order terms are included (e.g. Bennett 1974*b*). It is also true that

$$\boldsymbol{K}_\mathrm{B} = \Box_\mathrm{B} S, \tag{10}$$

in clear analogy with the result of differentiating the phase $\boldsymbol{K}.\boldsymbol{X}$ of the expression (9). We may refer to $\boldsymbol{K}_\mathrm{B}$ as the *local 4-vector wave number* and, correspondingly, to $\boldsymbol{k}_\mathrm{B}$ and $\omega_\mathrm{B}$ as the local wave number and local frequency respectively.

The preceding remarks are obvious from the standpoint of an approach via the equation of the eikonal and the method of characteristics. If, instead, $S$ is taken as being defined merely by the rays, i.e. by equations (2)–(6), then from equation (3) it is clear that (8) is satisfied. However, it is not obvious that $\boldsymbol{K}_\mathrm{B} = \Box_\mathrm{B} S$. The differential properties of the phase function in three dimensions have been discussed at great length (Bennett 1973). Precisely analogous results apply in four dimensions, and these may be used to establish directly the equality of $\boldsymbol{K}_\mathrm{B}$ and $\Box_\mathrm{B} S$. Concentrating on a ray through the point $\boldsymbol{X}_\mathrm{B}$ and leaving the initial surface at $\boldsymbol{X}_\mathrm{A}$, we find on taking differentials in equation (5)

$$\mathrm{d}S = \mathrm{d}S_0 + \int_\mathrm{A}^\mathrm{B} (\mathrm{d}\boldsymbol{K}.\boldsymbol{X}' + \boldsymbol{K}.\mathrm{d}\boldsymbol{X}')\,\mathrm{d}u\,. \tag{11}$$

Integrating by parts in the second term, this equation becomes

$$\mathrm{d}S = \mathrm{d}S_0 + \int_\mathrm{A}^\mathrm{B} (\mathrm{d}\boldsymbol{K}.\boldsymbol{X}' - \boldsymbol{K}'.\mathrm{d}\boldsymbol{X})\,\mathrm{d}u + \boldsymbol{K}_\mathrm{B}.\mathrm{d}\boldsymbol{X}_\mathrm{B} - \boldsymbol{K}_\mathrm{A}.\mathrm{d}\boldsymbol{X}_\mathrm{A}\,. \tag{12}$$

The increments in equation (12) all depend upon $\mathrm{d}\boldsymbol{X}_\mathrm{B}$ which is regarded as the increment of the independent variable. However, from equation (3) it follows that

$$\frac{\partial F}{\partial \boldsymbol{K}}.\mathrm{d}\boldsymbol{K} + \frac{\partial F}{\partial \boldsymbol{X}}.\mathrm{d}\boldsymbol{X} = 0\,. \tag{13}$$

Multiplying through by $\theta$ in equation (13) and making use of Hamilton's equations (2) we find that the integrand in (12) is zero. From the condition (6) on the value of $\boldsymbol{K}$ at the initial surface we have

$$\mathrm{d}S_0 - \boldsymbol{K}_\mathrm{A}.\mathrm{d}\boldsymbol{X}_\mathrm{A} = 0, \tag{14}$$

and only the end point term $\boldsymbol{K}_\mathrm{B}.\mathrm{d}\boldsymbol{X}_\mathrm{B}$ remains. Equation (10) then follows. This argument may be looked upon as providing the reason why the method of characteristics works. The characteristics are trajectories along which the gradient of the eikonal $S$ may be determined by the solution of a system of *ordinary* differential equations, the Hamilton or ray equations.

## 4. Transformation Laws

We consider a Lorentz transformation from the initial frame to a frame moving with a relative velocity $\boldsymbol{v}$. The origins are assumed to coincide at $t = 0$, and quantities in the new frame are distinguished by overhead bars. Thus, for example, $\boldsymbol{X} \rightarrow \bar{\boldsymbol{X}}$ or, in terms of components (using the summation convention),

$$\bar{X}_i = A_{ij} X_j \qquad \text{for} \qquad i, j = 1, \dots 4, \tag{15}$$

where $A_{ij}$ is orthogonal so that

$$A_{ij} A_{ik} = \delta_{jk}. \tag{16}$$

For the special case when $\boldsymbol{v}$ is in the direction of the $X_1$ axis, $A_{ij}$ takes the form (see e.g. Lawden 1962)

$$\begin{bmatrix} \cos\alpha & 0 & 0 & \sin\alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sin\alpha & 0 & 0 & \cos\alpha \end{bmatrix}, \tag{17}$$

with

$$v = -\mathrm{i}c \tan\alpha. \tag{18}$$

### (*a*) *Transformation of Dispersion Relation*

We consider a homogeneous medium and therefore write the dispersion relation as

$$F(\boldsymbol{K}) = 0, \tag{19}$$

on suppressing the dependence upon $\boldsymbol{X}$. It is well known, and easily proved using covariance principles, that $\boldsymbol{K}$ transforms as a vector for every possible plane wave. There therefore exists a transformed dispersion relation

$$\bar{F}(\bar{\boldsymbol{K}}) = 0 \tag{20}$$

which is satisfied if $\bar{\boldsymbol{K}}$ is the result of transforming a solution $\boldsymbol{K}$ of the original dispersion relation (19).

It is possible to write the dispersion relation so that we have

$$F(\boldsymbol{K}) = \bar{F}(\bar{\boldsymbol{K}}) \tag{21}$$

for every $\boldsymbol{K}$, that is, if every $\boldsymbol{K}$ transforms as a vector then $F(\boldsymbol{K})$ determines an *invariant field* (Lawden 1962, p. 29) over the $\boldsymbol{K}$ space. The truth of this statement may be demonstrated by example. We may write

$$F(\boldsymbol{K}) = \boldsymbol{K}.\boldsymbol{K} - [\omega^2 c^{-2}\{n^2(\hat{\boldsymbol{k}},\omega) - 1\}], \tag{22}$$

where $n(\hat{\boldsymbol{k}},\omega)$ is the refractive index for wave normal direction $\hat{\boldsymbol{k}}$ (that is, $\hat{\boldsymbol{k}}$ is a unit 3-vector) and frequency $\omega$. The refractive index represents the properties of the medium. More generally $F(\boldsymbol{K})$ is a product of terms like the right-hand side of equation (22). Now the scalar product

$$\boldsymbol{K}.\boldsymbol{K} = \boldsymbol{k}.\boldsymbol{k} - \omega^2 c^{-2} \tag{23}$$

is invariant. By definition of the refractive index, we have

$$\boldsymbol{k}.\boldsymbol{k} = \omega^2 n^2(\hat{\boldsymbol{k}},\omega)\, c^{-2} \tag{24}$$

for any plane wave solution of $F(\boldsymbol{K}) = 0$. Therefore from equation (23) it follows that

$$\omega^2 c^{-2}\{n^2(\hat{\boldsymbol{k}},\omega) - 1\}$$

is invariant if $\omega$ and $\hat{\boldsymbol{k}}$ are transformed in accordance with the transformation law for $\boldsymbol{K}$. (This invariance was used by Unz 1968.) Thus we have

$$\omega^2 c^{-2}\{n^2(\hat{\boldsymbol{k}},\omega) - 1\} = \bar{\omega}^2 c^{-2}\{\bar{n}^2(\hat{\bar{\boldsymbol{k}}},\bar{\omega}) - 1\}, \tag{25}$$

the left-hand side of which may be written

$$\gamma^2(\bar{\omega} - \bar{\boldsymbol{k}}.\boldsymbol{v})^2\{n^2(\hat{\boldsymbol{k}}(\bar{\omega},\bar{\boldsymbol{k}}), \gamma(\bar{\omega} - \bar{\boldsymbol{k}}.\boldsymbol{v})) - 1\}c^{-2}, \tag{26}$$

where we have substituted for $\omega, \hat{\boldsymbol{k}}$ in terms of $\bar{\omega}, \bar{\boldsymbol{k}}$. In this equation we have (Papas 1965)

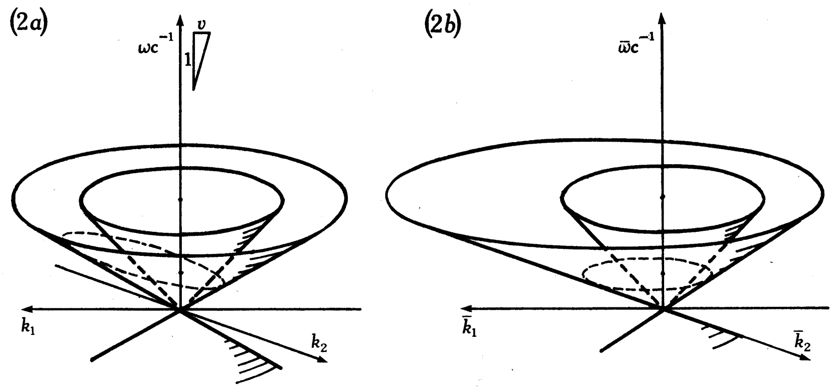
$$\gamma = (1 - v^2 c^{-2})^{-\frac{1}{2}} \quad \text{and} \quad \hat{\boldsymbol{k}}(\bar{\omega},\bar{\boldsymbol{k}}) = \frac{\bar{\boldsymbol{k}} + \gamma\bar{\omega}c^{-2}\boldsymbol{v} + (\gamma - 1)(\bar{\boldsymbol{k}}.\boldsymbol{v}/v^2)\boldsymbol{v}}{[\{\bar{\boldsymbol{k}} + \gamma\bar{\omega}c^{-2}\boldsymbol{v} + (\gamma - 1)(\bar{\boldsymbol{k}}.\boldsymbol{v}/v^2)\boldsymbol{v}\}^2]^{\frac{1}{2}}}. \tag{27,28}$$

Equation (28) represents the aberration, which is zero when $\bar{\boldsymbol{k}}$ is parallel to $\boldsymbol{v}$. From equations (22), (25) and (26) we may write

$$\bar{F}(\bar{\boldsymbol{K}}) = \bar{\boldsymbol{K}}.\bar{\boldsymbol{K}} - \gamma^2(\bar{\omega} - \bar{\boldsymbol{k}}.\boldsymbol{v})^2\{n^2(\hat{\boldsymbol{k}}(\bar{\omega},\bar{\boldsymbol{k}}), \gamma(\bar{\omega} - \bar{\boldsymbol{k}}.\boldsymbol{v})) - 1\}c^{-2}, \tag{29}$$

which makes $F(\boldsymbol{K})$ invariant in the sense of equation (21) for all $\boldsymbol{K}$, and not only those values which satisfy the dispersion relation.

The dispersion relation is a hypersurface, or a number of hypersurfaces, in $(\boldsymbol{k}, \omega c^{-1})$ space. Topology is maintained by the transformation from equation (19) to (20), e.g. distinct sheets remain distinct, lines of intersection transform into lines, etc. However, the transformation is nonlinear and nonorthogonal with respect to a Euclidean metric, so that the shape of the surface is changed. Fig. 2 represents in the initial frame (Fig. 2*a*) and the transformed frame (Fig. 2*b*) the three-dimensional projection of the simple case of a medium which is isotropic and nondispersive in the initial frame. A section of constant $\bar{\omega}$ is seen to be an ellipse, a simple conic section, in the initial frame since, for constant $\bar{\omega}$ surfaces, $\omega c^{-1} + \boldsymbol{k}.\boldsymbol{v}c^{-1}$ is a constant. This maps into a horizontal plane in the new frame. On interpreting equation (25), upon writing $n^2-1$ as $(n+1)(n-1)$, it can be seen that each point maps so that the product of its maximum and minimum distances from the unit cone (measured radially) is invariant. In other words the geometric mean of the distances is invariant.



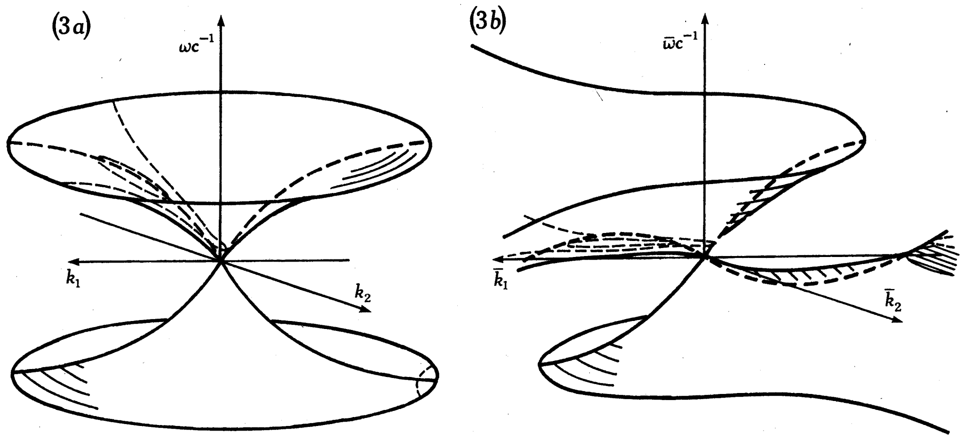




**Fig. 2.** Representation of a section ($k_3 = 0$) of the dispersion surface for a nondispersive isotropic medium showing the intersection (fine dashed curve) with a typical surface of constant $\bar{\omega}$, as it appears (*a*) in the initial frame and (*b*) for a frame moving at velocity $\boldsymbol{v}$ with respect to the medium. Also shown is the unit cone, the dispersion surface for free space.

**Fig. 3.** Representation of a section of the dispersion surface for very slow plasma waves illustrating the possibility of multiple intersections (fine dashed curves) with a surface of constant $\bar{\omega}$, as it appears (*a*) in the initial frame and (*b*) for a frame moving at velocity $\boldsymbol{v}$ with respect to the medium.

In Fig. 3 a more interesting situation is illustrated. The dispersion relation is assumed to flare out as it does when a plasma wave of very slow phase velocity and group velocity is generated. While topologically equivalent, the shape of the transformed surface is quite different. While in the rest frame the intersection with a surface of constant frequency is a single closed curve, in the transformed frame there may be multiple curves. This multiplicity is associated with the occurrence of the *complex Doppler effect*. The part of the sheet lying above the crosshatched region at the right-hand side of Fig. 3*b* is associated with the occurrence of *faster than light* radiation, and the boundary of the crosshatched region is associated with the Cerenkov effect (Frank 1960; Barsukov 1962; Bennett 1975). For a numerical example of a dispersion relation of this type, see Bitoun *et al.* (1970).

(*b*) *Invariance of S*

We now consider the inhomogeneous medium and apply the theory of the previous subsection to each point $X$. Thus equation (21) becomes

$$F(\boldsymbol{K},\boldsymbol{X}) = \bar{F}(\bar{\boldsymbol{K}},\bar{\boldsymbol{X}}), \tag{30}$$

and $F(\boldsymbol{K},\boldsymbol{X})$ determines an invariant field over both $\boldsymbol{K}$ and $\boldsymbol{X}$ spaces (while in equations (22) and (24) $n^2(\hat{\boldsymbol{k}},\omega)$ is replaced by $n^2(\hat{\boldsymbol{k}},\omega,\boldsymbol{X})$). Now, if $S(\boldsymbol{X})$ is a solution of $F(\Box S,\boldsymbol{X}) = 0$ with particular data for $S$ then, provided the data are invariant, $S(\bar{\boldsymbol{X}})$ is the corresponding solution of

$$\bar{F}(\overline{\Box S},\bar{\boldsymbol{X}}) = 0. \tag{31}$$

This follows by noting first that, from equation (30), the transformation of $\Box S$ as a 4-vector leads to a solution of the transformed eikonal equation. Assuming that the solution is uniquely determined by the data, the restriction on the data leads to the conclusion that $\overline{\Box S}$ is the 4-vector gradient of the transformed solution $\bar{S}$. (It is quite possible, but not physically reasonable, to specify data for $S$ that are not invariant.) Though unique in the small, the solution for $S$ is multiple valued, as noted previously. The order of multiplicity is also invariant.

(*c*) *Transformation Laws for Rays*

If $\boldsymbol{K}$ is a vector and $F(\boldsymbol{K},\boldsymbol{X})$ is written in an invariant form, i.e. one satisfying equation (30), then it follows immediately that $\partial F/\partial\boldsymbol{K}$ is a vector. (Henceforth we assume $F$ is so written without further comment.) Also, it is immediately seen that $\partial F/\partial\boldsymbol{X}$ is a vector. Thus, if in Hamilton's equations (2) $\theta$ is an invariant, the right-hand sides are vectors. Hence the left-hand sides are vectors and $u$ is an invariant. The ray equations in this form are vector equations.

In particular, by taking $u = \tau$, the time in a frame following the motion of a point along a ray, we can make $\mathrm{d}\boldsymbol{X}/\mathrm{d}\tau$ a 4-vector velocity. Let us put

$$\theta = \Theta \equiv \mathrm{i}c\{(\partial F/\partial\boldsymbol{K}).(\partial F/\partial\boldsymbol{K})\}^{-\frac{1}{2}}, \tag{32}$$

with the square root chosen so that $\Theta$ is positive. This expression is clearly an invariant. Consequently we have

$$\frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau} = \frac{\mathrm{i}c\,(\partial F/\partial\boldsymbol{k},\, -\mathrm{i}c\,\partial F/\partial\omega)}{\{(\partial F/\partial\boldsymbol{k}).(\partial F/\partial\boldsymbol{k}) - c^2(\partial F/\partial\omega)^2\}^{\frac{1}{2}}} \tag{33}$$

and, for those parts of the dispersion surface for which $\partial F/\partial\omega$ is negative, we have

$$\mathrm{d}\boldsymbol{X}/\mathrm{d}\tau = \gamma_{\mathrm{g}}(\boldsymbol{v}_{\mathrm{g}}, \mathrm{i}c), \tag{34}$$

where

$$\boldsymbol{v}_{\mathrm{g}} = -\frac{\partial F/\partial\boldsymbol{k}}{\partial F/\partial\omega} \tag{35}$$

is the group velocity and

$$\gamma_g = (1-v_g^2 c^{-2})^{-\frac{1}{2}}. \tag{36}$$

The expression (34) is in typical form for a 4-vector velocity and is thus the *4-vector group velocity* which we denote $\boldsymbol{V}_g$. It should be noted that

$$\Theta = -\gamma_g(\partial F/\partial\omega)^{-1}. \tag{37}$$

The former restriction on the sign of $\partial F/\partial\omega$ ensures that the rays enter the forward light cone as $\tau$ increases. If the other sign is chosen in equation (32) then we require $\partial F/\partial\omega$ to be positive.

Using (37) the ray equations (2) may be written

$$\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}\tau} = \gamma_g\frac{\partial F/\partial \boldsymbol{X}}{\partial F/\partial\omega} \quad\text{and}\quad \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}\tau} = -\gamma_g\frac{\partial F/\partial \boldsymbol{K}}{\partial F/\partial\omega}. \tag{38}$$

Choosing $u$ such that $\gamma_g\,\mathrm{d}\tau = -\mathrm{d}u(\partial F/\partial\omega)$, the ray equations become

$$-\mathrm{d}\boldsymbol{K}/\mathrm{d}u = \partial F/\partial\boldsymbol{X} \quad\text{and}\quad \mathrm{d}\boldsymbol{X}/\mathrm{d}u = \partial F/\partial\boldsymbol{K} \tag{39}$$

(cf. Synge 1954, equation 2.1.17; Rawer and Suchy 1967, equation 12.5) which are also vector forms. From the equations (38), making use of the fact that $\mathrm{d}t = \gamma_g\,\mathrm{d}\tau$, where $t$ is time measured in the frame of $\boldsymbol{X}$, we have

$$\frac{\mathrm{d}\boldsymbol{K}}{\mathrm{d}t} = \frac{\partial F/\partial \boldsymbol{X}}{\partial F/\partial\omega} \quad\text{and}\quad \frac{\mathrm{d}\boldsymbol{X}}{\mathrm{d}t} = -\frac{\partial F/\partial \boldsymbol{K}}{\partial F/\partial\omega}. \tag{40}$$

This last form is of course not a vector system but may still be convenient for computations.

Notice that, from the system (38) or (39) and the ray integral expression in equation (5) for $S$, it can be shown directly that $S$ is invariant if the data for $S$ are invariant, for then in addition the initial values of $\boldsymbol{K}$ for each ray transform as vectors. Hence $\boldsymbol{K}$ determined along each ray transforms as a vector, and both $\boldsymbol{K}.\boldsymbol{X}' = \Theta\boldsymbol{K}.\partial F/\partial\boldsymbol{K}$ and the integral in equation (5) are invariant.

## 5. Doppler Shift of Instantaneous Frequency

### (*a*) *Instantaneous Frequency*

We define the instantaneous frequency observed by a receiver at B as

$$\omega_R = |\mathrm{d}S/\mathrm{d}\tau_B|, \tag{41}$$

where $\tau_B$ is proper time in the frame of the receiver. This is a natural generalization to the special relativistic case of the idea of instantaneous frequency used by Ville (1948), Lewis and Pressman (1967) and others. However, since frequency is classically defined in terms of a spectral analysis involving an integral taken over all time, assuming certain characteristics of the signal are time stationary, a brief explanation may be useful.

Making use of the invariance of our formulae we may transform to a frame moving with the receiver (at least near the time of interest if the receiver should be accelerating). We see that the instantaneous frequency is a local frequency in the sense discussed in Section 3. If a wave corresponding to a single sheet of the equation of the eikonal is received, the instantaneous frequency is a measurable quantity, if not instantaneously, at any rate averaged over short time intervals. For example, the time between zero values of the wave field may be measured.

A practical spectral analyser will analyse only the signal arriving during a finite time interval. If, for a simple wave of the type described, the amplitude and instantaneous frequency of the wave are both constant during this time, the resulting spectrum would have a centre frequency $\omega_R$ and a width corresponding to the time interval characterizing the analyser. It would naturally be interpreted as corresponding to a line spectrum of frequency $\omega_R$. Variations of the amplitude or instantaneous frequency during the time interval would lead to spectral broadening.

If a number of such simple waves are received, the individual instantaneous frequencies $\omega_{Ri}$, say, are no longer directly measurable unless the waves may be distinguished by their polarization properties or directions of arrival. However, the practical spectrum is still made up of bands of power at the $\omega_{Ri}$. If the instantaneous frequency is not nearly constant, either because the time scale of the changes is less than the analyser time interval or because several waves with near-constant $\omega_{Ri}$ are combined to yield an 'effective instantaneous frequency', it must not be taken for granted that the time average of the instantaneous frequency is equal to the centre frequency of the spectrum (Ville 1948).

### (*b*) *Doppler Shift of Instantaneous Frequency*

From the definition (41) it immediately follows that

$$\omega_R = |(\Box_B S) \cdot \mathrm{d}\boldsymbol{X}_B/\mathrm{d}\tau_B|, \tag{42}$$

but $\mathrm{d}\boldsymbol{X}_B/\mathrm{d}\tau_B$ is the 4-vector velocity $\boldsymbol{V}_B = \gamma_B(\boldsymbol{v}_B, ic)$, say, of the receiver. Making use of equation (10) we thus have

$$\omega_R = |\boldsymbol{K}_B \cdot \boldsymbol{V}_B| \tag{43}$$

or, in terms of 3-vectors,

$$\omega_R = \gamma_B |\boldsymbol{k}_B \cdot \boldsymbol{v}_B - \omega_B|. \tag{44}$$

It should be noticed that, if $\boldsymbol{v}_B = 0$ so that $\boldsymbol{V}_B = (0, ic)$, equation (44) gives $\omega_R = \omega_B$. Clearly $\omega_R$ determined from equation (43) is invariant, as we would hope, since it is a proper frequency.

### (*c*) *Source Effects*

In order to be able to evaluate $\omega_R$ from equation (43) the ray paths must be determined. Although the ray equations must be integrated as a whole, the nature of the problem becomes clearer if the equation for $\boldsymbol{K}$ is formally integrated. Thus, taking the form (40) for Hamilton's equations, we have

$$\boldsymbol{K}_B = \boldsymbol{K}_A + \int_{t_A}^{t_B} \frac{\partial F/\partial \boldsymbol{X}}{\partial F/\partial \omega} \, \mathrm{d}t, \tag{45}$$

where the integral is taken along the ray from A (coordinates $\boldsymbol{X}_A$) to B. The fourth component of equation (45) becomes

$$\omega_{\mathrm{B}} = \omega_{\mathrm{A}} - \int_{t_{\mathrm{A}}}^{t_{\mathrm{B}}} \frac{\partial F/\partial t}{\partial F/\partial \omega}\, \mathrm{d}t \tag{46}$$

or, on evaluating the derivatives for $F$ as in equation (22) with $n$ a function of $\boldsymbol{X}$,

$$\omega_{\mathrm{B}} = \omega_{\mathrm{A}} - \int_{t_{\mathrm{A}}}^{t_{\mathrm{B}}} \frac{\omega\, \partial n/\partial t}{\partial(\omega n)/\partial \omega}\, \mathrm{d}t. \tag{47}$$

Clearly equation (47) is independent of the particular form in which the dispersion relation is written since it involves only partial derivatives of the refractive index. The derivatives are to be evaluated holding the wave-normal direction fixed. The same result was obtained earlier for the nonrelativistic case (Fante 1972; Bennett 1974*a*). The denominator $\partial(\omega n)/\partial\omega$ is usually known as the group refractive index. It should be emphasized that the integrals in equations (46) and (47) can only be evaluated if the system (40) is integrated as a whole so that both the path of integration and $\boldsymbol{K}$ along the path are known. Here we are particularly interested in specifying $\omega_{\mathrm{A}}$. For a *point source of time harmonic waves* of frequency $\omega_0$, clearly we have $\omega_{\mathrm{A}} = \omega_0$ (see Fig. 4*a* and Section 2). If more generally $\omega_0$ varies slowly with time so that

$$S_0(\zeta) = -\int^{\zeta} \omega_0\big(t(\zeta)\big)\,\{\mathrm{d}t(\zeta)/\mathrm{d}\zeta\}\, \mathrm{d}\zeta \tag{48}$$

then

$$\omega_{\mathrm{A}} = \omega_0(t_{\mathrm{A}}). \tag{49}$$

We may also consider negative frequencies but no information is gained thereby.

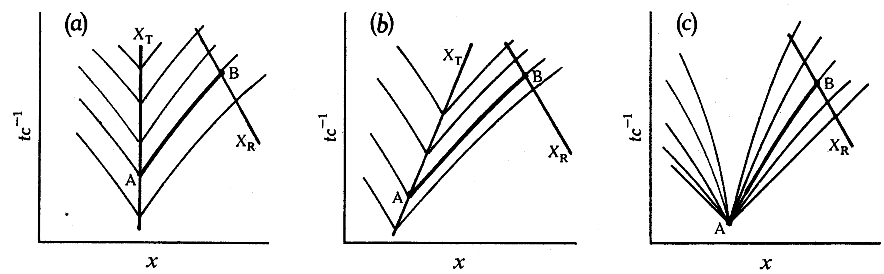



**Fig. 4.** Schematic representation of the world lines $X_{\mathrm{T}}$ and $X_{\mathrm{R}}$ of the source and receiver respectively and the rays necessary to determine the Doppler frequency, for the following cases: (*a*) stationary sinusoidal source, (*b*) moving sinusoidal source, (*c*) impulsive disturbance.

For a *moving transmitter* we suppose that a ray pattern of the same general character as that for the stationary source exists (see Fig. 4*b*). From the equivalent of equation (43) applied at the endpoint A it follows that the frequency in the frame of the transmitter is

$$\omega_{\mathrm{T}} = |\boldsymbol{K}_{\mathrm{A}} . \boldsymbol{V}_{\mathrm{A}}|, \tag{50}$$

where $\boldsymbol{V}_{\mathrm{A}}$ is the 4-vector velocity of the transmitter at A. (The modulus signs may be avoided if the source is considered to transmit both positive and negative frequencies; for negative frequencies the other square root must be taken in the expression (32) for $\Theta$.)

The total Doppler shift between transmitter and receiver may be written

$$\omega_R - \omega_T = |\boldsymbol{K}_B \cdot \boldsymbol{V}_B| - |\boldsymbol{K}_A \cdot \boldsymbol{V}_A| \tag{51}$$

or more explicitly, making use of equations (44) and (47),

$$\omega_R(t_B) - \omega_T(t_A) = \gamma_B \left| \boldsymbol{k}_B \cdot \boldsymbol{v}_B - \omega_A + \int_{t_A}^{t_B} \frac{\omega\, \partial n/\partial t}{\partial(\omega n)/\partial \omega}\, \mathrm{d}t \right| - \gamma_A \left| \boldsymbol{k}_A \cdot \boldsymbol{v}_A - \omega_A \right|. \tag{52}$$

Equation (52) is a natural generalization of the corresponding nonrelativistic formula (Bennett 1975). Indeed the same result was obtained earlier using special relativistic formulae at both endpoints of the ray (Bennett 1971), although at that time it was not realized that the treatment of nonstationarity of the medium was also relativistically correct.

The apparent simplicity of equation (51) should not be allowed to obscure the fact that data are given for $\omega_T$ and not $\omega_A$. The relation between the two is nonlinear and may be multiple valued. Even if $\omega_T$ is zero, a charged body may give rise to a nonzero $\omega_A$ through the Cerenkov effect (Bennett 1975). One method of proceeding is to transform to a frame that is stationary with respect to the transmitter using the results of Section 4*a*.

For an *impulsive disturbance* (Fig. 4*c*) $\omega_A$ is the frequency corresponding to the ray through B. It will generally vary with the time at which $\omega_R$ is evaluated. If the medium is time stationary and plane stratified then in the nonrelativistic limit the result reduces to that obtained by Fejer and Wu (1970). Assuming the dispersion relation does not depend explicitly on time and that the receiver is stationary, this case reduces to that studied by Whitham (1961).

## Conclusions

We have obtained ray equations, correct within the framework of special relativity, which describe the rapidly varying features of waves propagating in media that vary slowly in time and space. The transmitter, receiver and medium may be in relative motion. In particular, relativistically correct formulae for the Doppler frequency shifts have been obtained. The results take a particularly concise form in a 4-vector notation.

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