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Finite Amplitude Convection in a Compressible Layer with Polytropic Structure

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Abstract

In this paper are derived the basic equations for finite amplitude convection in a compressible medium with polytropic structure within the framework of the one-mode anelastic approximation. Only the case of non self-interacting planforms is considered, and the effects of viscous dissipation are not taken into account. These equations are solved numerically to illustrate the effects of compressibility and density stratification on the flow pattern and on the thermodynamic variables and their fluctuations.

1. Introduction

It is likely that convection will occur somewhere in most stars, and it is therefore important to understand convective transport processes and their effect on stellar structure. There is considerable interest in this problem and in recent years a number of review articles have appeared on this subject (e.g. Brindley 1967; Spiegel 1971, 1972). The theory which has been most extensively used in stellar convection problems is the mixing-length theory (for additional references, see e.g. Cox and Giuli 1968). Unfortunately there does not seem to exist a non-Boussinesq version of the theory, and it is therefore unlikely to be sufficiently accurate when applied to stellar convection zones in which the density varies by several orders of magnitude.

The linear theory of convection in a compressible medium has been studied extensively, although attention has been focused mainly on the determination of growth rates in inviscid models (e.g. Skumanich 1955; Böhm 1958; Böhm and Richter 1959; Spiegel and Unno 1962; Spiegel 1964). The effect of viscosity and thermal diffusivity in the fully compressible linear case has been studied by Spiegel (1965) and also in the anelastic approximation by Unno *et al.* (1960) and Kato and Unno (1960).

Unfortunately, when it comes to the nonlinear theory of convection in a compressible medium, very little has appeared in the literature so far. The full two-dimensional compressible equations were solved numerically in 1972 (E. Graham, personal communication), and a number of research workers such as E. A. Spiegel, J. Toomre, J. P. Zahn, E. Graham and J. Latour are actively working on the problem. Latour (1972) applied the theory of finite amplitude convection to the study of thermal convection in A stars. The one-mode fully compressible equations were derived by the present author some years ago (Van der Borght 1971). It is the purpose of the present paper to provide numerical solutions of these equations within the framework of the anelastic approximation in the case when the undisturbed compressible layer has a polytropic structure.

2. Basic Equations

The basic equations have been derived previously (Van der Borght 1971), taking into account the viscous dissipation terms in the energy equation. If we neglect this effect and restrict ourselves to rolls or convective cells with square or rectangular planform, these equations can be written:

Continuity equations

$$W_0 = -P\psi/\rho_0, \tag{1}$$

$$\mathrm{D}(W_0 P) + \rho_0(\mathrm{D}\psi - \mathrm{D}W) + \psi\,\mathrm{D}\rho_0 = 0. \tag{2}$$

Energy equations

$$p_0\,\mathrm{D}W_0 + \Pi(-\mathrm{D}W + \mathrm{D}\psi) + \rho_0\,C_v(F\mathrm{D}W + \psi\mathrm{D}F + W_0\,\mathrm{D}T_0) \\ + C_v\,P(\psi\mathrm{D}T_0 + W_0\,\mathrm{D}F) = K\mathrm{D}^2T_0, \tag{3}$$

$$\rho_0\,C_v(\psi\mathrm{D}T_0 + W_0\,\mathrm{D}F) + C_v\,P(EF\mathrm{D}W + W_0\,\mathrm{D}T_0 + 3E\psi\mathrm{D}F) \\ + p_0(-\mathrm{D}W + \mathrm{D}\psi) + \Pi\mathrm{D}W_0 = K(\mathrm{D}^2 - a^2)F. \tag{4}$$

Equations of motion

$$\rho_0(\psi\mathrm{D}W + \psi\mathrm{D}\psi + W_0\,\mathrm{D}W_0) = -\mathrm{D}p_0 + g\rho_0 + \tfrac{4}{3}\mu\,\mathrm{D}^2W_0, \tag{5}$$

$$\rho_0\,\mathrm{D}(\psi W_0) + P\{W_0\,\mathrm{D}W_0 + E(\psi\mathrm{D}W + 3\psi\mathrm{D}\psi)\} \\ = -\mathrm{D}\Pi + gP + \mu(\mathrm{D}^2 - a^2)\psi + \tfrac{1}{3}\mu(\mathrm{D}^2\psi - \mathrm{D}^2W), \tag{6}$$

$$\mathrm{D}[\rho_0\,W_0\,\mathrm{D}^2W + P\{\tfrac{1}{2}a^{-4}(\mathrm{D}W)^2(a^4E - \bar{H}) + E\psi\mathrm{D}^2W\}] \\ = -a^2\mathrm{D}\Pi + \mu(-a^2\mathrm{D}^2W + \mathrm{D}^4W) + \tfrac{1}{3}\mu a^2(\mathrm{D}^2\psi - \mathrm{D}^2W). \tag{7}$$

Equations of state

$$p_0 = R_*(\rho_0\,T_0 + PF), \tag{8}$$

$$\Pi = R_*(\rho_0 F + T_0 P). \tag{9}$$

It should be noted that the pressure, density, temperature and velocity have been written as

$$p = p_0 + \Pi f, \qquad \rho = \rho_0 + Pf, \qquad T = T_0 + Ff, \tag{10}$$

$$\boldsymbol{u} = (u, v, w,) = \left(\frac{\mathrm{D}W}{a^2}\frac{\partial f}{\partial x}, \ \frac{\mathrm{D}W}{a^2}\frac{\partial f}{\partial y}, \ W_0 + \psi f\right). \tag{11}$$

Here the quantities p_0, Π, ρ_0, P, T_0, F, W, W_0 and ψ are functions of z that are to be determined, while $D \equiv \mathrm{d}/\mathrm{d}z$, and f is a function of x and y which satisfies the differential equation

$$\partial^2 f/\partial x^2 + \partial^2 f/\partial y^2 = -a^2 f, \tag{12}$$

with a being the horizontal wave number. The actual functional dependence of f on x and y is determined by the shape of the convection cell. The quantities μ, K and C_v are the viscosity, conductivity and specific heat at constant volume of the medium. We also have

$$E = \tfrac{1}{3}\langle f^4\rangle \quad \text{and} \quad \bar{H} = \langle\{(\partial f/\partial x)^2 + (\partial f/\partial y)^2\}^2\rangle, \tag{13}$$

where

$$\langle\ \ \rangle \equiv K\iint [\ \ ]\,\mathrm{d}x\,\mathrm{d}y, \tag{14}$$

the integration being taken over one convective cell. For a particular cell shape, the constant K is determined by the normalizing condition

$$\langle f^2\rangle = K\iint f^2\,\mathrm{d}x\,\mathrm{d}y = 1. \tag{15}$$

It should also be noted that

$$C = \tfrac{1}{2}\langle f^3\rangle \tag{16}$$

has been put equal to zero, which is the case of rolls and square or rectangular planforms for the convective cells.

3. Anelastic Approximation

We assume, within the framework of the anelastic approximation (Ogura and Phillips 1962; Dutton and Fichtl 1969; Gough 1969), that the equations can be linearized with respect to the fluctuating parts in the thermodynamic variables P, F and Π. It then follows from equation (1) that the mean vertical velocity W_0 is small and that its square can be neglected.

Neglect of the term in $W_0 P$ in equation (2) gives

$$\rho_0(\mathrm{D}\psi - \mathrm{D}W) + \psi \mathrm{D}\rho_0 = 0. \tag{17}$$

This equation will be automatically satisfied if one introduces a new variable W_1 defined by

$$\mathrm{D}W_1 = \rho_0 \mathrm{D}W \quad \text{and} \quad W_1 = \rho_0 \psi. \tag{18}$$

Within the anelastic approximation, expression (9) for the pressure fluctuation remains unchanged but expression (8) for the mean pressure can be written

$$p_0 = R_* \rho_0 T_0. \tag{19}$$

Making use of equations (1), (18) and (19) it can be shown that equation (3) can be written

$$\begin{aligned} K\mathrm{D}^2 T_0 = {} & -R_* T_0 W_1 \rho_0^{-1} \mathrm{D}P - R_* T_0 P \rho_0^{-1} \mathrm{D}W_1 \\ & + 2R_* T_0 P W_1 \rho_0^{-2} \mathrm{D}\rho_0 - W_1 \Pi \rho_0^{-2} \mathrm{D}\rho_0 + C_v \mathrm{D}(FW_1). \end{aligned} \tag{20}$$

The second energy equation (4) can also be considerably simplified by linearizing the equation in the fluctuating parts of the thermodynamic variables. Keeping in mind that from equation (1) the average velocity W_0 is a first-order quantity, one finds

$$K(\mathrm{D}^2 - a^2)F = \rho_0 C_v \psi \mathrm{D}T_0 + p_0(-\mathrm{D}W + \mathrm{D}\psi). \tag{21}$$

In the anelastic approximation (e.g. Gough 1969) the density ρ on the left-hand side of the equations of motion is replaced by the average density ρ_0. This corresponds, in our notation, to neglecting the density fluctuation term P in equations (5)–(7) except in the buoyancy term gP. Neglecting also squares in the average velocity W_0, one can easily see that these equations reduce to the following

$$\mathrm{D}p_0 = g\rho_0 + \tfrac{4}{3}\mu \mathrm{D}^2 W_0 - \rho_0 \psi(\mathrm{D}W + \mathrm{D}\psi), \tag{22}$$

$$\mu(\mathrm{D}^2 - a^2)\psi = \mathrm{D}\Pi - gP - \tfrac{1}{3}\mu(\mathrm{D}^2\psi - \mathrm{D}^2 W) + \rho_0 \mathrm{D}(\psi W_0), \tag{23}$$

$$\mu(\mathrm{D}^2 - a^2)\mathrm{D}W = a^2 \Pi - \tfrac{1}{3}\mu a^2(\mathrm{D}\psi - \mathrm{D}W) + \rho_0 W_0 \mathrm{D}^2 W. \tag{24}$$

Equations (20)–(24) are the basic equations of the problem, within the anelastic approximation.

The right-hand side of equation (20) is not a total differential and does not give, after integration, a simple 'local' definition of the convective flux. It simplifies considerably if one applies to this equation the usual Boussinesq approximation:

(i) Neglecting the pressure fluctuation Π, one obtains from equation (9)

$$\rho_0 F + T_0 P = 0. \tag{25}$$

Differentiating this equation and eliminating $\mathrm{D}P$ from equation (20) one gets

$$K\mathrm{D}^2 T_0 = (C_\mathrm{v} + R_*)\mathrm{D}(FW_1) + R_* W_1 \rho_0^{-1}\{P\mathrm{D}T_0 - F\mathrm{D}\rho_0\}. \tag{26}$$

(ii) In the Boussinesq approximation both the fluctuation in the thermodynamic variables P and F and the density and temperature gradients $\mathrm{D}T_0$ and $\mathrm{D}\rho_0$ are assumed to be small. It then follows that the last term on the right-hand side of equation (26) is of second-order and can be neglected within the framework of this approximation.

Keeping in mind that

$$C_\mathrm{p} = C_\mathrm{v} + R_*, \tag{27}$$

one finds that equation (26) reduces to

$$C_\mathrm{p}\mathrm{D}(FW_1) = K\mathrm{D}^2 T_0, \tag{28}$$

and this is the same equation as the one for the incompressible case, except that the specific heat at constant volume has been replaced by the specific heat at constant pressure. If one eliminates W and ψ from equations (21)–(24) with the help of (18) and (19) one obtains, besides equation (20),

$$K(\mathrm{D}^2 - a^2)F = C_\mathrm{v} W_1 \mathrm{D}T_0 - R_* T_0 \rho_0^{-1} W_1 \mathrm{D}\rho_0, \tag{29}$$

$$\mathrm{D}p_0 = \frac{g\rho_0 - 2W_1\rho_0^{-1}\mathrm{D}W_1 - R_*\rho_0 \mathrm{D}T_0 - \tfrac{4}{3}\mu \mathrm{D}^2(P \quad {}_1\rho_0^{-2})}{R_* T_0 - W_1^2 \rho_0^{-2}}, \tag{30}$$

$$\begin{aligned}\mu\rho_0^{-1}\mathrm{D}^3 W_1 = {} & \mu(2\rho_0^{-2}\mathrm{D}^2 W_1 + \tfrac{1}{3}a^2 W_1 \rho_0^{-2})\mathrm{D}\rho_0 \\ & + \mu\rho_0^{-2}(\mathrm{D}W_1)(\mathrm{D}^2\rho_0) - 2\mu\rho_0^{-3}(\mathrm{D}W_1)(\mathrm{D}\rho_0)^2 \\ & + \mu a^2 \rho_0^{-1}\mathrm{D}W_1 + a^2 R_*(\rho_0 F + T_0 P) - PW_1\rho_0^{-1}\mathrm{D}(\rho_0^{-1}\mathrm{D}W_1),\end{aligned} \tag{31}$$

$$
\begin{aligned}
R_* T_0 \mathrm{D}P = (g - R_* \mathrm{D}T_0)P + \mu\rho_0^{-1}\mathrm{D}^2 W_1 - \tfrac{7}{3}\mu \mathrm{D}W_1 \rho_0^{-2}\mathrm{D}\rho_0 \\
- \tfrac{4}{3}\mu W_1 \rho_0^{-2}\mathrm{D}^2\rho_0 + \tfrac{8}{3}\mu W_1 \rho_0^{-3}(\mathrm{D}\rho_0)^2 - R_* F \mathrm{D}\rho_0 \\
- R_* \rho_0 \mathrm{D}F - a^2 \mu W_1 \rho_0^{-1} + \rho_0 \mathrm{D}(P W_1^2 \rho_0^{-3}).
\end{aligned} \tag{32}
$$

It should be noted that the above system of fundamental equations, together with equation (20), is a tenth-order system in the unknowns T_0, F, ρ_0, W_1 and P.

4. Boundary Conditions

We assume that there is no overshooting at the two boundaries $z = z_0$ (upper boundary) and $z = z_0 + d$ (lower boundary), where d is the thickness of the layer. It follows from the expression (11) for the velocity, the definition (18b) of W_1 and the definition (1) of W_0 that this condition can be written

$$W_1 = 0 \quad \text{at} \quad z = z_0 \quad \text{and} \quad z = z_0 + d. \tag{33}$$

We also assume that the temperature fluctuations vanish at the boundaries, i.e.

$$F = 0 \quad \text{at} \quad z = z_0 \quad \text{and} \quad z = z_0 + d. \tag{34}$$

In the present paper we consider the case of free boundaries, i.e. we assume that the tangential stresses vanish on the bounding surfaces. Therefore we have

$$P_{zx} = \mu\left(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}\right) = 0 \quad \text{and} \quad P_{zy} = \mu\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) = 0. \tag{35}$$

Using the expression (11) for the velocity, we may reduce these two conditions to

$$\frac{\mathrm{D}^2 W}{a^2}\frac{\partial f}{\partial x} + \psi\frac{\partial f}{\partial x} = 0 \quad \text{and} \quad \frac{\mathrm{D}^2 W}{a^2}\frac{\partial f}{\partial y} + \psi\frac{\partial f}{\partial y} = 0. \tag{36}$$

Since these conditions have to be satisfied on the two boundary surfaces, irrespective of the x and y values, the following condition must be satisfied

$$\mathrm{D}^2 W + a^2 \psi = 0. \tag{37}$$

It then follows from the equations (18) that this condition can be written as follows

$$\mathrm{D}^2 W_1 - (\mathrm{D}W_1)\rho_0^{-1}\mathrm{D}\rho_0 + a^2 W_1 = 0 \tag{38}$$

and, if the condition (33) of no overshooting applies, we also have at z_0 and $z_0 + d$

$$\mathrm{D}^2 W_1 - (\mathrm{D}W_1)\rho_0^{-1}\mathrm{D}\rho_0 = 0. \tag{39}$$

So far we have derived six boundary conditions. Since the nonlinear system of differential equations to be integrated is of the tenth order, we have to add four more boundary conditions. These are discussed in greater detail in Section 7.

5. Dimensionless Form of Equations

The basic equations derived in Section 4 are not in dimensionless form. Before starting on the numerical integrations we need to convert these equations into a more suitable form using a series of scalings. We transform the equations (20) and (29)–(32) into their dimensionless form with the aid of the following substitutions

$$z \to \phi d \qquad \text{and} \qquad a \to a/d. \tag{40}$$

If the convective layer extends between the limits $z = z_0$ to $z = z_0 + d$, the dimensionless variable ϕ has the range

$$\phi_0 \leqslant \phi \leqslant \phi_0 + 1, \qquad \text{where} \qquad \phi_0 = z_0/d. \tag{41}$$

In addition we scale the dependent variables in the following way

$$W_1 \to \rho_{00}(gd)^{\frac{1}{2}} W_1, \qquad T_0 \to (gd/R_*)T_0, \qquad \rho_0 \to \rho_{00}\,\rho_0, \qquad \text{(42a, b, c)}$$

$$F \to (\mu g^{\frac{1}{2}}/R_* \rho_{00}\, d^{\frac{1}{2}})F, \qquad P \to (\mu/g^{1/2} d^{3/2})P. \qquad \text{(42d, e)}$$

With the help of the above substitutions the fundamental system of equations can be written in the dimensionless form

$$\rho_0^{-1}\,\mathrm{D}\rho_0 = \frac{\rho_0^2 - 2W_1\,\mathrm{D}W_1 - \rho_0^2\,\mathrm{D}T_0 - \frac{4}{3}\sigma H^{-1}\mathrm{D}^2(PW_1\,\rho_0^{-2})}{\rho_0^2\, T_0 - W_1^2}, \tag{43}$$

$$\begin{aligned} \mathrm{D}^3 W_1 = {} & (2\rho_0^{-1}\,\mathrm{D}^2 W_1 + \tfrac{1}{3}a^2 \rho_0^{-1}\, W_1)\mathrm{D}\rho_0 \\ & + \rho_0^{-1}(\mathrm{D}W_1)\mathrm{D}^2\rho_0 - 2\rho_0^{-2}(\mathrm{D}W_1)(\mathrm{D}\rho_0)^2 + a^2 \mathrm{D}W_1 \\ & + a^2(\rho_0^2 F + T_0\,\rho_0\, P) - PW_1\{\rho_0^{-1}\,\mathrm{D}^2 W_1 - \rho_0^{-2}(\mathrm{D}W_1)\mathrm{D}\rho_0\}, \end{aligned} \tag{44}$$

$$\begin{aligned} T_0\,\mathrm{D}P = {} & (1 - \mathrm{D}T_0)P + \rho_0^{-1}\,\mathrm{D}^2 W_1 - \tfrac{7}{3}(\mathrm{D}W_1)\rho_0^{-2}\,\mathrm{D}\rho_0 \\ & - \tfrac{4}{3}W_1\,\rho_0^{-2}\,\mathrm{D}^2\rho_0 + \tfrac{8}{3}W_1\,\rho_0^{-3}(\mathrm{D}\rho_0)^2 - F\mathrm{D}\rho_0 - \rho_0\,\mathrm{D}F \\ & - a^2 W_1\,\rho_0^{-1} + \rho_0\,\mathrm{D}(PW_1^2\,\rho_0^{-3}), \end{aligned} \tag{45}$$

$$(\mathrm{D}^2 - a^2)F = H(\mathrm{D}T_0)W_1 - (\gamma - 1)HT_0\,\rho_0^{-1}\, W_1\,\mathrm{D}\rho_0, \tag{46}$$

$$\begin{aligned} \mathrm{D}^2 T_0 = {} & \{(\gamma - 1)/\gamma\}\sigma\{-T_0\, W_1\,\rho_0^{-1}\,\mathrm{D}P - PT_0\,\rho_0^{-1}\,\mathrm{D}W_1 \\ & \qquad + 2T_0\,\rho_0^{-2}P(\mathrm{D}\rho_0)W_1 - W_1\,\rho_0^{-1}(\mathrm{D}\rho_0)(\rho_0 F + T_0 P)\} \\ & + \sigma\gamma^{-1}\mathrm{D}(FW_1). \end{aligned} \tag{47}$$

In these equations we have

$$H = C_{\mathrm{v}} g d^3 \rho_{00}^2/\mu K \qquad \text{and} \qquad \sigma = \mu C_{\mathrm{v}}/K, \tag{48}$$

where σ is the Prandtl number and

$$\gamma = 1 + R_*/C_{\mathrm{v}} \tag{49}$$

is the ratio of specific heats.

6. Linear Case

In the linear case the above fundamental equations take a much simpler form:

(i) The structure equations are now independent of the velocity amplitude and can be written

$$D^2 T_0 = 0 \quad \text{and} \quad \rho_0^{-1} D\rho_0 = (1 - DT_0)/T_0 . \tag{50a, b}$$

They can be integrated separately subject to the following boundary conditions at $\phi = \phi_0$

$$T_0 = \phi_0/(s+1) , \qquad DT_0 = 1/(s+1) , \qquad \rho_0 = 1 , \tag{51a, b, c}$$

resulting in a polytropic structure (of polytropic index s) of the undisturbed layer.

(ii) The perturbation equations (44), (45) and (46) are already linear in W_1, F and P, and remain unchanged for the linear case.

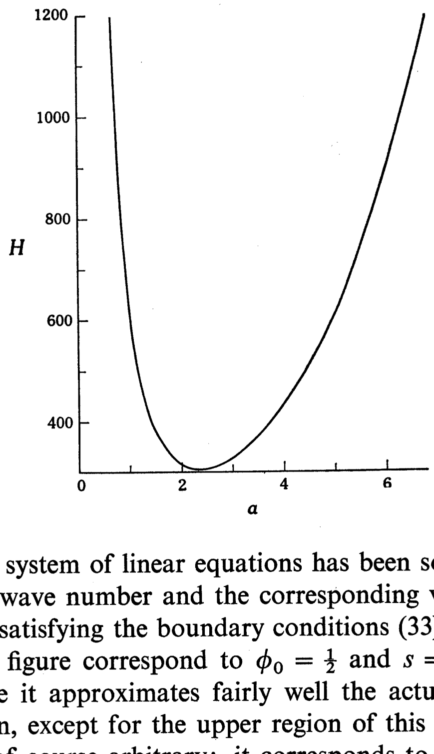


Fig. 1. Eigenvalue H in the linear problem (case B: $\phi_0 = \frac{1}{2}$, $s = 2$) plotted as a function of the horizontal wave number a.

The system of linear equations has been solved for a series of values of the horizontal wave number and the corresponding values of H are given in Fig. 1 for W_1 and F satisfying the boundary conditions (33), (34) and (39). The values of H given in this figure correspond to $\phi_0 = \frac{1}{2}$ and $s = 2$. This value of s has been chosen because it approximates fairly well the actual situation in the convective layer in the Sun, except for the upper region of this layer. The value of $\frac{1}{2}$ selected here for ϕ_0 is of course arbitrary; it corresponds to a temperature stratification of 3. This would correspond in the Sun to a layer extending from a depth of 1000 to 6000 km inside the convective zone.

The present theory could in fact be applied to the whole convective layer but the assumption of anelasticity introduced in Section 3 would have to be removed. More importantly, the effect of ionization and therefore the dependence of the specific heats on temperature and composition would have to be taken into consideration. In the following discussion, the values of C_v, C_p and γ are assumed to be constant throughout the layer which, at rest, is assumed to have a polytropic structure, i.e. we also assume that the conductivity K is a constant.

7. Nonlinear Solutions

The highly nonlinear tenth-order system of differential equations (43)–(47) has been solved by a finite-difference method coupled with a Newton–Raphson procedure, subject to the eight boundary conditions (33), (34), (39), (51a) and (51b), together with two additional boundary conditions that are introduced below. In order to illustrate the effect of initial density stratification on the character of steady-state stationary finite-amplitude convection, two families of solutions have been investigated: (A) those in which the polytropic index $s = 0{\cdot}2$ and the density varies by $2{\cdot}0406\%$ across the layer, and (B) those in which $s = 2$, corresponding to a 900% change in the density across the layer. In order to achieve such initial density stratifications, the parameter ϕ_0 must have the value of 10 in case A and $0{\cdot}5$ in case B.

In addition to the two parameters s and ϕ_0 which appear in the two boundary conditions (51a) and (51b) and which in essence determine the temperatures at the top and bottom of the layer, it is necessary to determine the values of the two parameters σ and γ. The value of σ is assumed here to be equal to one, which is the case for air. The ratio of specific heats γ will determine the initial buoyancy of the layer. In the linear case we have

$$\Delta = T_0^{-1} \mathrm{D} T_0 - (\gamma - 1)\rho_0^{-1} \mathrm{D}\rho_0 = \{1 - (\gamma - 1)s\}/\phi, \tag{52}$$

and in order to have a positive buoyancy it is necessary to choose γ in such a way as to ensure that the following inequality is satisfied:

$$\gamma < (1+s)/s. \tag{53}$$

In case A we take $\gamma = 1{\cdot}495$ and in case B we take $\gamma = 1{\cdot}4$.

We also have to select a value for the horizontal wave number a, and there is still some controversy about the way in which this value should be selected: whether the chosen value of a should correspond to maximum instability, i.e. to the minimum of the curve given in Fig. 1; or whether it should correspond to that particular value of a for which the Nusselt number is extremized. This particular value of a increases with the Rayleigh number and its determination requires a large number of numerical integrations. A compromise solution was adopted here, and the numerical integrations were carried out for $a = \pi$.

As was explained in Section 6, the linear system of equations can be decoupled into the structure equations (50) and the perturbation equations (44), (45) and (46). This system may be solved subject to the additional boundary condition (51c) and the resulting eigenvalues H are obtained. For case A we find that $H = 8909{\cdot}08$ and in case B that $H = 337{\cdot}884$.

It follows from the definition (48a) of the eigenvalue H that we have obtained in this way an expression relating the thickness of the layer d and the density ρ_{00} at the upper boundary, once the physical characteristics of the fluid such as C_v, μ and K are known. That ρ_{00} stands for the density at the upper boundary is a direct consequence of the boundary condition (51c) and the scaling (42c). Once a value of ρ_{00} has been selected, the eigenvalue H determines the minimum thickness of the layer for which convection is about to start.

The temperature gradient is everywhere the same and is given by equation (51b). The mass of the gas contained in the layer is given by

$$M = A\int_{z_0}^{z_0+d} \rho_0(z)\,\mathrm{d}z = Ad\rho_{00}\int_{\phi_0}^{\phi_0+1} \rho_0(\phi)\,\mathrm{d}\phi\,, \tag{54}$$

where A is the area of the base of the layer. In the linear case the mass of gas within the layer will be proportional to

$$m_l = \int_{\phi_0}^{\phi_0+1} [\rho_0(\phi)]_{\mathrm{lin}}\,\mathrm{d}\phi = \{(\phi_0+1)^{s+1} - \phi_0^{s+1}\}/(s+1)\phi_0^s\,. \tag{55}$$

The nonlinear calculations have been carried out for increasing values of the temperature gradient $\mathrm{D}T_0$ at the upper boundary where $\phi = \phi_0$. These values are given by

$$(\mathrm{D}T_0)_{\phi_0} = N/(s+1)\,. \tag{56}$$

The linear value of N is unity and we see that N is in fact the Nusselt number defined as follows

$$N = [\mathrm{D}T_0/(\mathrm{D}T_0)_{\mathrm{lin}}]_{\phi_0}\,. \tag{57}$$

The integrations that were carried out correspond to the case where the energy flux across the boundary is given, and this case corresponds more closely to the astronomical situation. Under laboratory conditions the convection is induced by increasing the temperature of the lower boundary. There is of course a one-to-one correspondence between the two approaches, and in Table 1 we can observe the

Table 1. Variation of ΔT_0 with N

ΔT_0	0·00	0·65	4·13	7·40	10·58	13·75	16·95	20·18
N	1·0	1·2	2·4	3·6	4·8	6·0	7·2	8·4
ΔT_0	23·42	26·65	29·88	32·97	35·98	38·86	41·53	44·78
N	9·6	10·8	12·0	13·2	14·4	15·6	16·8	18·4

relation between the Nusselt number and the percentage increase ΔT_0 of the temperature at the lower boundary above its critical or linear value, for case A, where we have

$$\Delta T_0 = 100[\{(s+1)/(\phi_0+1)\}T_0 - 1]\,. \tag{58}$$

In the Boussinesq approximation the system of differential equations is of the eighth-order and therefore the eight boundary conditions (33), (34), (39), (51a) and (51b) fully determine the problem, and the numerical integrations can be carried out on the assumption that the density is everywhere a constant except in the buoyancy term.

In the compressible case the situation is quite different, and this can be seen from the system of fundamental equations (43)–(47) which is now of the tenth order. We therefore need two extra boundary conditions which can be arrived at by the following considerations:

(i) We are interested here in finding stationary solutions. This leads to the requirement that the energy flux at the top and bottom of the convective layer must be the same, i.e. to the condition

$$(\mathrm{D}T_0)_{\phi_0} = (\mathrm{D}T_0)_{\phi_0+1} = N. \tag{59}$$

In the linear case, such a condition is automatically satisfied since the temperature gradient is everywhere constant. In the incompressible case the mean energy equation (47) is in fact replaced by

$$\mathrm{D}^2 T_0 = \mathrm{D}(FW_1) \qquad \text{or} \qquad \mathrm{D}T_0 = FW_1 + N. \tag{60}$$

Note that in the present paper the integrations are carried out from top to bottom and that the temperature gradient $\mathrm{D}T_0$ is in fact positive.

It follows from boundary conditions (33) that the condition (59) is automatically satisfied for an incompressible fluid. In the compressible case one has to consider the full equation (47), and it is seen that the first term on the right-hand side of this equation is no longer a total differential. The condition (59) becomes therefore an extra condition to be satisfied at the boundaries.

(ii) Since the fluid within the layer is now compressible we must make sure that the condition of conservation of mass is satisfied, i.e. we now require that

$$\int_{\phi_0}^{\phi_0+1} \rho_0(\phi)\,\mathrm{d}\phi = m_l. \tag{61}$$

This condition is of course automatically satisfied in the incompressible case.

The full system of equations is of the second order in the average density ρ_0. The extra boundary conditions (59) and (61) fix the density and its gradient at the upper boundary and consequently right across the layer.

8. Discussion of Results

As indicated in the previous section the basic system of nonlinear differential equations, which is of the tenth order in W_1, F, P, T_0 and ρ_0, has been integrated numerically by a finite difference method, subject to the appropriate boundary conditions. The results are illustrated in Figs 2–7 and we consider here the main characteristics of these solutions. Case A is concerned with the effects of nonlinear convection on a layer with small initial density and temperature stratification, whereas case B is concerned with the situation where there is initially a strong density and temperature stratification throughout the layer.

It is possible to push the numerical integrations for case A to much higher values of the Nusselt number than for case B. This is to be expected since highly nonlinear solutions of the more asymmetric case will require more integration points to achieve the same accuracy. Nevertheless the results obtained so far are sufficient to give a good indication of the general behaviour of the solutions and their dependence on the initial structure of the layer.

Vertical Velocity ψ (Fig. 2)

The vertical velocity component ψ behaves in much the same way as in the corresponding incompressible case. The asymmetry of the curve increases with the amount of stratification in the layer; a result which is also obtained if one uses the mixing-length theory. A deep and highly stratified layer, such as the convection layer in the Sun, would lead to high vertical velocities near the upper boundary.

As can be seen from the definition (11) of the velocity, the vertical component may be written

$$w = W_0(z) + \psi(z) f(x, y). \tag{62}$$

In the incompressible case the average vertical velocity W_0 vanishes and only the component ψ, which is modulated by the convection, remains. In the compressible case the average vertical velocity W_0 does not vanish everywhere in the layer, and its importance increases with the strength of the convection. This is discussed in Section 9.

Average Temperature T_0 (Fig. 3)

We see that increasing temperatures across the bottom layer result in stronger convection with increased Nusselt number. Case A shows already that there is an increased tendency for the average temperature T_0 to become constant across the layer.

Temperature Fluctuation F (Fig. 4)

The temperature fluctuation F behaves in much the same way as in the incompressible case. At high Nusselt number two peaks appear near the boundaries, indicating the growing importance of boundary layers.

Average Density ρ_0 (Fig. 5)

The average density increases above its linear value at the top of the convective layer with a corresponding reduction at the bottom of the layer. In case B there is already an indication that the ρ_0 curve is about to take a negative slope at the bottom of the convective layer. This, according to equation (50b), occurs when the temperature gradient $\mathrm{D}T_0$ exceeds 1 at the boundary. A fully developed situation of this type is illustrated in Fig. 5A which corresponds to the case of small initial stratification. An interesting feature is the appearance of a density inversion; the bottom layer, which is at a higher temperature, having a lower density than the top layer.

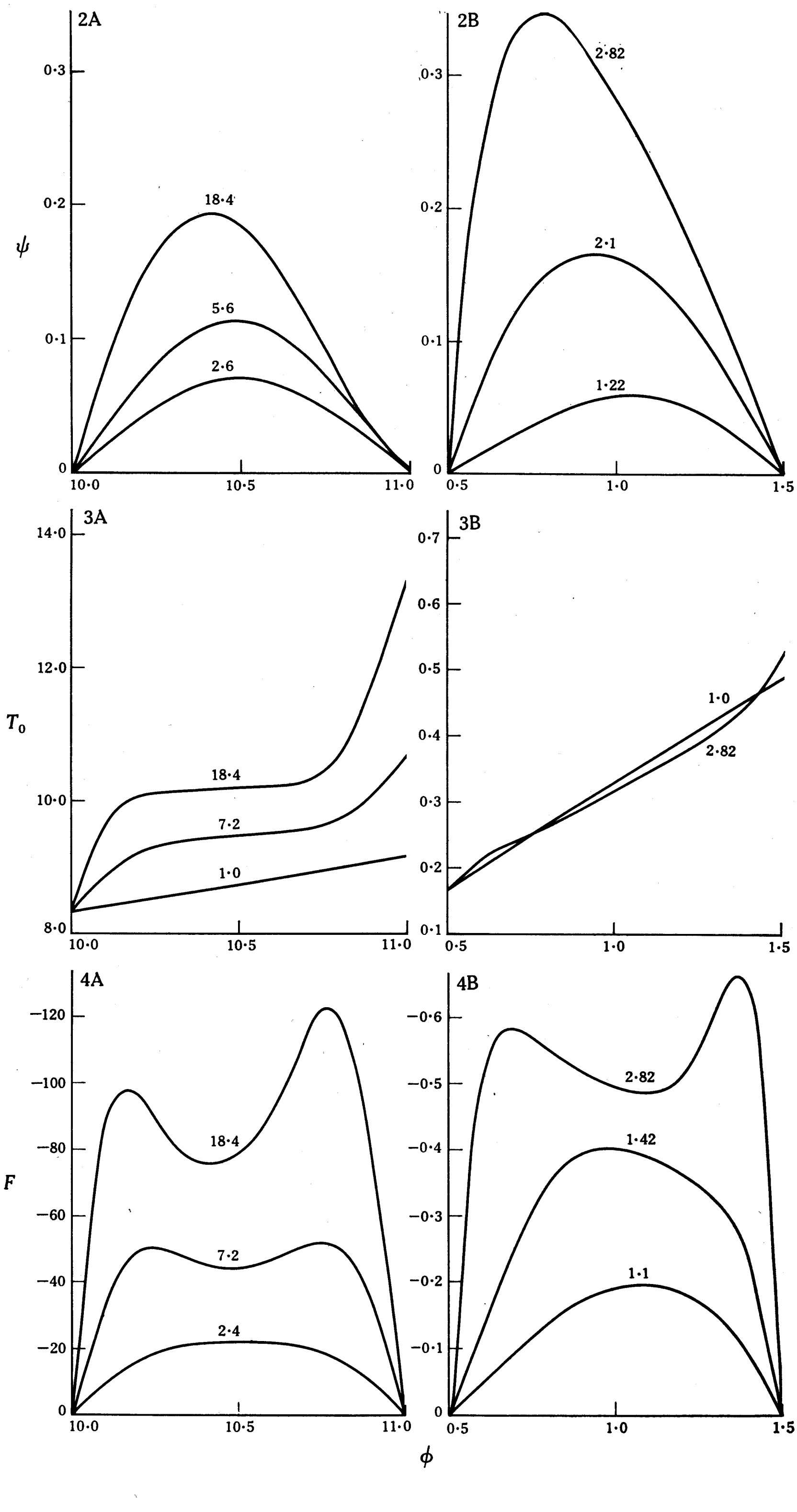
Figs 2–7 (pp. 448–9). Variation of the magnitudes of the following quantities as functions of the dimensionless position variable ϕ across the layer:

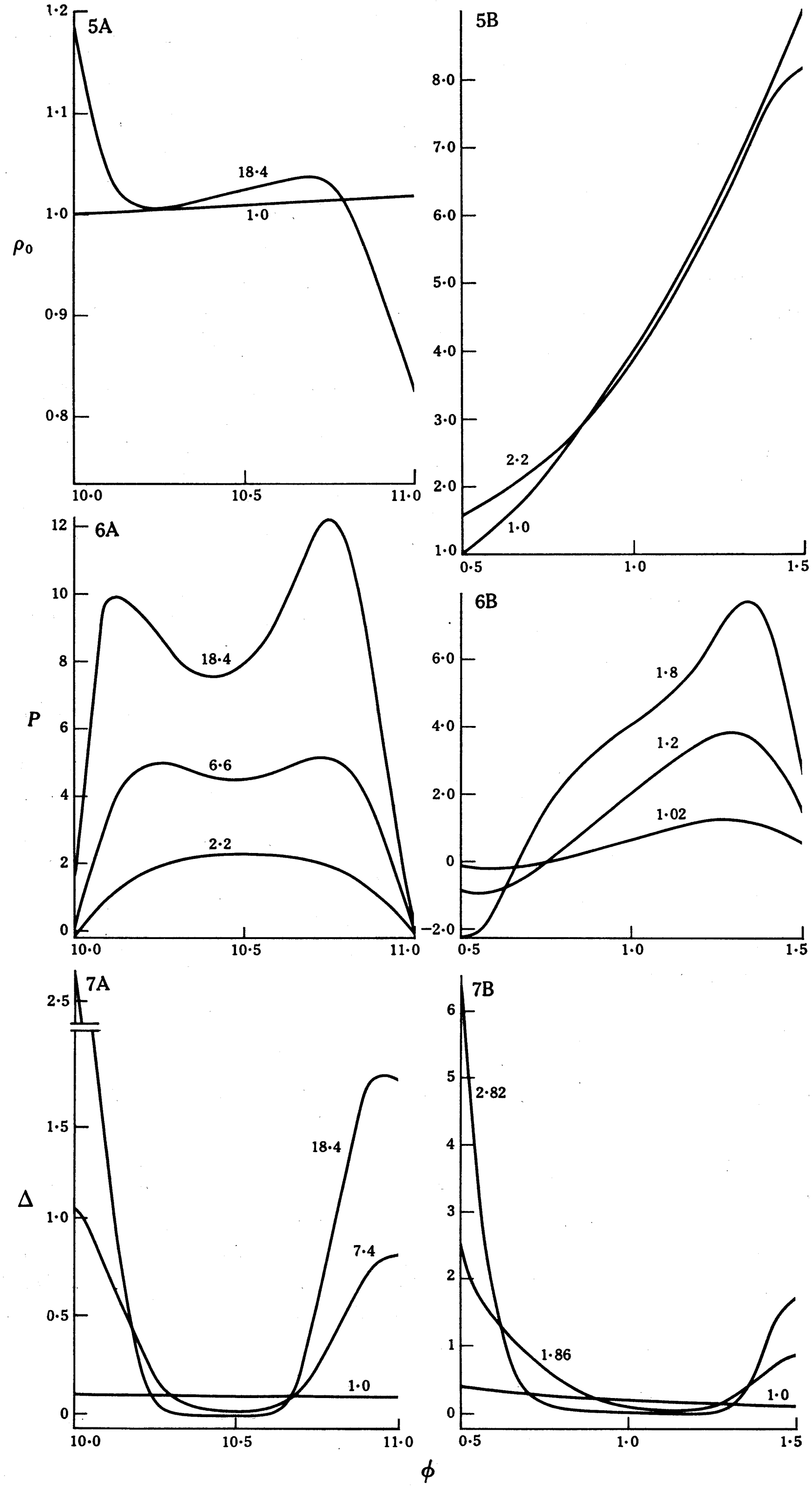
Fig. 2, vertical velocity component ψ; *Fig. 3*, average temperature T_0;

Fig 4, temperature fluctuation F; *Fig. 5*, average density ρ_0;

Fig. 6, density fluctuation P; *Fig. 7*, $\Delta = T_0^{-1}\mathrm{D}T_0 - (\gamma - 1)\rho_0^{-1}\mathrm{D}\rho_0$.

Case A is illustrated in the left-hand and case B in the right-hand parts of the figures. The quantities indicated on the curves are values of the Nusselt number N.


2A
0·3
0·2
0·1
0
ψ
18·4
5·6
2·6
10·0
10·5
11·0
2B
0·3
0·2
0·1
0
2·82
2·1
1·22
0·5
1·0
1·5
3A
14·0
12·0
10·0
8·0
T_0
18·4
7·2
1·0
10·0
10·5
11·0
3B
0·7
0·6
0·5
0·4
0·3
0·2
0·1
1·0
2·82
0·5
1·0
1·5
4A
−120
−100
−80
−60
−40
−20
0
F
18·4
7·2
2·4
10·0
10·5
11·0
4B
−0·6
−0·5
−0·4
−0·3
−0·2
−0·1
0
2·82
1·42
1·1
0·5
1·0
1·5
ϕ


5A
5B
ρ_0
1·2
1·1
1·0
0·9
0·8
18·4
1·0
10·0
10·5
11·0
8·0
7·0
6·0
5·0
4·0
3·0
2·0
1·0
2·2
1·0
0·5
1·0
1·5
6A
6B
P
12
10
8
6
4
2
0
18·4
6·6
2·2
10·0
10·5
11·0
6·0
4·0
2·0
0
−2·0
1·8
1·2
1·02
0·5
1·0
1·5
7A
7B
Δ
2·5
1·5
1·0
0·5
0
18·4
7·4
1·0
10·0
10·5
11·0
6
5
4
3
2
1
0
2·82
1·86
1·0
0·5
1·0
1·5
ϕ

Density Fluctuation P (Fig. 6)

The density fluctuation P behaves in a similar manner to the temperature fluctuation F in the case of low initial density stratification (case A). In the case of strong stratification (case B) there is a correspondingly strong asymmetry in the density fluctuation curve.

Adiabatic Excess of Temperature Gradient (Fig. 7)

It is generally assumed that in the case of strong convection the actual temperature gradient will be equal to the adiabatic one. In fact such an assumption is made when dealing with convection in the convective cores of massive stars. Figs 7A and 7B show plots of the values of Δ across the convective layer, where

$$\Delta = T_0^{-1}\{\mathrm{D}T_0-(\mathrm{D}T_0)_{\mathrm{ad}}\} \qquad \text{or} \qquad \Delta = T_0^{-1}\,\mathrm{D}T_0-(\gamma-1)\rho^{-1}_0\mathrm{D}\rho_0. \tag{63}$$

We see that this quantity does in fact tend to zero in the central regions of the convective layer and that this tendency increases with the Nusselt number. It is interesting to note that strong buoyancy establishes itself in the boundary layers. This is not surprising since a residual buoyancy is necessary to drive the convection.

Table 2. Variation across layer of $|P/\rho_0| = |W_0/\psi|$ and $|F/T_0|$

$\phi-\phi_0$	Case A ($N=18\cdot4$) $\lvert P/\rho_0\rvert$	Case A ($N=18\cdot4$) $\lvert F/T_0\rvert$	Case B ($N=1\cdot8$) $\lvert P/\rho_0\rvert$	Case B ($N=1\cdot8$) $\lvert F/T_0\rvert$	$\phi-\phi_0$	Case A ($N=18\cdot4$) $\lvert P/\rho_0\rvert$	Case A ($N=18\cdot4$) $\lvert F/T_0\rvert$	Case B ($N=1\cdot8$) $\lvert P/\rho_0\rvert$	Case B ($N=1\cdot8$) $\lvert F/T_0\rvert$
0·0000	0·0097	0·0000	0·0858	0·0000					
0·0625	0·0729	0·0736	0·0675	0·0419	0·5625	0·0882	0·0887	0·0546	0·0710
0·1250	0·1015	0·1019	0·0129	0·0703	0·6250	0·1014	0·1002	0·0560	0·0653
0·1875	0·1000	0·1024	0·0035	0·0877	0·6875	0·1153	0·1144	0·0575	0·0614
0·2500	0·0930	0·0929	0·0350	0·0961	0·7500	0·1259	0·1235	0·0607	0·0583
0·3115	0·0831	0·0844	0·0421	0·0972	0·8125	0·1170	0·1156	0·0616	0·0537
0·3750	0·0802	0·0798	0·0538	0·0932	0·8750	0·0874	0·0865	0·0569	0·0434
0·4375	0·0783	0·0790	0·0536	0·0861	0·9370	0·0435	0·0439	0·0412	0·0246
0·5000	0·0825	0·0818	0·0558	0·0782	1·0000	0·0012	0·0000	0·0181	0·0000

9. Accuracy of Anelastic Approximation

The basic equations describing finite-amplitude convection which have been studied in this paper have been derived under the assumption that the anelastic approximation holds. This approximation consists in neglecting higher powers and products of the thermodynamic fluctuations such as F and P but retaining the nonlinear terms in the velocity components. It is therefore necessary to investigate at this stage how good such an approximation has been in the present case. In order to do this we have to introduce a scaling somewhat different from the one used so far. Although the scalings introduced in the equations (42) appear to reduce the basic equations to their simplest form, they do not allow a direct comparison of the fluctuations in the thermodynamic variables with their average values. An appropriate scaling to carry out such a comparison is the following one:

$$\left.\begin{aligned} W_1 &\to (K/C_v d)W_1, \qquad \rho_0 \to \rho_{00}\rho_0, \qquad P \to \rho_{00}P, \\ T_0 &\to (K^2/R_* d^2C_v^2\,\rho_{00}^2)T_0, \qquad F \to (K^2/R_* d^2C_v^2\,\rho_{00}^2)F. \end{aligned}\right\} \tag{64}$$

No new integrations are required since the change from the old to the new variables can easily be accomplished by the following transformations

$$(H\sigma)^{\frac{1}{2}} W_1 \to W_1, \qquad (\sigma/H)^{\frac{1}{2}} P \to P, \qquad H\sigma T_0 \to T_0, \qquad H^{1/2}\sigma^{3/2} F \to F. \quad (65)$$

Once these changes have been made, and in fact they are done automatically in the author's computer program, it is a simple matter to evaluate the ratio of the thermodynamic fluctuations to their average values. These ratios are given in Table 2 for case A with $N = 18 \cdot 4$, that is, for low initial stratification and strong convection, and for case B with $N = 1 \cdot 8$, that is, for strong initial stratification and relatively weak convection. We see from this table that the ratios $|P/\rho_0|$ and $|F/T_0|$ can reach values as high as $12 \cdot 6\%$ somewhere in the convective layer for strong enough convection (case A), and a value of $9 \cdot 7\%$ for strong stratification and weak convection (case B).

Since squares and products of such terms have been neglected in the anelastic approximation, it appears that an error of $\sim 1\%$ has already appeared in the results due to such an approximation. If the integrations were to be pushed much higher it would be advisable to integrate the full equations (Van der Borght 1971) and the author hopes to do so in a subsequent paper.

An interesting by-product of the comparison we have just made is that we have

$$|P/\rho_0| = |W_0/\psi| \quad (66)$$

and, as a consequence, the previous computations give straight away an indication of the importance of the average vertical velocity W_0 as compared with the modulated component ψ. We see that this ratio already reaches the value of $12 \cdot 6\%$ in case A and that therefore the average vertical velocity W_0 cannot be neglected in the formulation of the problem.

That the anelastic approximation does in fact break down sooner in the case of strong initial stratification is not surprising in view of the fact that, in applying such an approximation, one has actually assumed that the scale height and thickness of the layer are of the same order of magnitude. In case B the density scale height is one-ninth the thickness of the layer, and the anelastic approximation cannot be expected to be sufficiently accurate for strong convection.

10. Conclusions

It has been shown above that the integration of the system of differential equations representing nonlinear convection within the framework of the one-mode anelastic approximation for non self-interacting planforms can throw some light upon the effect of density stratification and compressibility on steady convection. The equations to be solved are of course far more complex, and the resulting numerical integrations more difficult, than in the corresponding case of the Boussinesq approximation. Nevertheless, this paper shows that such an approach is possible and that it may be worth while to relax some of the assumptions underlying our calculations.

One important step would be to drop the anelastic assumption and to integrate the full equations. The present investigation shows that this would be essential if a study were to be carried out of strong convection in a medium that is highly stratified. This would certainly be essential if the results of this study were to be extended to astronomical applications. A study of convection in the outer layers of the Sun would necessitate the inclusion of the effects of ionization and variable conductivity in the basic equations.

An even more important point would be to have a closer look at the boundary conditions to see which of them need to be relaxed in order to approximate more closely the astrophysical situation. For instance, the assumption that there are no temperature fluctuations on the boundary is probably the correct one under laboratory conditions but it is unlikely to apply to any degree of accuracy in stellar convective regions. The assumption of zero modulation in the energy flux is more likely to give a better representation of the actual situation, and numerical integrations within the Boussinesq approximation have already been carried out (Van der Borght 1974). This is now being extended to the anelastic approximation.

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