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Deficiencies of the Asymptotic Solutions Commonly Found in the Quasilinear Relaxation Theory

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Abstract

Several authors have attempted to describe the dynamics of a suprathermal beam of electrons relaxing in a cold collisionless plasma by analytic asymptotic solutions of the quasilinear equations. These solutions are criticized on the grounds that they cannot correctly take account of the evolution of the plasma wave spectrum. A numerical solution is given which illustrates this point in the case of a homogeneous one-dimensional beam. Two important points can be made on the basis of the present work: (1) for self-consistency, the spontaneous emission terms cannot be neglected at any stage, and (2) the characteristic times of evolution in the early stages (accessible for the first time through the present numerical solutions) are found to be much longer than commonly quoted (i.e. several times the linear growth time) when discussing the reasons for neglecting higher-order interactions.

1. Introduction

Since Drummond and Pines (1962) and Vedenov *et al.* (1962) introduced the quasilinear theory, as derived from the collisionless Vlasov equation with a self-consistent field, the validity of this theory has been the subject of some controversy (see e.g. Fukai and Harris 1972; Kaufman 1972). However, the quantum derivation by Harris (1969) shows the quasilinear equations to be perfectly valid as long as it is understood that discrete modes and the phenomenon of trapping are not involved. In particular, the equations describe correctly the nonlinear growth and decay of waves. Particles, momentum and energy are conserved, and it is quite easy to obtain an 'H theorem' in the quantum-mechanical formalism. Following Kaufman, those quasilinear equations derived with the formalism of the second quantization can also be obtained through a more careful classical analysis (see also Rogister and Oberman 1968).

There are two fundamental improvements in the later quasilinear equations as compared with the original ones: (1) the inclusion of a diffusion tensor $\mathsf{D}(v,t)$ in velocity space which remains *positive definite* even in the domain where the growth rate γ_k of a wave with wavevector $\boldsymbol{k}$ is negative (damped wave), and (2) the presence of spontaneous emission terms (originally introduced by Pines and Schrieffer 1962). In the present study we take Harris's (1969) linear equations to be valid.

In the application of the quasilinear theory to the dynamics of a beam of suprathermal electrons relaxing in a cold collisionless plasma, the spontaneous emission terms are commonly neglected; the existence of a 'metastable state' known as a plateau distribution can then be formally established. Following Ivanov and Rudakov (1966), most authors have assumed that a large class of unstable initial distributions of electrons should reach this metastable state after a duration of the order of several times the characteristic time of plasma wave growth, as given in the linear approximation. This process, known as the quasilinear relaxation, would lead to a nearly instantaneous 'beam blow-up'. It is shown here that this 'asymptotic' theory of the quasilinear relaxation of a homogeneous beam and also the dynamical study of an inhomogeneous beam presented by Ryutov and Sagdeev (1970) suffer from many serious deficiencies. The quasilinear relaxation theory will therefore have to be considered anew.

2. Quasilinear Relaxation of a Homogeneous Beam

The quasilinear equations which describe the evolution of a one-dimensional beam of suprathermal electrons in a cold collisionless plasma can be written in the following dimensionless form (see Appendix 1 for a derivation from Harris's quasilinear equations):

$$\frac{\partial f}{\partial t} = \frac{\partial^2 J}{\partial t\,\partial v} \qquad \text{for} \qquad v \neq 0, \tag{1}$$

$$\frac{\partial J}{\partial t} = \operatorname{sign}(v)\left(v^2 J \frac{\partial f}{\partial v} + f\right) \qquad \text{for} \qquad v \neq 0, \tag{2}$$

where the first and second terms on the right-hand side of equation (2) refer to stimulated and spontaneous emission respectively.

The dimensionless distribution function of the electrons, $f = f(v,t)$, is (apart from a factor $\frac{3}{2}$) the number of electrons per unit cell of velocity space included in one Debye sphere. The dimensionless function $J = J(v,t)$ is related to the power spectrum P of the plasma oscillations through $v^3 J = P/KT_e$, where K is the Boltzmann constant and T_e is the electron temperature of the background plasma. The unit of velocity is taken to be $v_{TH} = (KT_e/m_e)^{\frac{1}{2}}$, the thermal velocity of the background plasma. The unit of time is $2/\zeta\omega_0$ (that is, $\sim 0\cdot 1$ s at 100 MHz), where ω_0 is the angular frequency of the plasma oscillations and ζ is the plasma parameter given by $\zeta = (N_0 L_e^3)^{-1}$, where N_0 is the background plasma density and $L_e = v_{TH}/\omega_0$ is the electron Debye length.

Equation (1) admits a first integral (which is well known as the quasilinear integral):

$$f(v,t) = \partial J(v,t)/\partial v + g(v), \tag{3}$$

where the function $g(v)$ is readily determined from the initial conditions

$$g(v) \equiv f_0(v) - \mathrm{d}J_0(v)/\mathrm{d}v. \tag{4}$$

Introducing the functions

$$\delta f(v,t) \equiv f(v,t) - f_0(v) \qquad \text{and} \qquad \delta J(v,t) \equiv J(v,t) - J_0(v),$$

we obtain from equations (3) and (4) the relation

$$\delta f(v,t) = \partial\{\delta J(v,t)\}/\partial v. \tag{5}$$

Then, given *any* differentiable function $\phi(v)$, we have the identity

$$\frac{\partial}{\partial v}\{\phi(v)\,\delta J(v,t)\} = \phi\frac{\partial \delta J}{\partial v} + \frac{\mathrm{d}\phi}{\mathrm{d}v}\delta J = \phi\,\delta f + \frac{\mathrm{d}\phi}{\mathrm{d}v}\delta J. \tag{6}$$

We now consider the integral of the above identity over an open set Ω in velocity space. To simplify the notation, we define the integral of a function $f(x)$ over an open set $\Omega =]a, b[$ as the limit (if it exists) of the integral of $f(x)$ over the closed interval $[\alpha, \beta] \subset \Omega$ when $\alpha \to a$ and $\beta \to b$, that is,

$$\int_\Omega f(x)\,\mathrm{d}x \equiv \lim_{\alpha\to a, \beta\to b} \int_{\alpha>a}^{\beta<b} f(x)\,\mathrm{d}x\,.$$

If $F(x)$ is a primitive function of $f(x)$ in Ω, we have for any $[\alpha, \beta] \subset \Omega$

$$\int_\alpha^\beta f(x)\,\mathrm{d}x = F(\beta) - F(\alpha)\,,$$

and the right-hand side is the algebraic sum of $F(x)$ over the boundary points of $[\alpha, \beta]$. By definition (if the limits exist) the expression

$$\lim_{\beta\to b_-} F(\beta) - \lim_{\alpha\to a_+} F(\alpha)$$

will be called the algebraic sum of $F(x)$ over the boundary points of Ω. This concept is easily extended when Ω is a numerable collection of open intervals and also when the open sets are multidimensional. It is only useful when considering piecewise continuous functions in order to avoid a cumbersome repetition of the 'lim' symbols.

Thus, the integral of the identity (6) over an open set Ω in velocity space is given by

$$\int_\Omega \{\phi\delta f + (\mathrm{d}\phi/\mathrm{d}v)\delta J\}\,\mathrm{d}v = \int_\Omega \{\partial(\phi\delta J)/\partial v\}\,\mathrm{d}v = 0\,,$$

provided that the integral exists whenever $t \geqslant 0$, and that the sum of $\phi\delta J$ over the boundary points of Ω (with the proper signs) always vanishes. This conservation theorem is valid for *any* pair of functions δf and δJ related by equation (5). It has interesting physical implications if we take $\Omega =]-\infty, 0_-[\cup]0_+, +\infty[$ and the following special functions $\phi(v)$:

(i) $\phi(v) = 1$ yields

$$\int_\Omega \delta f(v,t)\,\mathrm{d}v = 0 \qquad \text{(conservation of electrons);}$$

(ii) $\phi(v) = v$ yields

$$\int_\Omega (v\delta f + \delta J)\,\mathrm{d}v = 0 \qquad \text{(momentum equation);}$$

(iii) $\phi(v) = \frac{1}{2}v^2$ yields

$$\int_\Omega (\tfrac{1}{2}v^2\delta f + v\delta J)\,\mathrm{d}v = 0 \qquad \text{(energy equation); etc.}$$

We stress the fact that equation (2) plays *no* part either in the derivation of the quasi-linear integral, or in the conservation theorem. In particular, all the above results remain true whether the spontaneous emission terms are included or not.

In terms of the functions δf and δJ, the system of quasilinear equations (1) and (2) becomes

$$\delta f = \partial(\delta J)/\partial v\,, \tag{7}$$

$$\frac{\partial(\delta J)}{\partial t} = \operatorname{sign}(v)\left(v^2(J_0+\delta J)\frac{\partial^2(\delta J)}{\partial v^2} + \frac{\partial(\delta J)}{\partial v} + v^2\frac{\mathrm{d}f_0}{\mathrm{d}v}\delta J + v^2 J_0\frac{\mathrm{d}f_0}{\mathrm{d}v} + f_0\right), \tag{8}$$

where $f_0 = f_0(v)$ and $J_0 = J_0(v)$ are given functions. The initial conditions are

$$\delta f(v, t{=}0) = \delta J(v, t{=}0) = 0$$

and the boundary conditions are

$$\lim_{|v|\to\infty} \delta f(v,t) = \lim_{|v|\to\infty} \delta J(v,t) = 0\,.$$

Equation (8) is not defined at $v = 0$, and we must therefore impose a condition of continuity on $\delta f(v,t)$ and on the power spectrum $v^3\delta J(v,t)$ for $v \to 0_+$ and 0_-.

From the system (7) and (8) it is readily seen that no evolution will take place if and only if

$$v^2 J_0\, \mathrm{d}f_0/\mathrm{d}v + f_0 = 0\,.$$

In particular, if the initial distribution function is the Maxwellian function

$$f_0 = (2\pi)^{-\frac{1}{2}} N_0 \exp(-\tfrac{1}{2}v^2)\,,$$

it will remain undisturbed if and only if

$$J_0 = v^{-3}, \qquad \text{or} \qquad P_0 \equiv v^3 J_0 K T_\mathrm{e} = K T_\mathrm{e}\,,$$

i.e. for an initial Rayleigh–Jeans distribution of the plasma-wave power spectrum. As expected, the state of thermodynamic equilibrium is stable in the quasilinear theory with spontaneous emission terms (Fukai and Harris 1972). However, it is in general incompatible with the quasilinear integral (5) except for a special class of initial conditions satisfying (O. C. Elridge, quoted by Harris 1969)

$$f_0 - \mathrm{d}J_0/\mathrm{d}v = (2\pi)^{-\frac{1}{2}} N_0 \exp(-\tfrac{1}{2}v^2) - \mathrm{d}(v^{-3})/\mathrm{d}v\,.$$

It is possible that a collision term has to be added to equation (7). Then the dissipative term will prevent the existence of a quasilinear integral, and the otherwise 'forbidden' transition from an initially nonthermal state towards the thermodynamic 'ground level' will be permitted.

Not much can be said about the steady state of the quasilinear relaxation. The only qualitative knowledge derives directly from equation (2). If an asymptotic state exists and is described by the functions $f_\infty(v)$ and $J_\infty(v)$, these functions must be related by

$$v^2 J_\infty\, \mathrm{d}f_\infty/\mathrm{d}v + f_\infty = 0\,.$$

Now, since both the power spectrum $P = v^3 J_\infty K T_\mathrm{e}$ and the distribution function f_∞ must be positive everywhere, it is readily seen that the following inequalities hold:

$$\mathrm{d}f_\infty/\mathrm{d}v < 0 \quad \text{for} \quad v > 0 \qquad \text{and} \qquad \mathrm{d}f_\infty/\mathrm{d}v > 0 \quad \text{for} \quad v < 0\,.$$

However, this result is trivial. It simply states the obvious: that there is no unstable region in the asymptotic distribution function. It does not reveal anything about the characteristic time of relaxation or even about the existence of an asymptotic steady state.

Because the induced processes are much faster than the spontaneous ones, there must be a 'metastable' state reached, asymptotically and approximately, by the system when the induced emission ceases to be dominant. We therefore investigate this metastable state by seeking the steady state of the quasilinear equations without the spontaneous emission terms. These equations are

$$\partial f/\partial t = \partial^2 J/\partial t\,\partial v \qquad \text{and} \qquad \partial J/\partial t = \operatorname{sign}(v)\,v^2(\partial f/\partial v)J. \tag{9, 10}$$

It is quite obvious that the condition

$$\lim_{t\to\infty} J\,\partial f/\partial v = 0 \tag{11}$$

is both necessary and sufficient to yield

$$\lim_{t\to\infty} \partial f/\partial t = \lim_{t\to\infty} \partial J/\partial t = 0.$$

In other words, once the condition (11) is satisfied, the system is in a stable state and any further evolution is 'forbidden' unless we introduce additional terms in equations (9) and (10). Equation (11) admits a special (regular) solution:

$$\mathrm{d}f_\infty/\mathrm{d}v = 0 \quad \text{for} \quad J_\infty \neq 0 \qquad \text{and} \qquad \mathrm{d}f_\infty/\mathrm{d}v \neq 0 \quad \text{for} \quad J_\infty = 0, \tag{12}$$

together with continuity requirements. This solution is the so-called 'plateau' distribution introduced by Drummond and Pines (1962) and Vedenov *et al.* (1962). The quasilinear equations (9) and (10) without spontaneous terms are the basic equations used by Ivanov and Rudakov (1966) in their study of the dynamics of the quasilinear relaxation.

A serious difficulty arises with this approximation in the velocity intervals where $\mathrm{d}f_0/\mathrm{d}v \leqslant 0$ for $v > 0$ or $\mathrm{d}f_0/\mathrm{d}v \geqslant 0$ for $v < 0$. In these 'stable' regions, there is absorption and spontaneous emission but no induced emission. Neglecting spontaneous emission leads to very peculiar results. The most obvious deficiency is the instability of the thermodynamic equilibrium state as an initial state for the system of equations (9) and (10) (we have seen that it is stable, as it should be when the spontaneous emission terms are present). It is clear that we cannot take the conditions $J_\infty = 0$ or $J_\infty \neq 0$ too strictly. For instance, it is obvious that in the velocity interval $|v| \lesssim 1$, where the background plasma is dominant, we should have $\mathrm{d}f_\infty/\mathrm{d}v \neq 0$ and $J_\infty \approx J_{\mathrm{TH}} = v^{-3}$. One could then replace the solution (12) by

$$\mathrm{d}f_\infty/\mathrm{d}v = 0 \quad \text{for} \quad J_\infty > J_0 \qquad \text{and} \qquad \mathrm{d}f_\infty/\mathrm{d}v \neq 0 \quad \text{for} \quad J_\infty = J_0, \tag{13}$$

together with continuity requirements.

Using the quasilinear integral, we now complete the description of this 'solution', not essentially different from the plateau distribution (12), and show that a serious difficulty remains. Introducing the function

$$\delta J_\infty(v) \equiv J_\infty(v) - J_0(v) \qquad \text{such that} \qquad \lim_{|v|\to\infty} \delta J_\infty(v) = 0, \tag{14}$$

we can write the quasilinear integral (which is the integral of equation 9) as

$$f_\infty(v) - f_0(v) = \mathrm{d}\{\delta J_\infty(v)\}/\mathrm{d}v\,. \tag{15}$$

We define also the two functions

$$F_\infty(v) = \int_{-\infty}^{v} f_\infty(v')\,\mathrm{d}v' \quad \text{and} \quad F_0(v) = \int_{-\infty}^{v} f_0(v')\,\mathrm{d}v', \tag{16}$$

for which the conservation of electrons is expressed by

$$\lim_{v\to\infty} F_\infty(v) = \lim_{v\to\infty} F_0(v)\,.$$

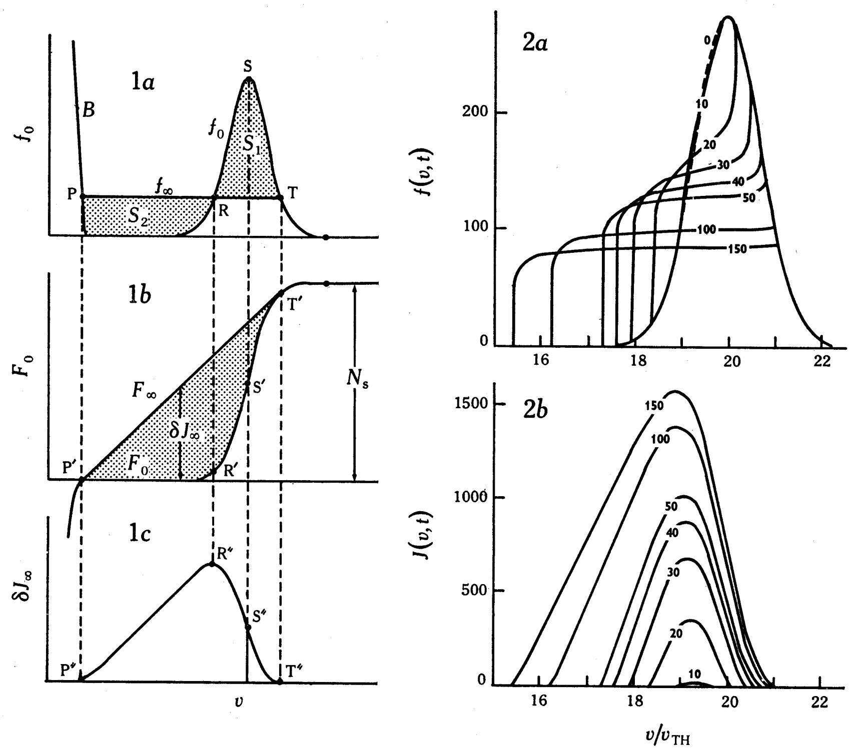

Fig. 1. Graphical representation of the 'plateau' solution, showing:

(*a*) the background plasma *B*, the initial beam distribution function $f_0(v)$ and the steady-state (plateau) distribution function $f_\infty(v)$;

(*b*) the primitive functions $F_0(v)$ and $F_\infty(v)$ corresponding to $f_0(v)$ and $f_\infty(v)$ (see equations 16);

(*c*) the plasma wave function δJ_∞ ($= J_\infty - J_0$) given by $F_\infty - F_0$ (equation 17). The δJ_∞ curve is tangential to the abscissa at P″ and T″.

Fig. 2. Numerical solutions for the quasilinear relaxation of a homogeneous beam, showing the evolution of (*a*) the electron distribution function $f(v, t)$ measured in electrons per Debye cube per unit velocity, and (*b*) the corresponding evolution of the normalized plasma wave function $J = P/KT_e v^3$. The initial form $f_0(v)$ is a gaussian given by equation (18) and the parameters listed in the text. Times are indicated on the curves in units of τ_0, the reciprocal of the maximum growth rate for $f_0(v)$ given by equation (19).

Now equation (15) can be written

$$F_{\infty}(v)-F_0(v) = \delta J_{\infty}(v). \tag{17}$$

Using the graph of $F_0(v)$ in the F_0, v plane (see Fig. 1b), a simple geometric construction will provide $F_{\infty}(v)$ and $\delta J_{\infty}(v)$. Indeed, where $\delta J_{\infty}(v) \equiv J_0(v) \neq 0$, the equations (13) give

$$\mathrm{d}f_{\infty}/\mathrm{d}v = 0 \qquad \text{and} \qquad F_{\infty}(v) = av+b,$$

the latter being a straight line segment in the F_0, v plane. Since $F_0(\pm\infty) = F_{\infty}(\pm\infty)$, this segment must join two points of the graph of $F_0(v)$, say P′ and T′. Furthermore, we have assumed in the equations (13) that the solution $f_{\infty}(v)$ is continuous, and therefore that $\mathrm{d}F_{\infty}/\mathrm{d}v = f_{\infty}$ is continuous, and the only solution possible for $F_{\infty}(v)$ is the segment P′T′ *tangential* to the graph of $F_0(v)$ at P′ and T′. If then we trace the horizontal segment between the corresponding points P and T in the f_0, v plane, we have found the usual plateau solution 'removing the hump (designated S_1 in Fig. 1a) and filling the valley (designated S_2)' (obviously the areas S_1 and S_2 are equal). The difference between the graphs of $F_{\infty}(v)$ and $F_0(v)$ gives directly $\delta J_{\infty}(v)$ according to equation (17). The graph of $\delta J_{\infty}(v)$ has been replotted in the $\delta J_{\infty}, v$ plane (see Fig. 1c). We note that $f_{\infty}(v)$ crosses $f_0(v)$ at P, R and T, while $\mathrm{d}\{\delta J_{\infty}(v)\}/\mathrm{d}v = 0$ holds at the corresponding points P″, R″ and T″. Furthermore, we have $\delta J_{\infty} = 0$ at P″ and T″, while R″ is seen to be the maximum of $\delta J_{\infty}(v)$.

The solution just described seems quite reasonable and indeed is the plateau solution of Drummond and Pines (1962) and Vedenov *et al.* (1962) except for the removal of a trivial difficulty concerning the background through the *ad hoc* replacement of the solution (12) by (13). However, a much more serious difficulty persists. If we consider the point S, where $\mathrm{d}f_0/\mathrm{d}v = 0$, it is clear that at this point $\mathrm{d}^2\delta J_{\infty}/\mathrm{d}v^2 = 0$, so that S″ is a point of inflexion in the graph of $\delta J_{\infty}(v)$. It follows then, that there exists an interval ST such that $\mathrm{d}f_0/\mathrm{d}v < 0$ and $\delta J_{\infty}(v) = J_{\infty}(v) > 0$ in a stable region where, according to the approximation, only absorption should take place (at least as long as the nonlinear diffusion described by the term $v^2(J_0+\delta J)\,\partial^2(\delta J)/\partial v^2$ in equation (8) is unimportant; see Appendix 2). In this region we should *not* have neglected the spontaneous emission terms and, in consequence, our treatment is physically inconsistent unless the regions where the initial distribution is *stable* are vanishingly small (i.e. the number of 'stable' electrons in the beam is very much less than the number of 'unstable' ones in the region where $\mathrm{d}f_{\infty}/\mathrm{d}v > 0$. This unlikely condition *severely* limits the usefulness of the quasilinear equations without the spontaneous emission terms.

Ivanov and Rudakov (1966) based their dynamic study entirely on a nonlinear diffusion equation (with source) derived from the quasilinear equations *without* the spontaneous emission terms, instead of using the correct equation (8). The validity of their treatment is thus *a priori* restricted to initial electron distribution functions which are unstable nearly everywhere. Any analysis based on the equations (9) and (10) instead of (1) and (2) suffers from the same serious deficiency.

The present author is currently working on a reappraisal of the dynamics of quasilinear relaxation, based on a numerical analysis of the quasilinear equations with the spontaneous emission terms. At present, it is possible to give a preliminary result of this numerical analysis applied to an initial gaussian beam (see Fig. 2) with an electron distribution function given by

$$f_0 = 2N_s \pi^{\frac{1}{2}}(\Delta v_s)^{-1} \exp\{-(v-v_s)^2/(\Delta v_s)^2\}, \tag{18}$$

where $N_s = 500$ electrons per Debye cube, $v_s = 20\,v_{TH}$ and $\Delta v_s = v_{TH}$. The characteristic time of plasma wave growth in the linear approximation is given initially by

$$\tau_0 = \{(v^2 \,|\, dg/dv \,|)_{max}\}^{-1}, \tag{19}$$

where $g = f_0 - dJ_0/dv$, and the initial plasma wave function is the thermodynamic one, namely, $J_0 = v^{-3}$.

Fig. 2 shows that it takes more than $10\,\tau_0$ for the plasma wave to grow to a level such that the beam is significantly altered. At the end of this 'linear' regime (see Appendix 2), two discontinuities appear in the distribution function at the boundary points of the unstable region in velocity space. One of them, starting at the lower point, propagates towards the lower velocities and was predicted and discussed by Ivanov and Rudakov (1966). However, spontaneous emission, which they neglected, provides a more natural mechanism for the propagation of this discontinuity than is possible with the background plasma waves that were invoked exclusively in their theory. The other discontinuity starts at the upper boundary point where $df_0/dv = 0$ (i.e. under the peak of the initial gaussian profile) and moves towards the higher velocities of the stable region. This phenomenon has never been considered before. Indeed its absence from the quasilinear theory, in the approximation where spontaneous emission is neglected, renders this theory physically inconsistent. Without spontaneous emission, the level of plasma waves in the stable region drops below that at thermodynamic equilibrium by the end of the linear regime (see Appendix 2) and the diffusion coefficient $D = v^2 J$ becomes so small that the propagation of the discontinuity across the stable region becomes virtually impossible. The spontaneous emission is necessary in order to prevent this absurd situation. Then the diffusion coefficient is never below its positive value at the thermodynamic equilibrium and the discontinuity can propagate to higher velocities, removing progressively the 'spike' in the electron distribution function which is left as the result of the rapid relaxation in the unstable region in velocity space. This spike becomes vanishingly small after $100\,\tau_0$, and we are left eventually with a π-like electron distribution which only at this stage resembles the *initial* condition considered by Ivanov and Rudakov (1966).

The presence of 'shock' waves in velocity space can be more readily understood by considering equation (8) as a *nonlinear thermal* equation ('temperature' $T \equiv \delta J$ and 'space coordinate' $x \equiv v$) of the form

$$\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(T^n \frac{\partial T}{\partial x}\right) + \ldots,$$

where the exponent $n = 1$. The theory of thermal waves (see e.g. Zel'dovich and Raizer 1967) shows that if $n > 0$ the heat δJ propagates from the source (i.e. from the unstable region U in velocity space where induced emission is possible) with a *finite* velocity, in such a manner that sharply defined boundaries exist between the heated region U and the cold regions not yet reached by the thermal disturbance (i.e. the regions in velocity space where there is no induced emission). In the vicinity of a boundary point of U, say x_f, the temperature distribution is of the form (for $n = 1$)

$$T \approx |x - x_f| \quad \text{for} \quad x \in \mathrm{U},$$

$$T \equiv 0 \quad \text{for} \quad x \notin \mathrm{U}.$$

The gradient of temperature ($\partial \delta J/\partial v \equiv f - f_0$) has then a *finite* discontinuity at x_f.

3. Ryutov–Sagdeev Asymptotic Theory of Quasilinear Relaxation of an Inhomogeneous Beam

In an early study, Ryutov and Sagdeev (1970) suggested an asymptotic solution of the problem posed by the diffusion of a hot electron cloud in a cold collisionless plasma. This theory has recently been applied (Zaitsev *et al.* 1972) to the dynamics of type III solar bursts. Their tentative explanation has been criticized on various physical grounds by Smith (1974) and Melrose (1974). However, we shall presently see a more fundamental deficiency of the asymptotic approach which is not unlike the shortcoming of the steady-state solution encountered in Section 2.

In the case of an inhomogeneous beam, equation (1) must be complemented by the advective term $v\,\partial f/\partial x$. On the other hand, the corresponding advective term in equation (2) may be safely neglected because the group velocity of the plasma waves is only of the order of v_{TH}. The unit of length must clearly be the distance travelled at the speed v_{TH} in the unit of time, that is,

$$L_0 = 2v_{\mathrm{TH}}/\zeta\omega_0 = 2(N_0 L_e^3)L_e.$$

The number of background electrons per Debye cube, $N_0 L_e^3$, acts as a scaling factor multiplying the Debye length to provide a natural macroscopic unit of length L_0 (at 100 MHz, we have $L_0 \approx 400$ km). The system of quasilinear equations then becomes

$$\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} = \frac{\partial^2 J}{\partial t\,\partial v}, \tag{20}$$

$$\frac{\partial J}{\partial t} = v^2 J \frac{\partial f}{\partial v} + f, \tag{21}$$

and we seek a solution $f = f(x, v, t)$ and $J = J(x, v, t)$ that is valid in the open set Ω defined by

$$\Omega = \{x \in]0, \infty[\} \times \{v \in]0, \infty[\} \times \{t \in]0, \infty[\}.$$

In the Ryutov and Sagdeev (1970) derivation, equation (21) plays virtually no role in the first-order solution, as one can see directly from the following argument. They considered the system at each space point x, only after a time $t_0(x)$ great enough for the π-like electron distribution to be already formed at this point. Then, in an expansion somewhat similar to the classical Chapman–Enskog method, equation (21) becomes automatically a second-order equation which provides the residual slope $\partial f/\partial v$ of the π-like distribution. An *a priori* hypothesis must be made concerning the respective orders of magnitude of $\partial^2 J/\partial t\,\partial v$ and $\partial J/\partial t$ to avoid a flagrant contradiction.

The above procedure is quite objectionable, since we are left only with equation (20) which, by itself, cannot provide any definite answer because the constraint imposed by the first order of approximation is very weak: $f^{(0)} = f^{(0)}(x, t)$. Ryutov and Sagdeev (1970) obviated this difficulty by the heuristic requirement that the π-like electron distribution should be self-similar. Self-similarity is compatible only with two classes of conditions, as they pointed out: (1) a continuous injection of electrons at the origin $x = 0$ (boundary value problem); (2) an initial 'burst-like' release of electrons at the origin (initial value problem). We shall only proceed with the latter condition.

In order to remove all doubts concerning the validity of the Ryutov and Sagdeev (1970) solution and its uniqueness under the above assumptions, we shall develop a rather more formal derivation, independent of their own. A self-similar solution of equation (20), in the first order of approximation where the velocity dependence of f is neglected, is a pair of functions $f^{(0)}$ and $J^{(0)}$ of the form

$$f^{(0)}(x, t) = q(\xi)\, s(t) \qquad \text{and} \qquad J^{(0)}(x, v, t) = h(\xi, v)\, s(t),$$

where

$$\xi = x/t,$$

$q(\xi)$ and $h(\xi, v)$ are universal functions and $s(t)$ is a common scaling factor varying with time. If such a solution exists, it will satisfy the equation

$$\frac{\mathrm{d}s}{\mathrm{d}t}\left(q - \frac{\partial h}{\partial v}\right) = \frac{s}{t}\left((\xi - v)\frac{\mathrm{d}q}{\mathrm{d}\xi} - \xi\frac{\partial^2 h}{\partial \xi\, \partial v}\right). \tag{22}$$

The scaling factor must therefore satisfy

$$\mathrm{d}s/\mathrm{d}t = \alpha s/t$$

and is of the form $s(t) = t^{\alpha}$, where α is a constant.

However, as we are assuming a *fixed* number of electrons to have been injected in the half-space $x \geqslant 0$, the integral

$$\int_0^\infty \int_0^\infty f(x, v, t)\, \mathrm{d}v\, \mathrm{d}x$$

should be time independent. If we require self-similarity for the distribution function, this integral becomes

$$t^{\alpha+1} \int_0^\infty \int_0^\infty f(\xi, v)\, \mathrm{d}v\, \mathrm{d}\xi,$$

and we are forced to take $\alpha = -1$. Then the equation (22) is reduced to

$$\frac{\partial}{\partial \xi}\left((\xi - v)\, q - \xi\frac{\partial h}{\partial v}\right) = 0. \tag{23}$$

The most general solution for $h(\xi, v)$, corresponding to any function $q(\xi)$, is seen to be

$$h(\xi, v) = v(1 - \tfrac{1}{2}v\xi^{-1})q(\xi) + C_1(v)\,\xi^{-1} + C_2(\xi),$$

where $C_1(v)$ and $C_2(\xi)$ are arbitrary functions. However, the term $C_1(v)x^{-1}$ in $J^{(0)}(x,v,t)$ is independent of time, and could only represent the background plasma waves, which are negligible when compared with the waves produced by the quasi-linear relaxation. Furthermore, it is clear that we want to impose the boundary condition $J^{(0)}(x,v{=}0,t) = 0$, and thus we have $C_2(\xi) \equiv 0$. Consequently, we can conclude that the *only* first-order plasma wave function compatible with self-similarity and a π-like electron distribution is given by the function

$$h(\xi,v) = v(1-\tfrac{1}{2}v\xi^{-1})\,q(\xi)\,. \tag{24}$$

No further restriction exists on $q(\xi)$. Therefore the most general solution of equation (22) compatible with our assumptions is:

$$f^{(0)}(x,t) = q(\xi)/t \qquad \text{and} \qquad J^{(0)}(x,v,t) = v(1-\tfrac{1}{2}v\xi^{-1})\,q(\xi)/t\,, \tag{25a}$$

where $q(\xi)$ is an arbitrary function (positive definite).

Obviously we have also to restrict the velocity interval to $v \in]0, u{=}2\xi[$, to obtain a plasma wave function which is positive definite. Therefore the solution (25a) is only valid in the open set $\Omega' \subset \Omega$, where

$$\Omega' = \{(x,v,t) \in \Omega;\;\; v < u(x,t) = 2x/t\}\,.$$

We complete the solution in Ω by taking the following (trivial) solution of equation (20):

$$f^{(0)}(x,t) = 0 \qquad \text{and} \qquad J^{(0)}(x,v,t) = 0\,, \tag{25b}$$

in the *open* set Ω'', complementing Ω' in Ω. The solution remains undefined on the surface

$$\Sigma = \{(x,v,t) \in \Omega;\;\; v = u(x,t) = 2x/t\}\,,$$

which forms the common boundary of Ω' and Ω''. Such a piecewise continuous solution of a differential equation is called a *weak* solution (see e.g. Courant and Hilbert 1962). It is worth while to recall what is meant exactly in this case.

Let us introduce the class D of all functions $\phi = \phi(x,v,t)$, continuous in Ω, such that $\partial\phi/\partial t$, $\partial\phi/\partial x$ and $\partial^2\phi/\partial t\,\partial v$ exist everywhere in Ω, and with $\phi = 0$ on the boundary of Ω (which includes the portion of the surface of a sphere with infinite radius). Any pair of regular functions f and J in Ω (i.e. such that f and J are continuous in Ω and $\partial f/\partial t$, $\partial f/\partial x$ and $\partial^2 J/\partial t\,\partial v$ exist everywhere in Ω), which is a solution of equation (20) in Ω, obviously satisfies the integral equation

$$\int_\Omega \phi\left(\frac{\partial f}{\partial t} + v\frac{\partial f}{\partial x} - \frac{\partial^2 J}{\partial t\,\partial v}\right) \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = 0$$

for all $\phi \in D$. Then repeated integration by parts yields

$$\int_\Omega \left\{f\left(\frac{\partial \phi}{\partial t} + v\frac{\partial \phi}{\partial x}\right) + J\frac{\partial^2 \phi}{\partial t\,\partial v}\right\} \mathrm{d}x\,\mathrm{d}v\,\mathrm{d}t = 0 \tag{26}$$

for all $\phi \in D$. *No* derivative of f or J appears in this expression, which is completely equivalent to equation (20) as long as f and J are regular. However, if f and J were discontinuous on a set of *negligible measure* in Ω, the integral equation (26) could be still satisfied for all $\phi \in D$, although the original differential equation (20) would be meaningless in the neighbourhood of the points of discontinuity. It is in that sense that the pair of functions f and J are said to be weak solutions of equation (20). For example, if we have a set of regular solutions of equation (20) in a (numerable) collection of nonintersecting open sets $\Omega_1, ..., \Omega_n, ...$ covering the open set Ω, this set of solutions will form a weak solution of equation (20), even if it is impossible to define the functions on the boundaries of $\Omega_1, ..., \Omega_n, ...$ (surfaces of discontinuity in Ω). It is now obvious that the pair of functions (25a) defined on Ω' and the trivial pair of functions (25b) defined on Ω'' together form a weak solution of equation (20) on Ω.

We now have to take the initial conditions into account in order to determine the function $q(\xi)$. This involves a time integration of the distribution function *from* $t = 0$. To calculate this time integral, Ryutov and Sagdeev (1970) replaced the function to be integrated by its asymptotic approximation. This procedure is clearly invalid, because the asymptotic solution approximates the exact solution at each point x only *after* a time $t_0(x) > 0$ (and *essentially unknown*). Incidentally, a similar operation also invalidates their derivation of the self-similar solution (see equation (6) of Ryutov and Sagdeev) and motivates our own, rather more lengthy, approach.

Before solving for $q(\xi)$ we need to establish a general conservation theorem that any pair of regular functions f and J satisfies if it is a solution of equation (20) in Ω. This is essentially a generalization of the corresponding theorem given in Section 2.

Given *any* test function $\phi(v) \in C^1(]0, \infty[)$ (that is, continuous and differentiable in the velocity interval $]0, \infty[$), and a pair of functions f and J, which are regular solutions of equation (20) in Ω, we have

$$\frac{\partial}{\partial t}\int_0^\infty \int_0^\infty \left(\phi f + \frac{\mathrm{d}\phi}{\mathrm{d}v} J\right) \mathrm{d}x\,\mathrm{d}v = \int_0^\infty \int_0^\infty \left\{\frac{\partial}{\partial v}\left(\phi \frac{\partial J}{\partial v}\right) - \phi v \frac{\partial f}{\partial x}\right\} \mathrm{d}x\,\mathrm{d}v = 0, \quad (27)$$

if we assume

$$\lim_{x\to 0} f(x, v, t) = \lim_{x\to\infty} f(x, v, t) = 0$$

and

$$\lim_{v\to 0} \phi(v)\,J(x, v, t) = \lim_{v\to\infty} \phi(v)\,J(x, v, t) = 0.$$

Thus if we introduce the initial conditions

$$f(x, t{=}0, v) = f_0(x, v) \qquad \text{and} \qquad J(x, t{=}0, v) = J_0(x, v),$$

the following integral equation applies whenever $t \geqslant 0$ for all functions $\phi(v) \in C^1(]0, \infty[)$:

$$\int_0^\infty \int_0^\infty \{\phi(v) f(x, v, t) + (\mathrm{d}\phi/\mathrm{d}v) J(x, v, t)\}\, \mathrm{d}x\,\mathrm{d}v$$
$$= \int_0^\infty \int_0^\infty \{\phi(v) f_0(x, v) + (\mathrm{d}\phi/\mathrm{d}v) J_0(x, v)\}\, \mathrm{d}x\,\mathrm{d}v. \quad (28)$$

No derivative of the solution f, J appears in this expression. Henceforth we shall make the obvious generalization: a *weak* solution of equation (20) satisfies the initial conditions if equation (28) holds good for all $\phi \in C^1(]0, \infty[)$.

However, the present initial conditions are themselves singular distributions (burst-like release of electrons at the origin). In spite of that, if we neglect the initial plasma wave (background), the right-hand side of equation (28) has still the meaning of a generalized moment:

$$\int_0^\infty \phi(v)\, f_0(x, v)\, \mathrm{d}v, \qquad \text{with} \qquad \phi \in C^1(]0, \infty[),$$

integrated throughout the half-space $x \geqslant 0$. Therefore, if we release in the half-space $x \geqslant 0$ a fixed number N_S of electrons with a normalized distribution $g_0(v)$ of velocities, such that

$$\int_0^\infty g_0(v)\, \mathrm{d}v = 1,$$

and which decreases for $v \to \infty$ at least exponentially, the initial generalized moment injected in the half-space $x \geqslant 0$ is certainly given by

$$N_\mathrm{S} \int_0^\infty \phi(v)\, g_0(v)\, \mathrm{d}v, \qquad \text{with} \qquad \phi \in C^1(]0, \infty[).$$

In consequence we shall say that the weak solution f, J satisfies the singular initial conditions if the integral equation

$$\int_0^\infty \int_0^\infty \{\phi(v)\, f(x, v, t) + (\mathrm{d}\phi/\mathrm{d}v)\, J(x, v, t)\}\, \mathrm{d}x\, \mathrm{d}v = N_\mathrm{S} \int_0^\infty \phi(v)\, g_0(v)\, \mathrm{d}v \tag{29}$$

is satisfied for *all* test functions $\phi(v) \in C^1(]0, \infty[)$. We now apply this concept to the weak solution (25) to derive the equation

$$\int_0^\infty \mathrm{d}\xi\, q(\xi) \int_0^{2\xi} \mathrm{d}v\, \{\phi + (\mathrm{d}\phi/\mathrm{d}v)\, v(1 - \tfrac{1}{2}v\xi^{-1})\} = N_\mathrm{S} \int_0^\infty g_0(\xi)\, \phi(\xi)\, \mathrm{d}\xi. \tag{30}$$

After repeated integration by parts this equation yields

$$\int_0^\infty \mathrm{d}\xi\, \{q(\xi)/\xi\} \int_0^{2\xi} v\, \phi(v)\, \mathrm{d}v = N_\mathrm{S} \int_0^\infty g_0(\xi)\, \phi(\xi)\, \mathrm{d}\xi.$$

At this stage we define the function $Q(\xi)$ such that $\mathrm{d}Q/\mathrm{d}\xi = q/\xi$ and we assume for the time being that

$$\lim_{\xi \to 0} \xi^2 Q(\xi) = 0,$$

and that $Q(\xi)$ decreases at least exponentially for $\xi \to \infty$. These assumptions have to be checked later for consistency. We then obtain

$$\int_0^\infty \{Q(\tfrac{1}{2}\xi)\, \xi + N_\mathrm{S}\, g_0(\xi)\}\, \phi(\xi)\, \mathrm{d}\xi = 0,$$

which must apply for all $\phi \in C^1(]0,\infty[)$. Therefore the function $Q(\xi)$ must be

$$Q(\xi) = -N_S g_0(2\xi)/(2\xi),$$

and we check that

$$\lim_{\xi\to 0} \xi^2 Q(\xi) = 0.$$

We also check that $Q(\xi)$ decreases at least exponentially for $\xi \to 0$ since, by hypothesis, $g_0(\xi)$ decreases at least exponentially for $\xi \to \infty$. Eventually we obtain for the function $q(\xi)$

$$q(\xi) \equiv \xi \frac{dQ}{d\xi} = -N_S \xi \frac{d}{d\xi}\left(\frac{g_0(2\xi)}{2\xi}\right), \tag{31}$$

i.e. the result obtained by Ryutov and Sagdeev (1970) through a more informal derivation.

However, the present formal derivation allows us to state the exact conditions under which equation (20) has a *uniquely defined* solution in Ω given by equations (25) and (31). These conditions are:

(1) in the first order of approximation of the system (20) and (21) in which the derivative $\partial f(x,v,t)/\partial v$ may be neglected (except on the surface $\Sigma = \{(x,v,t) \in \Omega;\ v = 2x/t\}$, where it is undefined);

(2) under the assumption of self-similarity;

(3) in the case of an instantaneous release at the origin of a fixed number of electrons with a distribution of velocities $g_0(v)$ ($v \in]0,\infty[$) decreasing at least exponentially for $v \to \infty$.

The solution is then:

(1) an asymptotic approximation valid only if we examine the system at each point x after a time $t_0(x)$ (essentially unknown) long enough for a π-like distribution of electrons to be already formed at this point;

(2) a *weak* solution in Ω (i.e. piecewise continuous) having a surface of discontinuity Σ in Ω;

(3) such that the initial conditions are satisfied in a weak sense, i.e. such that the theorem of conservation of generalized moments holds good (equation 29).

The asymptotic partition of the initial generalized moments between particles and waves can be worked out from equations (25) and (31) (see Appendix 3). It is found, for instance, that only one-third of the initial amount of particle energy can be dissipated into plasma waves, not two-thirds,* as is often stated (e.g. Melrose 1974). However, we are eventually confronted with the same kind of difficulty as that encountered in Section 2. In both cases a correct formal solution has been obtained, but it leads to an asymptotic plasma wave spectrum without explaining even qualitatively how this could have been formed.

In Section 2 the quasilinear integral provided a purely formal link between the initial conditions and the asymptotic solution across an undetermined 'transient'

* This result is referred to Shapiro (1963). However, Shapiro's result is only valid for a homogeneous *mono-energetic* beam, and if one assumes that the quasilinear equations still apply under these conditions.

region. However, a closer examination has shown that this transition was impossible without considering the spontaneous emission, and therefore it could not be as fast as is expected when we take account of induced mechanisms only. In the present section, and in the absence of a quasilinear integral, the conservation theorem (29) links the initial conditions and the asymptotic solution, again across an undetermined transition region: $\{(x, v, t) \in \Omega; \quad t \leqslant t_0(x)\}$. However, there is no proof that such a transition is indeed possible. This proof, and the evaluation of the function $t_0(x)$, would certainly require a numerical solution of the system (20) and (21), which is as yet unavailable. It is not difficult to predict that once again the spontaneous emission will play an important role. However, the issue is not as clear as it appears in Section 2 because of the possible importance of the additional dynamical effects.

4. Conclusions

A critical examination of the steady-state or asymptotic solutions commonly used in the applications of the quasilinear equations reveals a rather serious common failure which remains even after the introduction of obvious corrections to the original presentations. In short, the equation of evolution of the plasma waves is treated rather lightly in these approximations. No deeper information is derived from it than seemingly 'obvious' properties of the system. However, we have seen, at least in the case of a homogeneous beam, that these properties concern only the unstable electrons of the beam. The behaviour of the stable component is more subtle and has been completely overlooked. Its role, however, is far from negligible if the number of stable electrons is not small compared with the number of unstable ones.

On the other hand, once these 'obvious' properties of the system are admitted (whether they are valid or not), the equation of evolution of the electrons can be uniquely solved (for an inhomogeneous beam, self-similarity must be also assumed for this purpose). The initial conditions are also taken into account correctly, owing to a very general conservation theorem that this equation yields. At the same time, the fact that some global physical properties (conservation of electrons, momentum, energy etc.) are necessarily present in these solutions does not ensure their correctness.

Therefore the usefulness of the asymptotic solutions is severely limited. Their main usefulness lies in providing: (1) valuable tests for the algorithms used in the numerical analysis of a strongly nonlinear problem, and (2) analytical tools to understand, by comparison, some general features of the numerical solutions (necessarily limited to few specific cases). In the author's opinion, they can be very misleading as guidelines. It has been concluded also that spontaneous emission cannot be consistently neglected, at any stage, since it plays a leading role in the behaviour of the stable electrons. By the same token, the arguments presented by Zheleznyakov and Zaitsev (1970) to validate the approximation in which the higher-order nonlinear phenomena (wave–wave scattering) are neglected, will also have to be examined anew.

All the above fundamental problems require mathematically and physically consistent solutions before we can hope to go beyond speculations concerning the general theory of quasilinear relaxation and its application to the dynamics of the solar type III bursts. More than ever, this phenomenon appears to remain an interesting challenge offered by solar radio astronomy to the theory of plasma physics.

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Appendix 1. Dimensionless Quasilinear Equations

We start from the equations derived by Harris (1969)

$$\frac{\partial P_\lambda(\boldsymbol{k})}{\partial t} = 2\gamma_\lambda(\boldsymbol{k})\,P_\lambda(\boldsymbol{k}) + S_\lambda(\boldsymbol{k}), \tag{A1a}$$

$$\frac{\partial f_e(\boldsymbol{v})}{\partial t} = \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(\mathsf{D}(\boldsymbol{v})\cdot\frac{\partial f_e(\boldsymbol{v})}{\partial \boldsymbol{v}}\right) + \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(A(\boldsymbol{v})\,f_e\right), \tag{A1b}$$

$$\gamma_\lambda(\boldsymbol{k}) = 2\pi^2 e^2 m^{-1}\,k^{-2}\,\Omega_{\lambda k}\int \mathrm{d}^3v\,\boldsymbol{k}\,.(\partial f_e/\partial \boldsymbol{v})\,\delta(\boldsymbol{k}\,.\,\boldsymbol{v}-\Omega_{\lambda k}), \tag{A1c}$$

$$S_\lambda(\boldsymbol{k}) = 4\pi^2 e^2 k^{-2}\,\Omega_{\lambda k}^2\int \mathrm{d}^3v\,f_e(\boldsymbol{v})\,\delta(\boldsymbol{k}\,.\,\boldsymbol{v}-\Omega_{\lambda k}), \tag{A1d}$$

$$\mathsf{D}(\boldsymbol{v}) = 4\pi^2 e^2 m^{-2}\int \mathrm{d}^3k\,(2\pi)^{-3}\,P_\lambda(\boldsymbol{k})\,(\boldsymbol{k}\otimes\boldsymbol{k}/k^2)\,\delta(\boldsymbol{k}\,.\,\boldsymbol{v}-\Omega_{\lambda k}), \tag{A1e}$$

$$A(\boldsymbol{v}) = 4\pi^2 e^2 m^{-1}\int \mathrm{d}^3k\,(2\pi)^{-3}\,\Omega_{\lambda k}\,(\boldsymbol{k}/k^2)\,\delta(\boldsymbol{k}\,.\,\boldsymbol{v}-\Omega_{\lambda k}). \tag{A1f}$$

In these equations, $f_e(\boldsymbol{v})$ is the distribution function of electrons in velocity space and $P_\lambda(\boldsymbol{k})$ is the energy spectrum of the plasma oscillations. The spontaneous emission of plasma waves by the electrons is described by the terms

$$S_\lambda(\boldsymbol{k}) \qquad \text{and} \qquad \frac{\partial}{\partial \boldsymbol{v}}\cdot\left(A(\boldsymbol{v})\,f_e\right).$$

The stimulated emission is represented by $\gamma_\lambda(\boldsymbol{k})$, the growth coefficient of the plasma waves, and $\mathsf{D}(\boldsymbol{v})$, the diffusion tensor of the electron distribution function in velocity space.

Let us define the following parameters: $\omega_0 = (4\pi e^2 N_0/m_e)^{\frac{1}{2}}$, the angular frequency of oscillations in the background plasma of density N_0; $v_{\mathrm{TH}} = (KT_e/m_e)^{\frac{1}{2}}$, the thermal velocity of the background plasma; $L_e = v_{\mathrm{TH}}/\omega_0$, the electron Debye length; and $\zeta = N_0^{-1} L_e^{-3}$, the plasma parameter. We then introduce the dimensionless variables (they are made slightly different from those used by Harris (1969) in order to remove a factor $\zeta/2\pi$ in the final result):

$$\tau = \tfrac{1}{2}\zeta\omega_0 t, \qquad \boldsymbol{u} = \boldsymbol{v}/v_{\mathrm{TH}} \qquad \text{and} \qquad \boldsymbol{q} = L_e \boldsymbol{k}.$$

The dimensionless distribution function is thus

$$F(\boldsymbol{u}) = (2\pi v_{\mathrm{TH}}^3/\zeta N_0)\, f(\boldsymbol{v}) = \tfrac{3}{2}(\tfrac{4}{3}\pi L_e^3) v_{\mathrm{TH}}^3 f(\boldsymbol{v})$$

(i.e. apart from the factor $\frac{3}{2}$, the number of electrons per Debye sphere in a unit cell of velocity space). The dimensionless energy spectrum of the plasma oscillations will be

$$P(\boldsymbol{q}) = P(\boldsymbol{k})/KT_e,$$

which is the ratio between the actual spectrum and the Rayleigh–Jeans distribution (Harris 1969).

Then, with the approximation $\Omega_{\lambda k} \approx \omega_0$, the equations (A1a) and (A1b) reduce to the simpler forms

$$\frac{\partial P(\boldsymbol{q})}{\partial \tau} = \int G(\boldsymbol{q}, \boldsymbol{u})\, \mathrm{d}^3 u \qquad \text{and} \qquad \frac{\partial F(\boldsymbol{u})}{\partial \tau} = 2\pi \frac{\partial}{\partial \boldsymbol{u}} \cdot \int (2\pi)^{-3} \boldsymbol{q}\, G(\boldsymbol{q}, \boldsymbol{u})\, \mathrm{d}^3 q,$$

where

$$G(\boldsymbol{q}, \boldsymbol{u}) \equiv q^{-2}\{P(\boldsymbol{q})\, \boldsymbol{q} \cdot \partial F/\partial \boldsymbol{u} + F(\boldsymbol{u})\}\, \delta(\boldsymbol{q} \cdot \boldsymbol{u} - 1).$$

In one-dimensional form these equations reduce to (see Harris 1969)

$$\frac{\partial P(u,\tau)}{\partial \tau} = |u|^3 \left(\frac{P(u,\tau)}{u} \frac{\partial F(u,\tau)}{\partial u} + F(u,\tau) \right),$$

$$\frac{\partial F(u,\tau)}{\partial \tau} = \frac{\partial}{\partial u} \left\{ \mathrm{sign}(u) \left(\frac{P(u,\tau)}{u} \frac{\partial F(u,\tau)}{\partial u} + F(u,\tau) \right) \right\}.$$

We now introduce a new function

$$J(u,\tau) \equiv u^{-3} P(u,\tau) \qquad \text{for} \qquad u \neq 0,$$

from which we obtain

$$\frac{\partial J}{\partial \tau} = u^{-3} \frac{\partial P}{\partial \tau} = \mathrm{sign}(u) \left(u^{-1} P \frac{\partial F}{\partial u} + F \right) = \mathrm{sign}(u) \left(u^2 \frac{\partial F}{\partial u} J + F \right) \tag{A2}$$

and

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2 J}{\partial u\, \partial \tau} \tag{A3}$$

which, with appropriate changes of notation, are the equations (1) and (2).

If we had used the diffusion coefficient $D = P/u$ instead of J, we would have obtained the 'classical' quasilinear equations (with dimensionless variables)

$$\frac{\partial D}{\partial \tau} = \operatorname{sign}(u)\, u^2\left(D\frac{\partial F}{\partial u} + F\right) \quad \text{and} \quad \frac{\partial F}{\partial \tau} = \frac{\partial}{\partial u}\left\{\operatorname{sign}(u)\left(D\frac{\partial F}{\partial u} + F\right)\right\}. \tag{A4,5}$$

It is important to note that the first step in the 'classical analysis' of the system (A4) and (A5) consists in remarking that, from equation (A4), we have

$$\operatorname{sign}(u)\left(D\frac{\partial F}{\partial u} + F\right) = \frac{\partial}{\partial \tau}\left(\frac{D}{u^2}\right),$$

and substituting this expression in (A5), we obtain

$$\frac{\partial F}{\partial \tau} = \frac{\partial^2}{\partial u\, \partial \tau}\left(\frac{D}{u^2}\right), \tag{A6}$$

which is strictly equivalent to equation (A3). The main shortcomings of the classical analysis are that it solves only equation (A6) and that some extraneous conditions are imposed on the solutions, rather than determining which, if any, solutions of (A6) also satisfy (A4).

Appendix 2. 'Linear Regime' in Quasilinear Relaxation

The following analysis is quite crude and its only merit is to provide some help in understanding the early stage of the quasilinear relaxation. We write the equation of evolution of the plasma waves as

$$\partial J/\partial t = v^2 J\, \partial f/\partial v + \alpha f, \tag{A7}$$

where we have introduced a purely formal parameter α to keep track of the spontaneous emission: whenever spontaneous emission is taken into account $\alpha = 1$; otherwise $\alpha = 0$.

As long as the initial electron distribution $f_0(v)$ is not too much disturbed, equation (A7) can be approximated by the linear equation

$$\partial J/\partial t = v^2 J\, \mathrm{d}f_0/\mathrm{d}v + \alpha f_0. \tag{A8}$$

Its solution is given by

$$J(v,t) = J_0(v)\exp\{t/\tau_0(v)\} + \alpha f_0(v)\, \tau_0(v)\left[\exp\{t/\tau_0(v)\} - 1\right], \tag{A9}$$

where $J_0(v)$ is the initial plasma wave function (usually the thermal background $J_0 = v^{-3}$). We have also introduced the function

$$\tau_0(v) = (v^2\, \mathrm{d}f_0/\mathrm{d}v)^{-1}, \tag{A10}$$

which is, in modulus, the characteristic time of growth (or decay) of the plasma wave function at the point v in velocity space.

In the unstable region U′, the function $\tau_0(v)$ is positive and the plasma waves grow exponentially until the actual electron distribution becomes too different from the initial one to make the linear equation (A8) still acceptable as an approximation of (A7) ('end' of the linear regime). On the other hand, the function $\tau_0(v)$ is negative in the stable region U″. Then if we take $\alpha = 0$, the plasma wave energy density will decrease below the thermal level (KT_e), in violation of the second principle of thermodynamics. But, if the spontaneous emission is taken into account ($\alpha = 1$), the plasma waves will evolve towards some kind of local equilibrium, $J \to f_0(v) \,|\, \tau_0(v) \,|$, generally *above* the background level.

At the end of the linear regime, we are left with: (1) a 'peak' in the plasma wave function in the region U′, where the plasma wave growth is no longer exponential; (2) a level of plasma waves in the region U″, which has reached a pseudo-equilibrium above the thermal level with the corresponding stable component of the electron distribution; (3) an undisturbed thermal level of plasma waves elsewhere.

In the nonlinear regime which follows, the nonlinear diffusion of the plasma waves created in the unstable region plays the leading role. This diffusion, corresponding to the $v^2(J_0+\delta J)\partial^2(\delta J)/\partial v^2$ in equation (8), does *not* herald the onset of a new physical process but arises simply from the nonlinear character of the resonant wave–particle interaction.

Breakdown of Linear Regime

If the initial distribution function $f_0(v)$ is smooth, its derivative must vanish at the boundary points of the unstable region U′, and therefore we have at these points:

$$J = J_0 + \alpha f_0 t,$$

i.e. only a linear increase with time (if we choose $\alpha = 1$). If we assume the continuity of $\mathrm{d}f_0/\mathrm{d}v$, the continuous function $v^2\,\mathrm{d}f_0/\mathrm{d}v$, which is strictly positive in U′ and vanishes at the boundary points of U′, must reach a maximum in U′. Accordingly there is a point $v_\mathrm{M} \in$ U′ such that the characteristic growth time $\tau_0(v)$ is minimum at this point. After a few exponentiation times $\tau_0(v_\mathrm{M})$, the plasma wave function will have a well-defined maximum at this point: $J_\mathrm{M}(t) = J(v_\mathrm{M}, t)$. At the time $t = \tau_\mathrm{L}$, when the linear approximation (A9) breaks down, the value of the peak $J_\mathrm{M}(\tau_\mathrm{L})$ is given by

$$J_\mathrm{M}(\tau_\mathrm{L}) = J_0(v_\mathrm{M})\exp\{\tau_\mathrm{L}/\tau_0(v_\mathrm{M})\} + \alpha f_0(v_\mathrm{M})\,\tau_0(v_\mathrm{M})[\exp\{\tau_\mathrm{L}/\tau_0(v_\mathrm{M})\} - 1]. \quad (\mathrm{A11})$$

We make some crude estimates:

$$f_0(v_\mathrm{M}) \approx \tfrac{1}{2}f_0(v_0) \approx \tfrac{1}{2}N_\mathrm{u}/\Delta v_\mathrm{u}, \quad (\mathrm{A12a})$$

$$(\mathrm{d}f_0/\mathrm{d}v)_{v=v_\mathrm{M}} \approx f_0(v_\mathrm{M})/\Delta v_\mathrm{u} \approx \tfrac{1}{2}N_\mathrm{u}/(\Delta v_\mathrm{u})^2, \quad (\mathrm{A12b})$$

$$\tau_0^\mathrm{M} \equiv \tau_0(v_\mathrm{M}) = \{v_\mathrm{M}^2(\mathrm{d}f_0/\mathrm{d}v)_{v=v_\mathrm{M}}\}^{-1} \approx 2(\Delta v_\mathrm{u})^2/v_0^2 N_\mathrm{u}, \quad (\mathrm{A12c})$$

where v_0 is the velocity corresponding to the peak of the distribution function (upper boundary point of U′), N_u is the number of unstable electrons, and Δv_u is the half-width of the unstable component of the beam. The estimates (A12) amount to replacing the actual unstable component by a triangular profile (see Fig. 3). Eventually we obtain from equations (A11) and (A12):

$$J_M(\tau_L) \approx J_0(v_0)\exp(\tau_L/\tau_0^M) + \alpha(\Delta v_u/v_0^2)\{\exp(\tau_L/\tau_0^M) - 1\}. \qquad \text{(A13)}$$

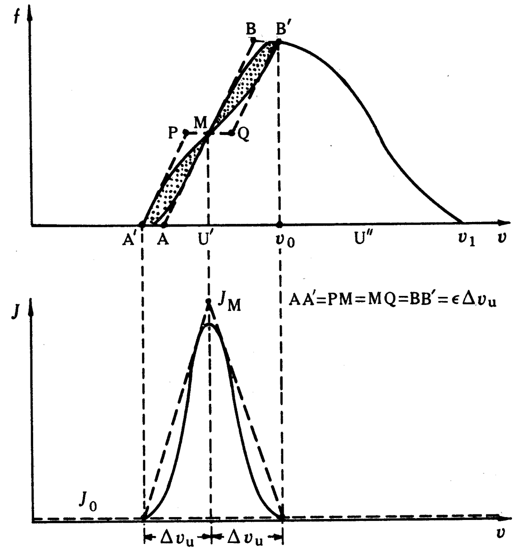

Fig. 3. Approximations used to describe the evolution of the distribution function during the 'linear' regime. The actual curves are shown by bold full lines, the approximations by bold dashed lines.

On the other hand, the actual distortion of the beam (see Fig. 2) occurs in such a fashion that, in the early stages at least, the point M ($v = v_M$) remains nearly fixed while the distribution function undergoes a kind of 'shearing' motion. We approximate this distortion by a shear with amplitude AA′ = BB′ = $\varepsilon\,\Delta v_u$, parallel to the abscissae and leaving M fixed (see Fig. 3). Then from the quasilinear integral, we deduce

$$J_M(\tau_L) = \text{area(AA'PM)} = \text{area(BB'QM)} \approx \varepsilon\,\Delta v_u f_0(v_M) = \tfrac{1}{2}\varepsilon N_u. \qquad \text{(A14)}$$

The parameter ε measures the amplitude of the distortion we are prepared to consider negligible during the linear regime. Combining equations (A13) and (A14), we obtain the following estimate:

$$\frac{\tau_L}{\tau_0^M} = \ln\left(\frac{\frac{1}{2}\varepsilon N_u + \alpha\,\Delta v_u/v_0^2}{J_0 + \alpha\,\Delta v_u/v_0^2}\right). \qquad \text{(A15)}$$

Let us compare this result with the numerical analysis (Fig. 2); i.e. when $N_u \approx 10^3$, $v_0 = 20$ and $\Delta v_u = 1$. We shall take the linear approximation (valid only if $\varepsilon \ll 1$) up to its complete breakdown at $\varepsilon = 1$. In the numerator of the argument on the right-hand side of equation (A15) the term in α can be neglected as compared with the first term (it amounts to neglecting $1 \ll \exp(\tau_L/\tau_0^M)$ in equation A13). However, in the denominator, we have

$$J_0 + \alpha\,\Delta v_u/v_0^2 \approx v_0^{-3}(1 + \alpha v_0\,\Delta v_u) = v_0^{-3}(1 + 20\alpha).$$

It is quite clear that, even in the *unstable* region, it would be incorrect to neglect the spontaneous emission (i.e. to set $\alpha = 0$) because the amplification of the background plasma waves by induced emission is less important than the 'self-induced' emission which results from the combined effects of spontaneous and induced emission. This result has been also derived by Melrose (1974). However, the formula (A15) is logarithmic and the order of magnitude of τ_L is not appreciably affected: (1) if we neglect the spontaneous emission ($\alpha = 0$)

$$\tau_L/\tau_0^M \approx \ln(\tfrac{1}{2}\varepsilon N_u v_0^3) \approx \ln(4\times 10^6) \approx 15 \quad \text{for} \quad \varepsilon = 1$$

or (2) if we neglect the amplification of the background

$$\tau_L/\tau_0^M \approx \ln(\tfrac{1}{2}\varepsilon N_u v_0^2/\Delta v_u) \approx \ln(2\times 10^5) \approx 12 \qquad \text{for} \qquad \varepsilon = 1 .$$

It is interesting to see that the numerical result of Section 2 shows that the nonlinear diffusion of the plasma waves starts to manifest itself sometime after $t = 10\,\tau_0^M$, in good agreement with the estimate that we have just made.

Pseudoequilibrium in Stable Region

As long as the linear approximation remains valid in the unstable region U′, it is *a fortiori* also correct in the stable region U″, where there is no induced emission. In this domain, the linear equation (A8) will fail as a good approximation only when the nonlinear diffusion of plasma waves from the unstable region becomes important. Until then, the distortion of the initially stable distribution function will be virtually nil, and the linear analysis will provide accurate results.

We take as the initial distribution in U″ the decreasing 'wing' of a gaussian function:

$$f_0(v) = \frac{2N_s}{\sqrt{\pi}\,\Delta v_s}\exp\left(\frac{-(v-v_0)^2}{(\Delta v_s)^2}\right) \qquad (v \in \,]v_0, v_1 = v_0 + a\Delta v_s[\,),$$

where N_s is the number of stable electrons and Δv_s is the dispersion of these electrons. The lower limit of U″ is v_0, the boundary point common to U′ and U″. The upper limit v_1 is written $v_1 = v_0 + a\,\Delta v_s$, and corresponds to the highest velocity above which it becomes meaningless to talk of 'beam' electrons. In the case in point, where $N_s \approx 10^3$, one can safely assume that $a \lesssim 3$.

The plasma wave function is thus given by

$$J(v,t) = J_0(v)\exp\left(\frac{-t}{|\tau_0(v)|}\right) + \frac{\alpha(\Delta v_s)^2}{2v^2(v-v_0)}\left\{1 - \exp\left(\frac{-t}{|\tau_0(v)|}\right)\right\}, \tag{A16}$$

where

$$|\tau_0(v)| = (\Delta v_s)^2/2f_0(v)\,v^2(v-v_0).$$

Obviously we cannot set $\alpha = 0$ (i.e. neglect the spontaneous emission), otherwise the plasma wave level will decrease exponentially far below its initial value and, if the latter is taken to be the thermal level, we will be in direct conflict with the second thermodynamic principle. On the other hand, when $\alpha = 1$, we can write equation (A16) as

$$\delta J \equiv J - J_0 = (J_* - J_0)\left\{1 - \exp\left(\frac{-t}{|\tau_0(v)|}\right)\right\},$$

where

$$J_* = (\Delta v_s)^2/2v^2(v-v_0). \tag{A17}$$

Consequently, absorption ($\delta J < 0$) will actually occur in U″ if and only if $J_0 > J_*$, and emission ($\delta J > 0$) when $J_0 < J_*$. One can see clearly that $J = J_*$ is a steady state in the region U″ for the linear regime. This equilibrium level will be *above* the thermal level when $J_* > v^{-3}$, that is, when

$$\tfrac{1}{2}(\Delta v_s)^2/(v - v_0) > v^{-1}, \tag{A18}$$

an inequality which is certainly satisfied whenever $\Delta v_s \geqslant \sqrt{2}$. When $\Delta v_s \leqslant \sqrt{2}$ we obtain from the inequality (A18)

$$(v - v_0)/\Delta v_s < \Delta v_s v_0/\{2 - (\Delta v_s)^2\}.$$

This inequality is certainly satisfied for all $v \in$ U″ if it is satisfied for the upper limit of U″: $v_1 = v_0 + a\Delta v_s$ ($a \lesssim 3$); that is, if

$$a \approx \Delta v_s v_0/\{2 - (\Delta v_s)^2\}.$$

As an example, if $v_0 = 20$ and $\Delta v_s = 1$, we get $a \leqslant 20$ when actually $a \lesssim 3$. In this case, there is no doubt that $J_* > J_{\mathrm{TH}}$ in a meaningful velocity range.

In general, the parameter a is fixed once the distribution of electrons is given. Then one has to check the condition, derived from the inequality (A18),

$$\Delta v_s \geqslant \tfrac{1}{2}\{(v_0^2 + 8a^2)^{\frac{1}{2}} - v_0\}/a. \tag{A19}$$

Usually $a \ll v_0$ and the inequality (A19) yields $\Delta v_s \geqslant 2a/v_0$, which is not too restrictive a condition on the dispersion. However, for a very steep negative slope such that $\Delta v_s < 2a/v_0 \ll 1$, the quasilinear theory will be clearly inadmissible. Otherwise, the equilibrium reached during the linear regime in the stable region will be above the thermal level.

Appendix 3. Generalized Moments Carried by Particles and Waves

Having obtained the self-similar asymptotic solution for an inhomogeneous beam given by equations (25) and (31), we can work out separately the generalized moments M_e and M_w, carried by the particles and the waves respectively. These moments are defined by the functionals

$$M_e[\phi] \equiv \int_0^\infty \int_0^\infty f(x, v, t)\, \phi(v)\, \mathrm{d}x\, \mathrm{d}v$$

and

$$M_w[\phi] \equiv \int_0^\infty \int_0^\infty J(x, v, t)\, (\mathrm{d}\phi/\mathrm{d}v)\, \mathrm{d}x\, \mathrm{d}v,$$

with argument $\phi = \phi(v) \in C^1(]0, \infty[)$. In the asymptotic self-similar solution, these functionals are time independent. After some calculations we find from equations (25) and (31):

$$M_e[\phi] = \tfrac{1}{2}M_0[\phi] + \tfrac{1}{2}N[\phi] \qquad \text{and} \qquad M_w[\phi] = \tfrac{1}{2}M_0[\phi] - \tfrac{1}{2}N[\phi],$$

where

$$M_0[\phi] \equiv N_s \int_0^\infty g_0(\xi)\,\phi(\xi)\,d\xi \qquad \text{and} \qquad N[\phi] \equiv N_s \int_0^\infty \{g_0(\xi)/\xi\}\left(\int_0^\xi \phi(v)\,dv\right) d\xi\,.$$

The functional $M_0[\phi]$ is, of course, the *initial* generalized moment carried by the particles. It is also obvious that $M_e + M_w = M_0$. Furthermore, if we set $\phi = 1$, we get

$$N[1] = N_s \int_0^\infty g_0(\xi)\,d\xi = N_s = M_0[1]\,.$$

Hence we have

$$M_e[1] = \tfrac{1}{2}N_s + \tfrac{1}{2}N_s = N_s \qquad \text{and} \qquad M_w[1] = \tfrac{1}{2}N_s - \tfrac{1}{2}N_s = 0\,,$$

a result which just expresses the conservation of particles. More generally, if we set $\phi = v^n/n$ $(n \neq 0)$, we get

$$M_e[v^n/n] = \tfrac{1}{2}\left(\frac{n+2}{n+1}\right)M_0[v^n/n] \qquad \text{and} \qquad M_w[v^n/n] = \tfrac{1}{2}\left(\frac{n}{n+1}\right)M_0[v^n/n]\,.$$

These expressions give us respectively for $n = 1, 2, \ldots$ the asymptotic distribution of the initial momentum, energy, ... between particles and waves. We could then write in general

$$M_e(n) = \tfrac{1}{2}\left(\frac{n+2}{n+1}\right)M_0(n) \qquad \text{for} \qquad n = 0, 1, 2, \ldots$$

and

$$M_w(n) = \tfrac{1}{2}\left(\frac{n}{n+1}\right)M_0(n)\,, \qquad n = 0, 1, 2, \ldots\,.$$

For momentum, $n = 1$ gives

$$M_e = \tfrac{3}{4}M_0 \qquad \text{and} \qquad M_w = \tfrac{1}{4}M_0\,.$$

For energy, $n = 2$ gives

$$E_e = \tfrac{2}{3}E_0 \qquad \text{and} \qquad E_w = \tfrac{1}{3}E_0\,.$$

Therefore, the total amount of particle energy that can be dissipated into plasma waves cannot exceed $\tfrac{1}{3}E_0$.

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