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Effect of Relaxation Parameters on the Quantum Theory of an Inhomogeneously Broadened Laser

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Abstract

The results of the Riska–Stenholm quantum theory for an inhomogeneously broadened laser are re-derived without using the Doppler limit approximation which restricts the utility of this theory. The photon distribution peak is seen to occur at a significantly lower photon number and the threshold condition is dependent on γ_{ab}/ku, where γ_{ab} is the relaxation parameter and ku the Doppler parameter. The dimensionless intensity parameter is obtained and the results are compared with the existing theories of Riska and Stenholm and of Scully and Lamb.

1. Introduction

Lamb's (1964) theory of optical masers, which describes the atoms quantum mechanically while treating the electromagnetic field in the cavity by classical Maxwell equations, gives a qualitative explanation for most of the laser phenomena, and its validity in the Doppler limit has been demonstrated by Stenholm and Lamb (1969). We have extended this theory to the case where the natural line width is comparable with the Doppler line width (Mohanty and Nayak 1974).

In further calculations Scully and Lamb (1967) incorporated the quantum nature of electromagnetic radiation while staying within the basic framework of semi-classical theory where the atoms are assumed to be stationary. The atomic motion was subsequently taken into account by Riska and Stenholm (1970) for the resonant case, i.e. where the transition frequency ω of the lasing atoms is in resonance with the cavity eigenfrequency Ω. However, the usefulness of the latter analysis is limited by the Doppler approximation, and it is unable to describe the exact effect of relaxation parameters which have a considerable influence on the performance of a gas laser (Mohanty and Nayak 1974). The main purpose of the present paper is to incorporate the effect of relaxation parameters into the quantum theory of the laser.

2. Model and Equations of Motion

The basic model, which follows closely that of Riska and Stenholm (1970), consists of lasing atoms with an upper level $|a\rangle$, a lower level $|b\rangle$ and a transition frequency between levels of $\omega = (E_a - E_b)/\hbar$. These atoms which have velocity v and which are all in the upper state are injected into the cavity at a rate r_a. The transition frequency is in resonance with the single cavity eigenfrequency Ω. In order to describe the loss mechanism, atoms in the lower state β of the two nonresonant broad levels α and β are introduced at a rate r_β into the cavity. All these atoms have a decay rate

γ_η for the state η. The atomic motion is assumed to cause a Doppler shift in the transition frequency so that $\omega = \Omega + (\Omega/c)v = \Omega + kv$.

The complete system is described by the density matrix

$$\rho_{\alpha n,\beta n'} = \langle \alpha n | \rho | \beta n' \rangle, \tag{1}$$

$|\alpha n\rangle$ being the field–atom condition for the nth photon with the atom in the state $|\alpha\rangle$. The calculations of Riska and Stenholm (1970) give the equation of motion for the diagonal element ρ_{nn} as

$$\mathrm{d}\rho_{nn}/\mathrm{d}t = -A_{n+1}\,\rho_{nn} + A_n\,\rho_{n-1,n-1} + C(n+1)\,\rho_{n+1,n+1} - Cn\,\rho_{nn}, \tag{2}$$

where A_n and C represent the integrals

$$A_n = \int_{-\infty}^{+\infty} \frac{2g^2 r_a \gamma_b \gamma_{ab}\, n\, W(v)}{\gamma_a \gamma_b(\gamma_{ab}^2 + k^2 v^2) + 4g^2 \gamma_{ab}^2\, n}\,\mathrm{d}v, \tag{3a}$$

$$C = 2g^2\, r_\beta\, \gamma_{\alpha\beta} \int_{-\infty}^{+\infty} \frac{W(v)}{\gamma_\beta(\gamma_{\alpha\beta}^2 + k^2 v^2)}\,\mathrm{d}v, \tag{3b}$$

with $\gamma_{ab} = \frac{1}{2}(\gamma_a + \gamma_b)$ and g the coupling constant. The velocity distribution for the atoms $W(v)$ is assumed to be Maxwellian with the form

$$W(v) = (u\sqrt{\pi})^{-1} \exp(-v^2/u^2).$$

The quantity A_n may be evaluated by rearranging the integral in equation (3a) to give

$$A_n = \frac{2g^2 r_a \gamma_{ab}\, n}{\gamma_a u \sqrt{\pi}} \int_{-\infty}^{+\infty} \frac{\exp(-v^2/u^2)\,\mathrm{d}v}{(kv)^2 + \gamma_{ab}^2(1 + 4g^2 n/\gamma_a \gamma_b)}$$

$$= -\frac{2g^2 r_a \gamma_{ab}\, n}{\gamma_a (ku)^2 \sqrt{\pi}} \frac{1}{\mathrm{i}z_n} \int_{-\infty}^{+\infty} \frac{\exp(-t^2)\,\mathrm{d}t}{\mathrm{i}z_n - t}$$

$$= \frac{2\sqrt{\pi}\, g^2 r_a \gamma_{ab}\, n}{\gamma_a (ku)^2\, z_n}\, w(\mathrm{i}z_n)$$

where

$$z_n = (\gamma_{ab}/ku)\{1 + (4g^2/\gamma_a \gamma_b) n\}^{\frac{1}{2}}.$$

Defining

$$A = 2\sqrt{\pi}\, g^2 r_a/\gamma_a ku \qquad \text{and} \qquad B = (4g^2/\gamma_a \gamma_b) A,$$

we finally obtain

$$A_n = A n (1 + Bn/A)^{-\frac{1}{2}}\, w(\mathrm{i}z_n), \tag{4a}$$

where the function

$$w(\mathrm{i}z_n) = \frac{\mathrm{i}}{\pi} \int_{-\infty}^{+\infty} \frac{\exp(-t^2)\,\mathrm{d}t}{\mathrm{i}z_n - t}$$

is known as the probability integral for complex arguments and is widely tabulated (Faddeyeva and Terent'ev 1961).

By a similar method to the above we may evaluate the integral C from equation (3b) to obtain

$$C = (2g^2 r_\beta \sqrt{\pi}/\gamma_\beta ku)\, w(\mathrm{i}z'), \tag{4b}$$

where $z' = \gamma_{\alpha\beta}/ku$. Although the exact value of C may be easily incorporated (see Section 4), for the moment we will only consider the approximation for broad levels, $\gamma_{\alpha\beta} \gg ku$. In this case C is given by

$$C \approx 2g^2 r_\beta/\gamma_\beta \gamma_{\alpha\beta}\,. \tag{4c}$$

Equation (2) now becomes

$$\begin{aligned} \mathrm{d}\rho_{nn}/\mathrm{d}t = & -A\,w(\mathrm{i}z_{n+1})\,(n+1)\{1+B(n+1)/A\}^{-\frac{1}{2}}\rho_{nn} \\ & +A\,w(\mathrm{i}z_n)\,n\{1+Bn/A\}^{-\frac{1}{2}}\rho_{n-1,n-1}+C(n+1)\rho_{n+1,n+1}-Cn\,\rho_{nn}\,. \end{aligned} \tag{5}$$

3. Steady State Solution

In the steady state, the solution of equation (5) takes the form

$$\rho_{nn} = (A/C)^n N \prod_{\nu=0}^{n} w(\mathrm{i}z_\nu)(1+B\nu/A)^{-\frac{1}{2}}\,, \tag{6}$$

where N is a normalization constant. From this expression the threshold condition can be evaluated (the threshold being the state where the losses of the cavity are equal to the pumping rate). It is clear then that ρ_{nn} will have a peak at $n = 0$. Since the rate of change near the peak is very small, we can approximate $\rho_{1,1}$ by $\rho_{0,0}$, which gives

$$A/C = (1+B/A)^{\frac{1}{2}}/w(\mathrm{i}z_1)\,. \tag{7}$$

This condition is slightly greater than the threshold condition $A/C = 1$ assumed by Riska and Stenholm (1970), which was obtained by inspection.

Since $w(\mathrm{i}z_n)$ is a real function, equation (6) can be easily solved for any combination of A/C and B/A. For small values of z_ν we have

$$w(\mathrm{i}z_\nu) \approx 1-(2\gamma_{ab}/ku\sqrt{\pi})(1+B\nu/A)^{\frac{1}{2}}$$

and, under this approximation, equation (6) takes the form

$$\rho_{nn} = \left(\frac{A}{C}\right)^n N \prod_{\nu=0}^{n}\left[\frac{1}{(1+B\nu/A)^{\frac{1}{2}}}-\frac{2\gamma_{ab}}{ku\sqrt{\pi}}\right]. \tag{8}$$

The only difference between equation (8) and that of Riska and Stenholm (1970) is the presence of the second term in the square brackets, but this has an appreciable effect on the photon statistics. Above threshold, the peak value of ρ_{nn} occurs at $n = n_\mathrm{p}$, which may be approximately calculated by assuming $\rho_{n_\mathrm{p}n_\mathrm{p}} = \rho_{n_\mathrm{p}-1,n_\mathrm{p}-1}$. We then have

$$1 = \frac{A}{C}\left(\frac{1}{(1+Bn_\mathrm{p}/A)^{\frac{1}{2}}}-\frac{2\gamma_{ab}}{ku\sqrt{\pi}}\right),$$

which gives

$$n_\mathrm{p} = \frac{A}{B}\left(\frac{(A/C)^2}{\{1+(2\gamma_{ab}/ku\sqrt{\pi})(A/C)\}^2}-1\right). \tag{9}$$

It may be noted here that by substituting $n_\mathrm{p} = 0$ in equation (9) we obtain the threshold condition

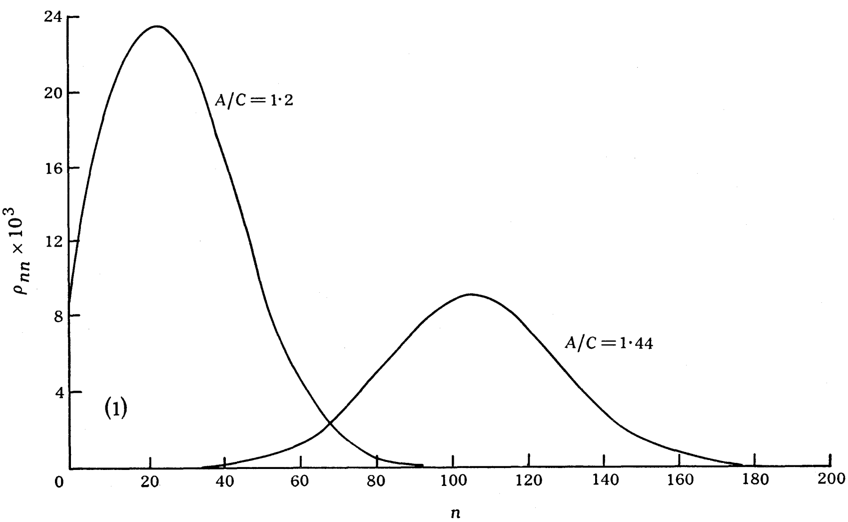
$$A/C = (1-2\gamma_{ab}/ku\sqrt{\pi})^{-1}\,.$$

Equation (7) reduces to this form for small z_1 and $B/A \ll 1$.

Keeping in mind the condition $\gamma_{ab} \ll ku$, we can write equation (9) approximately as

$$n_{\mathrm{p}} \approx (n_{\mathrm{p}})_{\mathrm{RS}} - \left(\frac{2\gamma_{ab}A^2}{\sqrt{\pi}\,ku\,C^2}\right)\frac{A}{B},$$

which shows that the peak occurs at a considerably lower photon number than the $(n_{\mathrm{p}})_{\mathrm{RS}}$ deduced by Riska and Stenholm (1970). As A/C increases so does n_{p}, and the peak thus shifts towards a higher photon number. This is to be expected since an increase in A/C means that pumping is greater than the loss by which amplification occurs and so there is a higher probability for more photons to be present. This broadening of the distribution curve with increase in A/C can be clearly seen in Fig. 1, where the exact distribution ρ_{nn} is shown for two values of A/C.



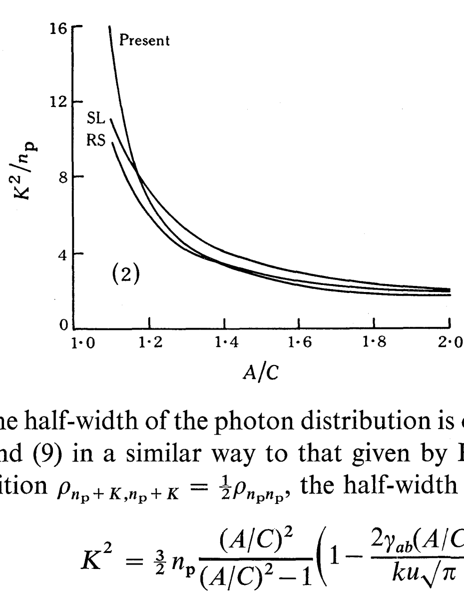


Fig. 1 (*above*). Exact photon distributions for the two indicated values of A/C with $B/A = 0 \cdot 005$ and $\gamma_{ab}/ku = 0 \cdot 1$.

Fig. 2 (*left*). Approximate variation of the width K^2/n_{p} as derived in the present paper compared with previous results:

SL, Scully and Lamb (1969);

RS, Riska and Stenholm (1970).

The half-width of the photon distribution is obtained approximately from equations (8) and (9) in a similar way to that given by Riska and Stenholm (1970). From the condition $\rho_{n_{\mathrm{p}}+K,n_{\mathrm{p}}+K} = \frac{1}{2}\rho_{n_{\mathrm{p}}n_{\mathrm{p}}}$, the half-width is given by

$$K^2 = \tfrac{3}{2}n_{\mathrm{p}}\frac{(A/C)^2}{(A/C)^2-1}\left(1-\frac{2\gamma_{ab}(A/C)}{ku\sqrt{\pi}}\right)\left(1+\frac{4\gamma_{ab}}{ku\sqrt{\pi}}\frac{A/C}{(A/C)^2-1}\right). \tag{10}$$

This relation gives a wider photon distribution than that of Riska and Stenholm. Neglecting the term involving $(\gamma_{ab}/ku)^2$ in equation (10), we have

$$K^2 \approx \tfrac{3}{2}n_{\mathrm{p}}\frac{(A/C)^2}{(A/C)^2-1}\left\{1+\frac{2\gamma_{ab}(A/C)}{ku\sqrt{\pi}}\left(\frac{2}{(A/C)^2-1}-1\right)\right\}.$$

This again brings out the interesting fact that γ_{ab}/ku has a marginal effect at threshold. The approximate value of K^2/n_{p} is compared with that of Riska and Stenholm (1970) and of Scully and Lamb (1967) in Fig. 2. It may be noted that the approximate value of K^2/n_{p} in the present case compares well with the exact values in the two other cases. However, our exact values, as derived from Fig. 1, are still higher.

4. Discussion

As noted above, Fig. 1 shows that the peak of the photon distribution occurs at a lower value of n than that predicted by Riska and Stenholm (1970). For $A/C = 1\cdot44$, the peak of the distribution found by Scully and Lamb (1967) occurs at $n_{\mathrm{p}} = 90$, which is the value at the peak predicted by Riska and Stenholm (1970) for $A/C = 1\cdot2$. The peak for ρ_{nn} in the present work occurs at $n_{\mathrm{p}} = 105$ for $A/C = 1\cdot44$. This shows that the effect of γ_{ab}/ku is more pronounced at higher photon numbers. The width of the distribution curve increases as A/C increases because ρ_{nn} depends heavily on γ_{ab}/ku and its effects become stronger for higher values of A/C.

The expression for A/C given by Riska and Stenholm (1970) is

$$(A/C)_{\mathrm{RS}} = \sqrt{\pi}\,(\gamma_\beta/\gamma_a)(\gamma_{\alpha\beta}/ku)(r_a/r_\beta)\,. \tag{11}$$

Since γ_β and $\gamma_{\alpha\beta}$ are much greater than ku, the ratio r_a/r_β is not very important. However, if atoms are pumped to the upper lasing level at a much higher rate than to the lower level ($r_a \gg r_\beta$) and if γ_β is assumed to be somewhat larger than γ_a, then $\gamma_{\alpha\beta}$ need not be too large compared with ku. When the previous stringent condition $\gamma_{\alpha\beta} \gg ku$ is lifted, we obtain the expression (4b) above for the integral C, and we then have

$$A/C = (\gamma_\beta/\gamma_a)(r_a/r_\beta)\,\{w(\mathrm{i}\gamma_{\alpha\beta}/ku)\}^{-1}\,, \tag{12}$$

which is dependent on ku only through $w(\mathrm{i}\gamma_{\alpha\beta}/ku)$. For a system with $\gamma_{\alpha\beta} \ll ku$, equation (12) shows the well-known property that the amplification is the ratio of the number of atoms r_a/γ_a $(=\mathcal{N}_a)$ in state a to the number of atoms r_β/γ_β $(=\mathcal{N}_\beta)$ in state β. If $\mathcal{N}_a$ is greater than $\mathcal{N}_\beta$ then $\gamma_{\alpha\beta}$ need not be larger than ku, and so we have $A/C = \mathcal{N}_a/\mathcal{N}_\beta$ when the loss level is not broad. However for $\gamma_{\alpha\beta} \gg ku$ equation (12) reduces to the form (11).

Finally, the dimensionless intensity parameter I can be calculated from the relation $I = (B/A)n_{\mathrm{p}}$. Substituting the expression for n_{p} from equation (9), we have

$$I = \frac{(A/C)^2}{\{1+(2\gamma_{ab}/ku\sqrt{\pi})(A/C)\}^2} - 1\,. \tag{13}$$

In Fig. 3, I is plotted against A/C for two values of γ_{ab}/ku and the results are compared with those of Scully and Lamb (1967) and Riska and Stenholm (1970). It is seen that the rate of increase of I as predicted by Riska and Stenholm (RS curve) is reduced as A/C increases, showing a saturation effect. This is an important departure from the results of Riska and Stenholm. For $\gamma_{ab}/ku = 0\cdot1$, I increases from zero at $A/C = 1\cdot15$ while for $\gamma_{ab}/ku = 0\cdot05$ it starts at $A/C = 1\cdot05$. This shows that at

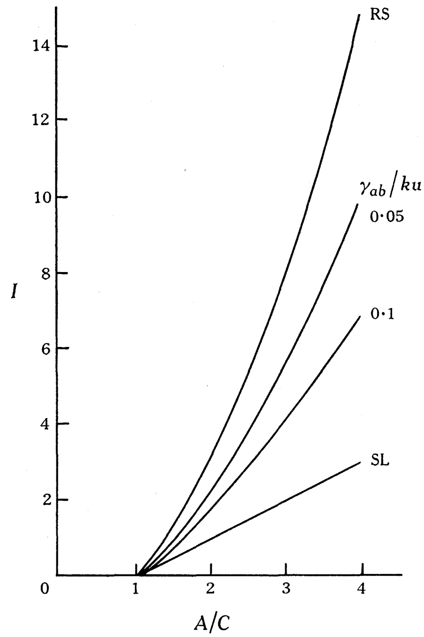


Fig. 3. Dependence of the dimensionless intensity parameter I on A/C for $\gamma_{ab}/ku = 0\cdot1$ and $0\cdot05$. The present results are compared with those of:
SL, Scully and Lamb (1969);
RS, Riska and Stenholm (1970).

threshold A/C is greater than unity and is a function of γ_{ab}/ku. In contrast, the RS curve starts from $I = 0$ at $A/C = 1\cdot0$. It is also seen from Fig. 3 that as γ_{ab}/ku decreases I approaches the RS result and coincides with it at $\gamma_{ab}/ku = 0$.

References

Faddeyeva, V. N., and Terent'ev, N. M. (1961). 'Table of the Probability Integral for Complex Argument' (Pergamon: New York).
Lamb, W. E., Jr (1964). *Phys. Rev.* **134**, A1429.
Mohanty, B. K., and Nayak, N. (1974). Paper presented at All India Symposium on Applied Optics, Bangalore.
Riska, D. O., and Stenholm, S. (1970). *J. Phys.* A **3**, 189.
Scully, M. O., and Lamb, W. E., Jr (1967). *Phys. Rev.* **159**, 208.
Stenholm, S., and Lamb, W. E., Jr (1969). *Phys. Rev.* **181**, 618.

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