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Effects of Rotation on a Test Particle in the NUT Field

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Abstract

The geodetic motion of a test particle in the NUT (Newman *et al.* 1963) field is investigated and the rotational effects are deduced. The behaviour of particles and light near the Schwarzschild singularity points is also considered.

1. Introduction

The problem of investigating the motion of a freely falling test particle in a specific physical situation brings out some of the inherent physical interactions defining a field. This fact has stimulated many workers in the discipline of general relativity to carry out a study of particle motion for numerous physical situations. We mention here only some examples: Hilton (1965), by investigating the motion of a test particle and of light in the Schwarzschild space–time, concluded that the singularity at $r = 2m$ represents a real barrier that is irremovable on physical grounds. Rosen (1970) also arrived at the same view from a study of geodetic motion in the neighbourhood of $r = 2m$. However, we note that the investigations of Finkelstein (1958), Fronsdal (1959) and Graves and Brill (1960) lead to the contrary viewpoint that the barrier is not a physical one. In an unpublished work on the motion of a test particle in the gravitational field of a charged particle, we obtained results which coincided with those of Hilton and Rosen. We note also that Markley (1973) has utilized Hamiltonian methods to investigate geodetic motion in a Schwarzschild field and that he deduced some interesting features for the limiting velocity of a test particle. Since the Hamiltonian approach yields a comprehensive attack on the problem, we adopt it here to investigate rotational effects in a NUT (Newman *et al.* 1963) field. In Section 2, the motivation for introducing the NUT field is briefly explained. The geodetic motion of a test particle and of light is then examined in Section 3, and the results of Hilton, Rosen and Markley are shown to emerge as special cases when the angular parameter a is made zero.

2. NUT Space–Time

Newman *et al.* (1963), in an attempt to solve the empty space–time field equations of general relativity, $R_{mn} = 0$, arrived at the solution widely known as the NUT metric, which is given by

$$\mathrm{d}s^2 = -A^{-1}\,\mathrm{d}r^2 - (r^2+a^2)(\mathrm{d}\theta^2 + \sin^2\theta\,\mathrm{d}\phi^2) + A\{\mathrm{d}t + 4a\sin^2(\tfrac{1}{2}\theta)\,\mathrm{d}\phi\}^2\,, \tag{1}$$

where

$$A = (r^2 - 2mr - a^2)/(r^2+a^2)\,.$$

The metric (1) belongs to the Petrov class: type-I degenerate, and admits a four-parameter group of motion. It is also a member of a class of stationary and axially symmetric metrics which possess nonshearing but curling geodesic rays with nonzero divergence. It contains two parameters m and a, and reduces to the Schwarzschild metric for $a = 0$. Furthermore, we find that the space–time depicted by the metric (1) is singular at $r = m \pm b$, where $b = (m^2+a^2)^{\frac{1}{2}}$, and also along the line of symmetry $\theta = 0$, $\theta = \pi$. It should be noted though that the metric is regular at $r = 0$.

Misner (1963) introduced a periodic coordinate time and then showed that the metric (1) possesses the strange property that every observer at rest in the coordinate system has a closed time-like worldline. Bonner (1963) also found an interesting interpretation for the NUT metric, according to which, the metric describes the field of a spherically symmetric mass (m) together with a semi-infinite massless source of angular momentum (a) along the axis of symmetry. Bonnor removed the singularity at $\theta = 0$ by a coordinate transformation, and he tentatively maintained the view that $\theta = \pi$ is a physical singularity representing a source of the field. He attributed a linear source of pure angular momentum along $\theta = \pi$ to the presence of the term in $\mathrm{d}\phi\,\mathrm{d}t$.

Having thus inquired into the physical aspects of the NUT metric, we now express it in a form which is advantageous for our discussion. We transform the curvature coordinate r to R in such a way that

$$r = r(R) \qquad \text{and} \qquad \mathrm{d}r/\mathrm{d}R = A = (r^2-2mr-a^2)/(r^2+a^2)\,. \tag{2a, b}$$

Then equation (1) can be written as

$$\begin{aligned}\mathrm{d}s^2 = {} & A(R)(\mathrm{d}t^2-\mathrm{d}R^2) - \{r^2(R)+a^2\}(\mathrm{d}\theta^2+\sin^2\theta\,\mathrm{d}\phi^2) \\ & + 8\,A(R)\,a\sin^2(\tfrac{1}{2}\theta)\,\mathrm{d}\phi\,\{2a\sin^2(\tfrac{1}{2}\theta)\,\mathrm{d}\phi+\mathrm{d}t\}\,.\end{aligned} \tag{3}$$

For convenience we now write r and A in place of $r(R)$ and $A(R)$ respectively.

For radial motion of light we have $\mathrm{d}s = \mathrm{d}\theta = \mathrm{d}\phi = 0$, so that equation (3) yields $\mathrm{d}R/\mathrm{d}t = 1$. Thus in the new coordinate system the radial speed of light is unity. which we now take to be the unit of measurement of speed along the radial direction.

Integration of equation (2b) produces

$$R = r + m\ln\big((r-m)^2-b^2\big) + b\ln\big((r-m-b)/(r-m+b)\big) + c\,, \tag{4}$$

where c is a constant of integration. Choosing c such that at some point we have $r = R = h$, say, we then obtain from equation (4)

$$R = r + m\ln\left(\frac{(r-m)^2-b^2}{(h-m)^2-b^2}\right) + \ln\left(\frac{(r-m-b)(h-m+b)}{(r-m+b)(h-m-b)}\right). \tag{5}$$

From this equation we observe that:

(i) $r(R)$ is a monotonic increasing function, and

(ii) $\lim\limits_{R\to-\infty} r(R) = m+b$ and $\lim\limits_{R\to+\infty} r/R = 1$.

Thus the domain of $r \geqslant m+b$ is to be replaced by $-\infty \leqslant R \leqslant \infty$.

3. Radial Motion of Test Particle

The motion of a test particle satisfies

$$\delta \int \mathrm{d}s = 0, \tag{6}$$

and so from equations (3) and (6) its path is described by

$$\delta \int f^{\frac{1}{2}}\,\mathrm{d}t = 0,$$

where

$$f = A(1-\dot{R}^2) - (r^2+a^2)\{\dot{\theta}^2 + \sin^2(\theta)\,\dot{\phi}^2\} + 8aA\sin^2(\tfrac{1}{2}\theta)\,\dot{\phi}\,\{2a\sin^2(\tfrac{1}{2}\theta)\,\dot{\phi}+1\},$$

with an overhead dot signifying differentiation with respect to time. The Lagrangian for the moving particle is then given by

$$L = -\alpha f^{\frac{1}{2}}, \tag{7}$$

where the constant α characterizes the moving particle and is always positive (Landau and Lifshitz 1971).

The equations of motion for θ and ϕ are

$$\dot{P} = \dot{\phi} f^{-\frac{1}{2}}[(r^2+a^2)\,\dot{\phi}\sin(\theta)\cos(\theta) - 2aA\sin(\theta)\{4a\,\dot{\phi}\sin^2(\tfrac{1}{2}\theta)+1\}], \tag{8a}$$

where

$$P = \dot{\theta}(r^2+a^2)f^{-\frac{1}{2}}, \tag{8b}$$

and

$$\dot{Q} = 0, \tag{9a}$$

where

$$Q = f^{-\frac{1}{2}}[(r^2+a^2)\sin^2(\theta)\,\dot{\phi} - 4aA\sin^2(\tfrac{1}{2}\theta)\{4a\,\dot{\phi}\sin^2(\tfrac{1}{2}\theta)+1\}]. \tag{9b}$$

If we take the initial conditions

$$\theta = \tfrac{1}{2}\pi, \quad \dot{\theta} = 0 \qquad \text{and} \qquad \phi = 0, \quad \dot{\phi} = 0 \tag{10}$$

then, from equations (8)–(10), we get $\ddot{\theta} = \ddot{\phi} = 0$, which implies that the particle continues to move along the radial direction. If we now write the Lagrangian L using equations (7) and (10) as

$$L = -\alpha\{A(1-\dot{R}^2)\}^{\frac{1}{2}}, \tag{11}$$

the canonical momentum p_R conjugate to R is then given by

$$p_R = \partial L/\partial \dot{R} = \alpha\dot{R}\{A/(1-\dot{R}^2)\}^{\frac{1}{2}}.$$

The Hamiltonian H now becomes

$$H = p_R\dot{R} - L = \alpha\{A/(1-\dot{R}^2)\}^{\frac{1}{2}}. \tag{12}$$

For sufficiently large values of R, we have

$$H \approx \frac{\alpha}{(1-\dot{R}^2)^{\frac{1}{2}}}\left(1-\frac{m}{r}-\frac{a^2}{r^2}-\frac{m^2}{2r^2}\right). \tag{13}$$

The right-hand side of this relation can be interpreted (on removing the bracket) as follows: The first term resembles the special relativistic rest energy plus the kinetic energy of the test particle, if α is considered to be the particle's rest mass. The second term denotes the Newtonian potential of a gravitational field acting on a gravitational mass of $\alpha/(1-\dot{R}^2)^{\frac{1}{2}}$, with the radial variable r in place of R. The third term is the contribution to the potential (for a repulsive force) arising from the presence of a source of angular momentum. The fourth term plays the role of the potential of a repulsive force due to the mass m, this situation having no Newtonian analogue. It is clearly seen that the source of angular momentum increases the potential for a repulsive force.

From equation (13) we find that the Hamiltonian does not involve the time t explicitly, so that H represents a constant of the motion equal to the total energy (including rest energy), which we denote by αE. Thus we have from equation (12)

$$E = \{A/(1-\dot{R}^2)\}^{\frac{1}{2}}. \tag{14}$$

Since E is always positive and A is nonnegative for the metric (1), it is evident that equation (14) restricts $\dot{R}$ such that $\dot{R} < 1$, that is, the radial velocity of a particle is always less than the local speed of light; when $\dot{R} = 1$, E is infinite. The solution of equation (14), namely

$$\dot{R}^2 = 1-A/E^2, \tag{15}$$

reveals the alternative possibility that $\dot{R} = 1$ when $A = 0$, which implies that $r = m+b$ and that this value of r corresponds to the metric singularity (we neglect the situation corresponding to $r = m-b$ because, for $r < m+b$, r becomes a time-like coordinate). Therefore we conclude that, at the singularity, the velocity of the particle approaches the velocity of light. In this situation $\dot{R}$ coincides with the metric velocity V defined by Rosen (1970).

If we now rewrite the speed of the particle in the original coordinate r, we have

$$\dot{r}^2 = A^2(1-A/E^2).$$

Denoting $\dot{r}_{\mathrm{p}}$ and $\dot{r}_l$ as the radial speeds of the particle and light respectively, we find

$$\dot{r}_l = A \qquad \text{and} \qquad \dot{r}_{\mathrm{p}}^2 = \dot{r}_l^2(1-A/E^2).$$

At $r = m+b$, we have $\dot{r}_{\mathrm{p}} = \dot{r}_l = 0$. Furthermore, it can easily be seen that, at $r = m+b$, we have $\ddot{r}_{\mathrm{p}} = 0$ and $\ddot{r}_l = 0$. Therefore, at the singularity $r = m+b$, the coordinate velocity and acceleration of both the test particle and light vanish. Thus the observer using r and t as coordinates may conclude that both the particle and light stop at $r = m+b$. However, for a proper observer the situation will be different: he will conclude that the particle approaches the velocity of light asymptotically as $r \to m+b$.

To get a better picture of the situation at $r = m+b$, we examine the motion of the test particle in the neighbourhood of this point. Defining a new radial coordinate u by

$$u = r-(m+b) \tag{16}$$

and restricting ourselves to radial motion, we may describe the space–time in the neighbourhood of $u = 0$ (that is, $r = m+b$) by

$$ds^2 = -(m+b)u^{-1}\,du^2 + u(m+b)^{-1}\,dt^2. \tag{17}$$

If alternatively we take v as the radial variable defined by

$$r-(m+b) = v^2/4(m+b) \tag{18}$$

then we obtain

$$ds^2 = -dv^2 + e^2v^2\,dt^2, \qquad \text{with} \qquad 2e(m+b) = 1. \tag{19}$$

Incidently the transformation (18) is similar to that of Einstein and Rosen (1935). It is also interesting to note that the space–time expressed in the two different forms (17) and (18) is the same as that considered by Rosen (1970) and hence, in respect to the motion of particles and light rays, his discussion holds good in our situation too.

By suitably adjusting the initial conditions, the trajectories of a particle and a photon have the following expressions:

Particles

$$4Bu = D^2 - (s-2nD)^2 \qquad \text{for} \qquad (2n-1)D \leqslant s \leqslant (2n+1)D, \tag{20a}$$

$$\left.\begin{aligned} v &= (-1)^n\{q^2-(s-2nq)^2\}^{\frac{1}{2}} \qquad \text{for} \qquad (2n-1)q \leqslant s \leqslant (2n+1)q, \\ v &= q\,\mathrm{sech}(et). \end{aligned}\right\} \tag{20b}$$

Photons

$$\left.\begin{aligned} 4Bu &= D^2-2gp \qquad \text{for} \qquad 0 \leqslant 2gp \leqslant D^2, \\ 4Bu &= 2gp-D^2 \qquad\qquad\quad D^2 \leqslant 2gp \leqslant 2D^2, \end{aligned}\right\} \tag{20c}$$

$$\left.\begin{aligned} v^2 &= q^2-2kp \qquad \text{for} \qquad 0 \leqslant 2kp \leqslant q^2 \quad \text{and} \quad v \geqslant 0, \\ v^2 &= 2kp-q^2 \qquad\qquad\quad q^2 \leqslant 2kp^2 \leqslant 2q^2 \quad \text{and} \quad v \leqslant 0, \\ v &= q\exp(-et). \end{aligned}\right\} \tag{20d}$$

In the above expressions D, e, g, k and q are constants, p is the parameter in place of s, $B = m+b$, and n is a nonzero integer. Thus we see that particles trace segments of parabolas and semicircles (as shown in Figs 3 and 1 respectively of Rosen 1970), while the paths of light rays are composed of the two straight lines and parabolas (as depicted in Figs 4 and 2 respectively of Rosen 1970). All the geodesics describing the motion of particles and light rays are seen to be reflected at the boundary separating the physical and nonphysical regions. Furthermore, the proper time required for both a particle and a photon to reach $r = m+b$ from $r > m+b$ (sufficiently large) is finite, whereas the coordinate time required is infinite for both. Hilton's (1965) interpretation of this phenomenon on physical grounds is thus convincing. Returning to equation (15), we get

$$\begin{aligned} \ddot{R} &= -(1-\dot{R}^2)(mr^2+2a^2r-ma^2)/(r^2+a^2)^2, \\ &\approx (1-\dot{R}^2)\left(-\frac{m}{r^2}-\frac{2a^2}{r^3}+\frac{3ma^2}{r^4}\right). \end{aligned}$$

If we restrict ourselves to the second order of approximation, we find that the acceleration is always negative and has a Newtonian analogue; the extra terms containing the angular momentum parameter a are correction terms without a Newtonian counterpart.

By analogy with special relativity, we define the mechanical momentum of the moving particle as

$$p_{\mathrm{m}} = \alpha\dot{R}/(1-\dot{R}^2)^{\frac{1}{2}} = \beta\dot{R},$$

where β is the relativistic mass of the particle. The gravitational force on the particle is then given by

$$\begin{aligned} F &= \mathrm{d}p_{\mathrm{m}}/\mathrm{d}t, \\ &\approx \beta\left(-\frac{m}{r^2}\right)+\beta\left(-\frac{2a^2}{r^3}+\frac{3ma^2}{r^4}\right). \end{aligned} \tag{21}$$

From equation (20) it is easily seen that, in addition to the Newtonian attractive force, there are other attractive and repulsive forces associated with the source of angular momentum. However, if a is set equal to zero, the metric (1) reduces to the Schwarzschild metric, and all our results then reduce to the corresponding results obtained by Markley (1973).

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