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Nonstatic Spherically Symmetric Solutions for a Perfect Fluid in General Relativity

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Abstract

A general method is described by which exact solutions of Einstein's field equations are obtained for a nonstatic spherically symmetric distribution of a perfect fluid. In addition to the previously known solutions which are systematically derived, a new set of exact solutions is found, and the dynamical behaviour of the corresponding models is briefly discussed.

Nonstatic solutions for spherically symmetric systems containing a perfect fluid of inhomogeneous density and pressure have been obtained in isotropic coordinates by several authors previously (McVittie 1967; Nariai 1967; Faulkes 1969; Banerjee and Banerji 1975). Faulkes found a very simple solution that gave a collapsing model by solving the differential equation resulting from isotropy of pressure. Kuchowicz (1972) utilized such an equation to obtain exact solutions corresponding to static fluid spheres. In the present note we have applied a special technique to solve this equation and have systematically derived the previous solutions of Faulkes, Nariai and Banerjee and Banerji as special cases. In addition, under more general conditions, we have obtained a new set of exact solutions satisfying Einstein's field equations for a spherically symmetric distribution of a perfect fluid which can be matched with the exterior Schwarzschild solution at the boundary. It is found that the models may collapse, bounce or oscillate, depending on the boundary conditions.

Integration of Field Equations

We consider the isotropic form of the line element

$$\mathrm{d}s^2 = \mathrm{e}^{\nu}\,\mathrm{d}t^2 - \mathrm{e}^{\omega}(\mathrm{d}r^2 + r^2\,\mathrm{d}\theta^2 + r^2\sin^2\theta\,\mathrm{d}\varphi^2),$$

where ν and ω are functions of r and t. Assuming that the fluid is perfect and using comoving coordinates, we find that Einstein's field equations give (Faulkes 1969)

$$R_{11} = \Gamma(x)\,R^2, \qquad \mathrm{e}^{\nu} = A(t)\,\dot{\omega}^2, \tag{1a, b}$$

where the first relation is obtained from isotropy of pressure and the second follows from the condition $T_{14} = 0$. In equations (1), $x = r^2$, $R = \mathrm{e}^{-\frac{1}{2}\omega}$, the subscript 1 denotes differentiation with respect to x and a dot indicates a time derivative. A substitution of the form

$$R = \{\xi(r,t)+\theta\}v(x)$$

transforms equation (1a) to

$$v\xi_{11}+2v_1\xi_1+(\xi+\theta)v_{11} = (\xi+\theta)^2 v^2 \Gamma(x). \tag{2}$$

A first integral for this equation may be assumed to have the form

$$\xi_1^2 = B(x)\xi^2(\xi+p), \tag{3}$$

with

$$B(x) = v^{-4} = \tfrac{2}{3}v\Gamma(x), \tag{4}$$

provided the constant θ assumes either of the values 0 or $\frac{2}{3}p$. Then, v satisfies the relation

$$v_1^2 = \pm p/v^2+q, \tag{5}$$

the plus sign corresponding to the case $\theta = 0$ and the minus sign to $\theta = \frac{2}{3}p$. Depending on the nature of the constants p and q, various solutions may be obtained:

Case A. $p = 0, q = 0$

It is evident from equations (4) and (5) that $\Gamma(x)$ is constant in this case. This is the solution discussed by Faulkes (1969).

Case B. $p = 0, q \neq 0$

For these values we have

$$\Gamma(x) = 3/2z^5, \qquad \text{with} \qquad z = \pm\tau x\pm\varepsilon.$$

Here and in what follows τ, ε and λ will denote arbitrary constants. The solution in this case is

$$R = 4\tau^2 z^3/(1+Tz)^2,$$

where $T = T(t)$. This is the solution of Banerjee and Banerji (1975).

Case C. $p \neq 0, q = 0$

In this case we have

$$\Gamma(x) = 3/2z^{5/2}, \qquad \text{with} \qquad z = \pm\tau x+\varepsilon.$$

Here we have assumed $p = \frac{1}{4}\tau^2$ for $\theta = 0$ and $p = -\frac{1}{4}\tau^2$ for $\theta = \frac{2}{3}p$. The solutions in the respective cases are given by

$$R = \tau^2 Tz/(1-Tz^{\frac{1}{2}})^2,$$

which is the one derived by Nariai (1967), and

$$R = \tfrac{1}{4}\tau^2(\tfrac{1}{3}+\tan^2\eta)z^{\frac{1}{2}}, \tag{6}$$

where $\eta = \ln Tz^{\frac{1}{2}}$ with $T = T(t)$.

Case D. $p \neq 0,\ q \neq 0$

This case gives rise to three forms of $\Gamma(x)$:

$$\Gamma_a(x) = (3\varepsilon^5/2\tau^5)\sinh^5\psi\,, \quad \text{with} \quad \pm\varepsilon^2 x+\lambda = z = \tau\coth\psi; \tag{7a}$$

$$\Gamma_b(x) = (3\varepsilon^5/2\tau^5)\cos^5\psi\,, \quad \pm\varepsilon^2 x+\lambda = z = \tau\tan\psi; \tag{7b}$$

$$\Gamma_c(x) = (3\varepsilon^5/2\tau^5)\cosh^5\psi\,, \quad \pm\varepsilon^2 x+\lambda = z = \tau\tanh\psi. \tag{7c}$$

Case Da, $\Gamma(x) = \Gamma_a(x)$. We introduce now a new function $\eta(x,t)$ defined by

$$2\eta(x,t) = \phi(t)+\psi(x),$$

where $\phi(t)/\tau$ is an arbitrary function obtained on integrating equation (3) and $\psi(x)$ is as defined in (7a). We obtain for $\theta = 0$

$$R = (\tau^3/\varepsilon)\operatorname{cosech}\psi\operatorname{cosech}^2\eta\,, \tag{8a}$$

$$R = -(\tau^3/\varepsilon)\operatorname{cosech}\psi\operatorname{sech}^2\eta\,, \quad \varepsilon = -|\varepsilon|\,, \tag{8b}$$

and for $\theta = \frac{2}{3}p$

$$R = (\tau^3/\varepsilon)\operatorname{cosech}\psi\,(\tfrac{1}{3}+\tan^2\eta)\,. \tag{8c}$$

Case Db, $\Gamma(x) = \Gamma_b(x)$. Here we obtain for $\theta = 0$

$$R = (\tau^3/\varepsilon)\sec\psi\sec^2\eta\,, \tag{9a}$$

$$R = (\tau^3/\varepsilon)\sec\psi\operatorname{cosec}^2\eta\,, \tag{9b}$$

where in this case $2\eta(x,t) = \phi(t)+\psi(x)$ with $\psi(x)$ as defined in (7b), and for $\theta = \frac{2}{3}p$

$$R = (\tau^3/\varepsilon)\sec\psi\,(\tanh^2\eta-\tfrac{1}{3})\,. \tag{9c}$$

Case Dc, $\Gamma(x) = \Gamma_c(x)$. Here we obtain for $\theta = 0$

$$R = (\tau^3/\varepsilon)\operatorname{sech}\psi\operatorname{cosech}^2\eta\,, \tag{10a}$$

$$R = -(\tau^3/\varepsilon)\operatorname{sech}\psi\operatorname{sech}^2\eta\,, \quad \varepsilon = -|\varepsilon|\,, \tag{10b}$$

where in this case $\eta(x,t) = \phi(t)-\psi(x)$ with $\psi(x)$ as defined in (7c), and for $\theta = \frac{2}{3}p$

$$R = (\tau^3/\varepsilon)\operatorname{sech}\psi\,(\tfrac{1}{3}+\tan^2\eta)\,. \tag{10c}$$

Discussion of New Solutions

From Einstein's field equations we derive the expressions for the density ρ^* and the pressure p^* in the general forms

$$8\pi\rho^* = R^2\{8r^2R\Gamma(x)-3M^2\psi'^2+6M\psi'/r\}+3\dot{\phi}^2\,, \tag{11}$$

$$8\pi p^* = R^2\{MN\psi'^2-(M+N)\psi'/r\}-3\dot{\phi}^2+2\ddot{\phi}/L\,, \tag{12}$$

where a prime denotes differentiation with respect to r and expressions for the quantities L, M and N for each of the solutions (6), (8), (9) and (10) are given in Table 1.

Table 1. Definitions of L, M and N for new solutions

Soln	L	M	N
(6)	$\dfrac{\tan\eta\,(1+\tan^2\eta)}{\tan^2\eta+\frac{1}{3}}$	$1+\dfrac{\tan\eta\,(1+\tan^2\eta)}{\tan^2\eta+\frac{1}{3}}$	$1+\dfrac{\tan^2\eta-\frac{1}{3}}{\tan\eta\,(\tan^2\eta+\frac{1}{3})}$
(8a)	$-\coth\eta$	$-(\coth\psi+\coth\eta)$	$-(\coth\psi+\tanh\eta)$
(8b)	$-\tanh\eta$	$-(\coth\psi+\tanh\eta)$	$-(\coth\psi+\coth\eta)$
(8c)	$\dfrac{\tan\eta\,(1+\tan^2\eta)}{\tan^2\eta+\frac{1}{3}}$	$-\left(\coth\psi-\dfrac{\tan\eta\,(1+\tan^2\eta)}{\tan^2\eta+\frac{1}{3}}\right)$	$\coth\psi-\dfrac{\tan^2\eta-\frac{1}{3}}{\tan\eta\,(\tan^2\eta+\frac{1}{3})}$
(9a)	$\tan\eta$	$\tan\psi+\tan\eta$	$\tan\psi-\cot\eta$
(9b)	$-\cot\eta$	$\tan\psi-\cot\eta$	$\tan\psi+\tan\eta$
(9c)	$\dfrac{\tanh\eta\,(1-\tanh^2\eta)}{\tanh^2\eta-\frac{1}{3}}$	$\tan\psi+\dfrac{\tanh\eta\,(1-\tanh^2\eta)}{\tanh^2\eta-\frac{1}{3}}$	$\tan\psi+\dfrac{\tanh^2\eta+\frac{1}{3}}{\tanh\eta\,(\tanh^2\eta-\frac{1}{3})}$
(10a)	$-\coth\eta$	$-(\tanh\psi-\coth\eta)$	$-(\tanh\psi-\tanh\eta)$
(10b)	$-\tanh\eta$	$-(\tanh\psi-\tanh\eta)$	$-(\tanh\psi-\coth\eta)$
(10c)	$\dfrac{\tan\eta\,(1+\tan^2\eta)}{\tan^2\eta+\frac{1}{3}}$	$-\left(\tanh\psi+\dfrac{\tan\eta\,(1+\tan^2\eta)}{\tan^2\eta+\frac{1}{3}}\right)$	$-\left(\tanh\psi+\dfrac{\tan^2\eta-\frac{1}{3}}{\tan\eta\,(\tan^2\eta+\frac{1}{3})}\right)$

The metric corresponding to each of the above solutions can be matched with the exterior Schwarzschild metric across the moving boundary provided that $p^*(r_0, t) = 0$ and

$$\dot{\phi}^2 = R_0^2(M_0^2\psi_0'^2 - 2M_0\psi_0'/r_0 + 2mR_0/r_0^3), \tag{13}$$

where the zero subscript indicates values at the boundary $r = r_0$ and m denotes the usual Schwarzschild mass. The dynamical behaviour of the models may be interpreted with the help of equation (13). In fact the condition $\dot{\phi} = 0$ corresponds to a reversal of motion. In addition we note that for all values of t we have $\dot{\phi}^2(t) \geqslant 0$.

The condition $\dot{\phi}(t) = 0$ reduces for the solutions (8a), (8b), (9a), (9b), (10a) and (10b) to the form

$$AX^2 + BX + C = 0, \tag{14}$$

where A (>0), B and C are constants. A and C depend on m while X is a specified variable of t, namely $X = \coth\eta_0$ for solutions (8a) and (10a), $\tan\eta_0$ for (9a), $\cot\eta_0$ for (9b) and $\tanh\eta_0$ for (8b) and (10b). In these cases the condition $\dot{\phi}^2 \geqslant 0$ suffices to show that the variable X possesses either a lower bound or an upper bound and therefore that the models either collapse to a singularity of zero proper volume and infinite matter density or bounce back from a minimum proper volume thereby avoiding an ultimate catastrophe.

For the solutions (6), (8c), (9c) and (10c) the condition $\dot{\phi} = 0$ leads in each case to a sixth-degree equation in X in which the coefficient of the highest power is greater than zero, and X has the value $\tan\eta_0$ for (6), (8c) and (10c) and the value $\tanh\eta_0$ for (9c). The coefficients of X^6, X^4, X^2 and the constant term depend on m and hence so do the roots of the equation. Here also the greatest and smallest roots may be considered as the lower and upper bounds respectively for X so as to be consistent with the condition $\dot{\phi}^2 > 0$, and similarly collapsing and bouncing models follow.

One of the interesting aspects of the present results is the possibility of obtaining oscillatory models for the solutions (6), (8c), (9c) and (10c) apart from their collapse and bounce behaviours already discussed. In fact it can be shown after detailed calculations that, for a suitable choice of parameters such as r_0 and m, there exist intervals between two successive roots of the same sign in which the variable $X(t)$ may lie for all times t and be consistent with the condition $\dot{\phi}^2 > 0$. Thus the proper radius will then oscillate between values corresponding to these bounds of $X(t)$.

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