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The Combined Effect of Rotation and Magnetic Field on Finite-amplitude Thermal Convection with Hexagonal Planform

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Abstract

In this paper a study is made of the combined effect of rotation and magnetic field on the convective transfer in a horizontal Boussinesq layer of fluid heated from below, in the case when the convection cells have a hexagonal planform. The numerical integrations of the basic equations have been carried out to high values of the Rayleigh number, and two families of solutions have been discovered. A discussion of the main properties of these solutions is given.

Introduction

Most stars have convective layers somewhere in their interior, and in particular there exists an extensive convective region in the Sun's outer layer. Quite often, as in the case of the Sun, the star is rotating and a magnetic field is present. It is therefore of some importance to study the combined effect of rotation and magnetic field on thermal convection and, in particular, to study the nonlinear aspects of the problem. An initial investigation (Van der Borght and Murphy 1973) restricted itself to rolls and convection cells with square or rectangular planforms. In this way one could avoid the numerical integration of complex systems of nonlinear equations and, in addition, it was possible to develop perturbation and asymptotic methods which lead to fairly accurate analytical solutions at high Rayleigh number.

The present paper extends the above work to the case of convection cells with hexagonal planform. This fomulation is more realistic for two reasons. The actual convection cells observed in granulation and supergranulation and, for that matter, in mesoscale convection in the atmosphere possess a hexagonal planform. In addition, when hexagonal planforms are adopted, a larger number of nonlinear terms are retained in the basic equations, and the formulation of the problem is therefore more realistic. In particular, the inertia terms in the equation of motion are fully taken into account and the results are no longer independent of the Prandtl number.

Since the problem is highly nonlinear, one would not expect the solution to be unique, and in fact we find two families of solutions, whose characteristics are described below. No asymptotic or perturbation methods have been developed for the present case, although it is likely that the recently published work of Gough *et al.* (1975) could be extended to include both the effects of rotation and magnetic field.

Basic Equations

The basic equations of the problem, within the Boussinesq approximation, have been derived elsewhere (Van der Borght and Murphy 1973) and they are as follows:

$$(\mathrm{D}^2-a^2)Z = (C/\sigma)(W\mathrm{D}Z-Z\mathrm{D}W)-\mathscr{T}^{\frac{1}{2}}\mathrm{D}W+Q\tau C(X\mathrm{D}H-H\mathrm{D}X)-Q\tau \mathrm{D}X, \quad (1)$$

$$\begin{aligned}(\mathrm{D}^2-a^2)^2W = {} & Ra^2F+(C/\sigma)\{W\mathrm{D}(\mathrm{D}^2-a^2)W+2\mathrm{D}W(\mathrm{D}^2-a^2)W+3Z\mathrm{D}Z\} \\ & +\mathscr{T}^{\frac{1}{2}}\mathrm{D}Z-CQ\tau\{2\mathrm{D}H(\mathrm{D}^2-a^2)H+H\mathrm{D}(\mathrm{D}^2-a^2)H+3X\mathrm{D}X\} \\ & -Q\tau \mathrm{D}(\mathrm{D}^2-a^2)H, \end{aligned} \quad (2)$$

$$\tau(\mathrm{D}^2-a^2)H = C(W\mathrm{D}H-H\mathrm{D}W)-\mathrm{D}W, \quad (3)$$

$$\tau(\mathrm{D}^2-a^2)X = -C\{2(Z\mathrm{D}H-X\mathrm{D}W)+(H\mathrm{D}Z-W\mathrm{D}X)\}-\mathrm{D}Z, \quad (4)$$

$$(\mathrm{D}^2-a^2)F = W\mathrm{D}T_0+C(F\mathrm{D}W+2W\mathrm{D}F), \quad (5)$$

$$\mathrm{D}^2T_0 = \mathrm{D}(FW). \quad (6)$$

Here $\mathrm{D} \equiv \mathrm{d}/\mathrm{d}z$, and Z, W, H, X, F and T_0 are functions of z to be determined subject to the boundary conditions:

$$W = \mathrm{D}^2W = F = \mathrm{D}Z = X = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad 1; \quad (7)$$

$$T_0 = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad T_0 = -1 \quad \text{at} \quad z = 1; \quad (8)$$

$$\mathrm{D}H-aH = 0 \quad \text{at} \quad z = 0 \quad \text{and} \quad \mathrm{D}H+aH = 0 \quad \text{at} \quad z = 1. \quad (9)$$

It should be noted that the system of differential equations is in dimensionless form. The actual values of the temperature T, velocity $\boldsymbol{u}$ and magnetic field $\boldsymbol{H}$ are given by

$$T = \Delta T(T_0+Ff), \quad (10)$$

$$\boldsymbol{u} = \frac{\kappa}{d}\left(\frac{\mathrm{D}W}{a^2}\frac{\partial f}{\partial x}+\frac{Z}{a^2}\frac{\partial f}{\partial y}, \ \frac{\mathrm{D}W}{a^2}\frac{f}{\partial y}-\frac{Z}{a^2}\frac{\partial f}{\partial x}, \ Wf\right), \quad (11)$$

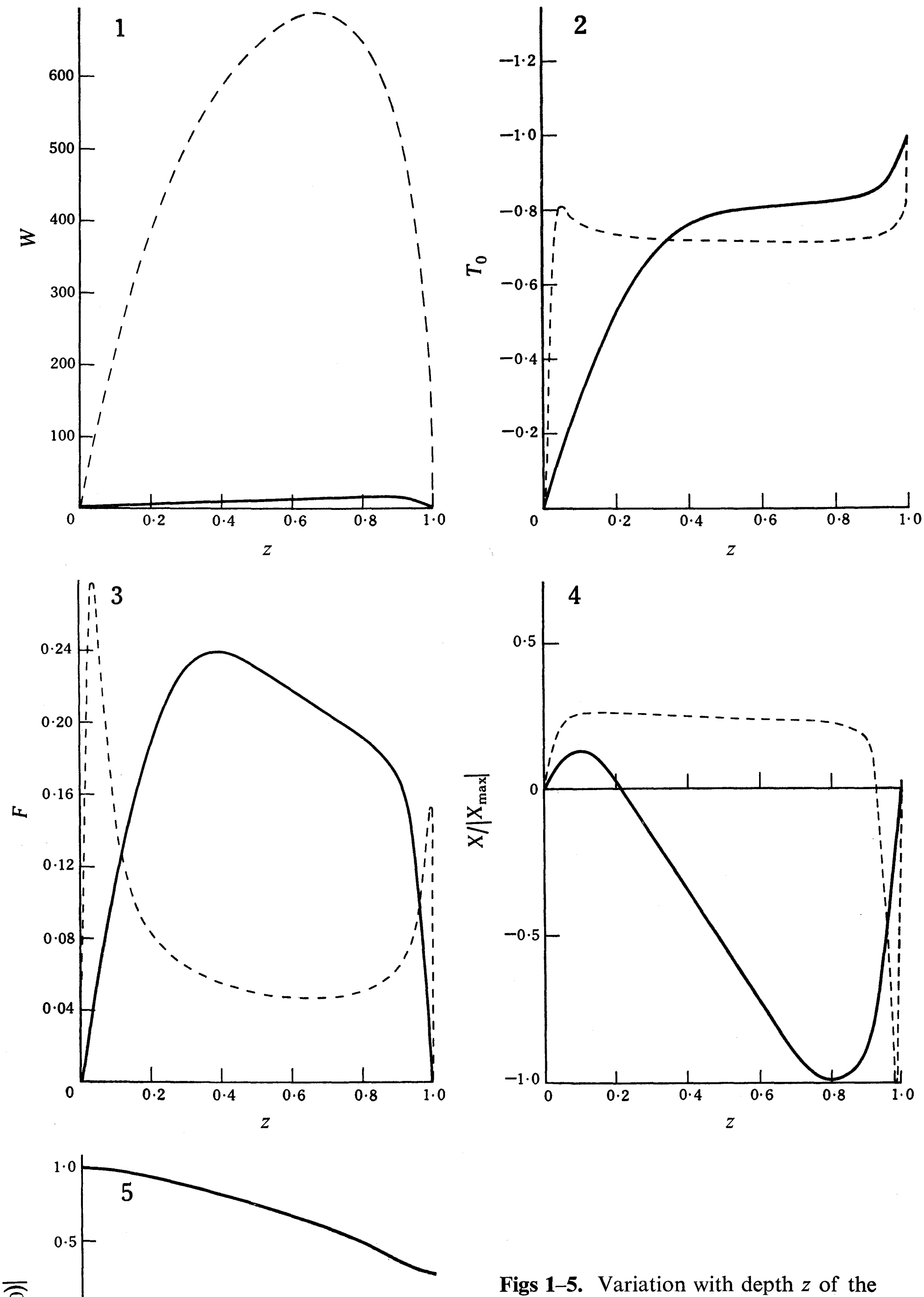
$$\boldsymbol{H} = H_0\left(\frac{\mathrm{D}H}{a^2}\frac{\partial f}{\partial x}+\frac{X}{a^2}\frac{\partial f}{\partial y}, \ \frac{\mathrm{D}H}{a^2}\frac{\partial f}{\partial y}-\frac{X}{a^2}\frac{\partial f}{\partial x}, \ 1+Hf\right), \quad (12)$$

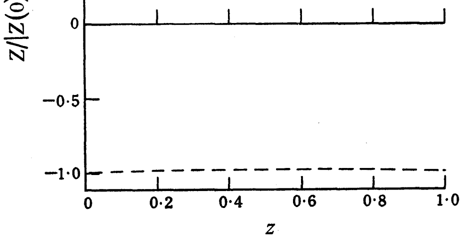
where ΔT is the temperature drop across the layer, κ the thermal diffusivity, d the thickness of the layer, a the horizontal wave number and H_0 the imposed magnetic field.

In the basic equations (1)–(6), σ is the Prandtl number ν/κ; $Q = \mu^* d^2 H_0^2/4\pi\mu\eta$ is the Chandrasekhar number, which depends on the permeability μ^*, the viscosity μ and the resistivity η of the fluid; $\tau = \eta/\kappa$ is the magnetic Prandtl number and $C = 1/\sqrt{6}$. In addition, R is the Rayleigh number defined by

$$R = g\alpha d^3\Delta T/\kappa\nu, \quad (13)$$

$\zeta = \kappa\mathscr{T}^{\frac{1}{2}}Zf/d^2$ is the z component of the vorticity and $\xi = H_0\mathscr{T}^{\frac{1}{2}}Xf/4\pi d$ is the





Figs 1–5. Variation with depth z of the vertical velocity W, the mean temperature T_0, the temperature fluctuation F, the normalized current density $X/|X_{max}|$ and the normalized vorticity $Z/|Z(0)|$ at Rayleigh number $R = 10^7$ for solutions of type I (dashed curves) and type II (full curves).

z component of the current density. Finally, $\mathscr{T}$ is the Taylor number defined by

$$\mathscr{T} = 4d^4\Omega_0^2/\nu^2, \tag{14}$$

Ω_0 being the angular velocity about the z axis and ν the kinematic viscosity.

Numerical Results and Discussion

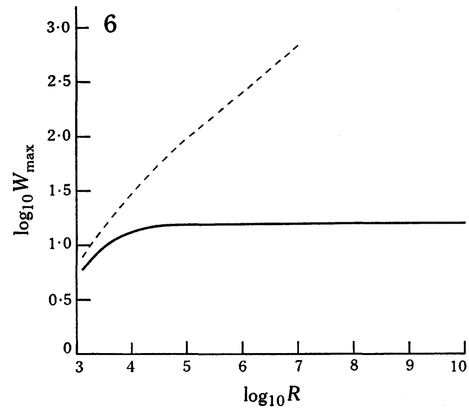
Solutions to the basic system of equations (1)–(6), satisfying boundary conditions (7)–(9), were found numerically with the help of a finite difference method. The numerical integrations were carried out for $\sigma = \eta = 1$. The results illustrated in Figs 1–10 are for $Q = 1$ and $\mathscr{T} = 10$. Two types of solutions were found and their characteristics are illustrated in the figures. The dashed curves correspond to a solution which we shall refer to as type I and the full curves correspond to a type II solution. The z dependence of the variables is given for both solutions in Figs 1–5 for $R = 10^7$.

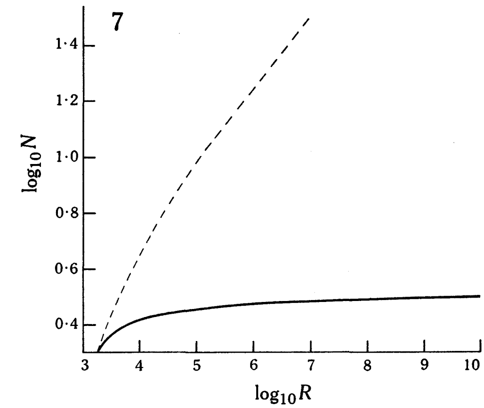
Table 1. Type II solution at $R = 1{\cdot}443\times10^{10}$

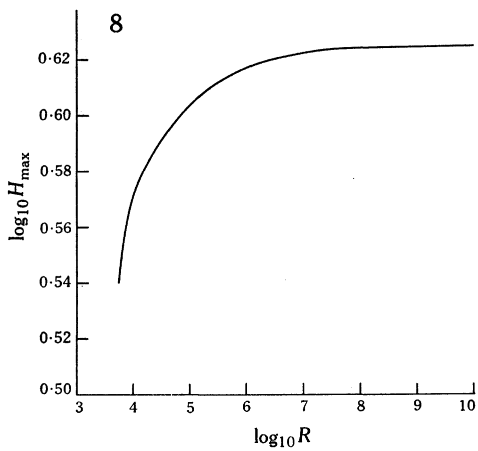
z	T_0	F	$X/\sqrt{R}$	H	$Z/\sqrt{R}$	W
0·0000	0·0000	0·0000	0·0000	2·8901	0·5671	0·0000
0·1005	−0·2994	0·1136	0·0111	3·6317	0·5588	3·0136
0·2010	−0·5368	0·1961	0·0026	4·0496	0·5353	5·9361
0·3015	−0·6873	0·2363	−0·0128	4·2064	0·5011	8·4562
0·4020	−0·7665	0·2437	−0·0299	4·1731	0·4611	10·5268
0·5025	−0·8030	0·2342	−0·0464	3·9994	0·4184	12·2105
0·6030	−0·8194	0·2197	−0·0619	3·7023	0·3740	13·5771
0·7035	−0·8383	0·2060	−0·0750	3·2589	0·3275	14·6632
0·8040	−0·8360	0·1942	−0·0817	2·5847	0·2755	15·4816
0·9045	−0·8536	0·1720	−0·0686	1·4811	0·2094	15·7623
1·0000	−1·0000	0·0000	0·0000	0·3703	0·1524	0·0000

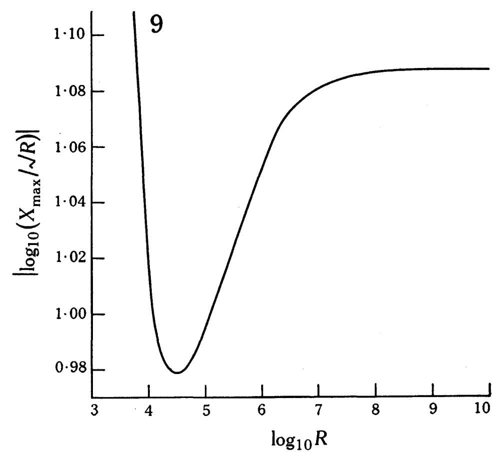
It is seen that the type I solutions behave very much like the solutions found by Van der Borght and Murphy (1973). They have a high value of the vertical velocity W (Fig. 1), an extended isothermal region in the main stream (Fig. 2) and sharp peaks in the thermal fluctuation F in the boundary layers (Fig. 3). Even the current density X exhibits a sharp peak near the upper boundary, whereas the vorticity Z is nearly constant across the layer but has a negative value, i.e. is of opposite sign to the general background rotation.

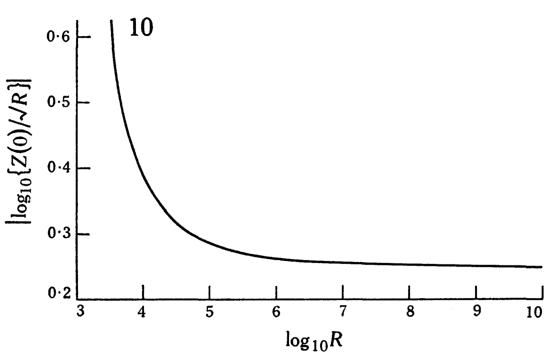
On the other hand, solutions of type II (also for $R = 10^7$) show a surprisingly low value of the vertical velocity (Fig. 1), less tendency to the establishment of an isothermal region (Fig. 2) and the absence of peaks in the temperature fluctuation (Fig. 3). Furthermore, the current density varies in a very different manner (Fig. 4) and the vorticity is now positive and shows definite variation across the layer. A further example of a type II solution for extremely large values of the Rayleigh number ($R = 1{\cdot}443\times10^{10}$) is given in Table 1. It can be seen that type II solutions are characterized by high values of the current density and vorticity, low values of the vertical velocity (i.e. weak convective motions) and the absence of peaks in the temperature fluctuation curve. In contrast to the type I solutions, which show very











Figs 6–10. Variation with Rayleigh number R of the maximum vertical velocity W_{max}, the Nusselt number N, the magnetic field strength H_{max}, the scaled current density $(X_{max}/\surd R)$ and the scaled vorticity $Z(0)/\surd R$ for solutions of type I (dashed curves) and type II (full curves).

strong convective motions, the vorticity for type II solutions is positive, and it seems that the latter are more closely related to solutions of a cyclonic type.

The behaviour of the solutions with increasing Rayleigh number is illustrated in Figs 6–10. It is seen that type I solutions behave in the normal manner and show a marked increase of the maximum vertical velocity W_{max} and the Nusselt number N with increasing Rayleigh number. Type II solutions on the other hand behave in a different way. The vertical velocity, Nusselt number and perturbation of the magnetic field H_{max} tend to a constant value as R increases (Figs 6–8), whereas the current density and vorticity keep increasing as $\sqrt{R}$ (Figs 9 and 10).

One could characterize the type I solutions as having increasingly strong convective motions as the Rayleigh number increases, with moderate vorticity and current density. Type II solutions on the other hand show a levelling off in the strength of the convective motions but a marked increase in vorticity and strength of the current density with Rayleigh number.

Conclusions

Two different stationary solutions have been found for the basic equations describing finite amplitude convection in a rotating medium in the presence of a magnetic field. One of these displays strong convective motions, whereas the other is characterized by a marked increase in the vorticity and current density with Rayleigh number. The nature of the initial disturbance would ultimately decide which solution would become established, and a numerical investigation of this problem would require the solution of the time-dependent equations. The existence of two stationary solutions is perhaps not surprising, since instability can set in as overstable oscillations which, in turn, may be transformed into convective motions due to the existence of metastable states.

References

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