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The Polarization Limiting Region—An Empty Concept

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Abstract

Polarization changes in stratified anisotropic media are shown to occur at a rate which is substantially independent of the degree of inhomogeneity of the medium, in contrast to the work of earlier authors. As a consequence, the concept of a 'polarization limiting region' introduced by the earlier authors is not useful.

Introduction

It has been generally agreed that there are two types of wave propagation in an inhomogeneous anisotropic medium (e.g. Zhelezniakov 1970, Section 24; Budden 1961, Chapter 19). In one type, the effect of the inhomogeneity is unimportant and the anisotropy causes polarization changes (Faraday rotation). In the other, the effect of the anisotropy is masked by the effect of the inhomogeneity, and no polarization changes occur. These two types of propagation correspond to weak and strong coupling, which are expressed in media stratified in the z direction by the conditions $Q \ll 1$ and $Q \gg 1$ respectively, where the coupling ratio Q is given by (Cohen 1960)

$$Q = \frac{|R_o' R_e'|^{\frac{1}{2}}}{\frac{1}{2}|R_o - R_e|\,|\mu_o - \mu_e|\,(\omega/c)}. \tag{1}$$

Here R_o (R_e) is the ratio $E_x : E_y$ of the electric fields of waves in the ordinary (extraordinary) mode, μ_o (μ_e) is the refractive index for the ordinary (extraordinary) mode, and primes are used throughout the present paper to denote differentiation with respect to z. Regions in which $Q \gg 1$ are said to be separated from regions in which $Q \ll 1$ by the 'polarization limiting region' (PLR). On one side of the PLR the inhomogeneity is unimportant while on the other the anisotropy is unimportant.

The concept of a PLR has been used in the following type of argument. In the interplanetary medium the inequality $Q \gg 1$ usually holds for radio signals of astronomical interest. Thus the polarization of radio waves arriving at the Earth is characteristic of their polarization in the last PLR that they encountered. From a knowledge of wave properties one can then infer information concerning the nature of the medium at the PLR. Arguments following this pattern have been used in connection with Jupiter's decametric radio emission (e.g. Goertz 1974) and type III solar radio bursts (Melrose 1975). In the present paper it is shown that there is only one type of propagation and that the concept of a PLR (in the foregoing sense) is empty.

Conventional Treatment of Polarization Limiting

In a medium stratified in the z direction, the equations of magnetoionic theory may be taken in the form (e.g. Budden 1961, p. 386)

$$e' + (\mathrm{i}\omega/c)\mathbf{T}e = 0\,, \tag{2}$$

where e is a column matrix defined by

$$e^{\mathrm{T}} = (e_1,\ e_2,\ e_3,\ e_4) = (E_x,\ -E_y,\ B_x,\ B_y)\,,$$

and $\mathbf{T}$ is a 4×4 matrix whose elements depend on the values of the background quantities (such as the plasma density or the magnetic field strength) but not on their gradients.

Beginning with an equation similar to equation (2), previous authors made a number of changes of variable and arrived at an equation which they interpreted as showing the existence of the two forms of propagation. However, in each case, the interpretation of the equation in the limit of strong inhomogeneity was based on the neglect of a significant term. For example, Cohen (1960) derived the equation (in a slightly different notation)

$$\begin{pmatrix} U_{\mathrm{o}}' \\ U_{\mathrm{e}}' \end{pmatrix} = \begin{pmatrix} (-\mathrm{i}\omega/2c)(\mu_{\mathrm{o}}-\mu_{\mathrm{e}}) & \psi_{\mathrm{e}} \\ \psi_{\mathrm{o}} & (\mathrm{i}\omega/2c)(\mu_{\mathrm{o}}-\mu_{\mathrm{e}}) \end{pmatrix} \begin{pmatrix} U_{\mathrm{o}} \\ U_{\mathrm{e}} \end{pmatrix}, \tag{3}$$

where

$$\psi_{\mathrm{o}} = \frac{R_{\mathrm{o}}'}{R_{\mathrm{o}}-R_{\mathrm{e}}} \exp\!\left(\int \mathrm{d}z \, \frac{R_{\mathrm{o}}'+R_{\mathrm{e}}'}{R_{\mathrm{e}}-R_{\mathrm{o}}} \right),$$

$$\psi_{\mathrm{e}} = \frac{R_{\mathrm{e}}'}{R_{\mathrm{e}}-R_{\mathrm{o}}} \exp\!\left(\int \mathrm{d}z \, \frac{R_{\mathrm{o}}'+R_{\mathrm{e}}'}{R_{\mathrm{o}}-R_{\mathrm{e}}} \right).$$

He argued that if $|\psi_{\mathrm{o}}\psi_{\mathrm{e}}|^{\frac{1}{2}} \gg \frac{1}{2}|\mu_{\mathrm{o}}-\mu_{\mathrm{e}}|$ then one could neglect the $(\pm\mathrm{i}\omega/2c)(\mu_{\mathrm{o}}-\mu_{\mathrm{e}})$ terms in equation (3). He produced solutions to the resultant simplified equations and showed that these corresponded to waves propagating without change in polarization. However, the neglected terms $(\pm\mathrm{i}\omega/2c)(\mu_{\mathrm{o}}-\mu_{\mathrm{e}})$ affect the polarization of waves significantly over scale lengths of the order of $(2c/\omega)|\mu_{\mathrm{o}}-\mu_{\mathrm{e}}|^{-1}$. Since these are precisely the scale lengths over which the polarization of waves in a homogeneous medium changes, it is unreasonable to infer the existence of a second, new form of propagation in which the polarization of waves is fixed.

Alternative Treatment

In this section it is argued on general grounds that in a stratified anisotropic medium the equation describing the way E_x and E_y vary is formally identical to the familiar equation describing this variation in a homogeneous anisotropic medium. As a result, the equation may be written down almost immediately.

Only one assumption is needed to carry out the procedure outlined in the previous paragraph: the desired solutions of equation (2) are assumed to be expressible as a linear combination of two of the eigenvectors of $\mathbf{T}$. Physically, this requirement corresponds to assuming that each natural mode of the medium couples strongly to at most one other natural mode. The previous authors have all made this assumption in one form or another. Melrose (1974) has given the following mathematical criterion for the assumption to be justifiable:

$$|\mu_o'|/\mu_o^2 \ll 2\omega/c \quad \text{and} \quad |\mu_e'|/\mu_e^2 \ll 2\omega/c. \tag{4}$$

After making the assumption, a solution $\boldsymbol{e}$ of equation (2) may be written as

$$\boldsymbol{e} = \lambda \boldsymbol{a} + \mu \boldsymbol{b}, \tag{5}$$

where $\boldsymbol{a}$ and $\boldsymbol{b}$ are eigenvectors of $\mathbf{T}$ corresponding to natural modes which couple together strongly.

Using equation (5), expressions for λ and μ may be obtained as linear combinations of e_1 and e_2 with coefficients depending on a_1, a_2, b_1 and b_2. Upon substituting these expressions back into equation (5), e_1, e_2, e_3 and e_4 may all be expressed as linear combinations of e_1 and e_2. The coefficients of these linear combinations depend on the background quantities through the dependence of $\boldsymbol{a}$ and $\boldsymbol{b}$ on the background quantities, but they do not depend on the gradients in the background quantities because $\boldsymbol{a}$ and $\boldsymbol{b}$ do not depend on the gradients.

The expressions for e_3 and e_4 thus obtained may be substituted into the first two equations of the system (2) to yield an equation of the type

$$\boldsymbol{f}' + (\mathrm{i}\omega/c)\mathbf{M}\boldsymbol{f} = 0, \tag{6}$$

where the column matrix $\boldsymbol{f}$ is defined by

$$\boldsymbol{f}^{\mathrm{T}} = (f_1, \quad f_2) = (e_1, \quad e_2) \quad [= (E_x, \quad -E_y)],$$

and $\mathbf{M}$ is a 2×2 matrix. From their construction, the elements of $\mathbf{M}$ do not depend on the gradients of the background quantities and hence they may be calculated assuming the medium to be homogeneous. The elements of $\mathbf{M}$ are found explicitly by firstly Fourier transforming equation (5) to obtain a system of linear equations. The nontrivial solutions of this system are known from ordinary magnetoionic theory (e.g. Budden 1961, Chapter 5), as they are just $(E_x, -E_y)$ for the appropriate modes of a homogeneous medium. From these solutions, the elements of the original matrix $\mathbf{M}$ may be determined. For simplicity the elements of $\mathbf{M}$ are obtained here only for vertical incidence ($k_x = k_y = 0$, where k_x and k_y are Fourier variables associated with x and y). The previous authors have also almost exclusively concentrated on this case. We then have

$$\mathbf{M} = \tfrac{1}{2}(\mu_o + \mu_e)\mathbf{I} + \tfrac{1}{2}(\mu_o - \mu_e)\mathbf{N}, \tag{7}$$

where $\mathbf{I}$ is the 2×2 unit matrix and $\mathbf{N}$ is defined by

$$\mathbf{N} = (1 + |R_o|^2)^{-1}\begin{pmatrix} |R_o|^2 - 1 & 2R_o \\ 2R^* & 1 - |R_o|^2 \end{pmatrix}, \tag{8}$$

with asterisks denoting complex conjugation. After making the change of variable (Cohen 1960)

$$\boldsymbol{g} = \boldsymbol{f}\exp\left((\mathrm{i}\omega/2c)\int \mathrm{d}z\,(\mu_{\mathrm{o}}+\mu_{\mathrm{e}})\right),$$

equation (6) becomes

$$\boldsymbol{g}' + (\mathrm{i}\omega/2c)(\mu_{\mathrm{o}}-\mu_{\mathrm{e}})\mathbf{N}\boldsymbol{g} = 0\,. \tag{9}$$

Conclusions

From equation (9) it is clear that polarization changes occur over scale lengths of the order of $(c/\omega)\,|\,\mu_{\mathrm{o}}-\mu_{\mathrm{e}}\,|^{-1}$ irrespective of the degree of inhomogeneity of the medium, subject of course to the restriction that the inequalities (4) be satisfied. Thus for no region is it reasonable to say that waves propagate without change in polarization—Faraday rotation always occurs, although it may be negligible over short enough distances. Accordingly, the concept of a polarization limiting region as a region where the polarization of a wave becomes fixed is empty, because no such regions exist.

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