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Vorticity of a Perfect Fluid Undergoing a Gravitational Collapse

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Abstract

The vorticity propagation equation for a perfect fluid in general relativity is derived in a form which is the same as that of Maxwell's equation for the magnetic field four-vector in relativistic magnetohydrodynamics. Starting from this result, an expression for the change of vorticity during a gravitational collapse is obtained in terms of the spatial geometry, using a procedure similar to that introduced by Cocke (1966) in relativistic magnetohydrodynamics. It is assumed that the equation of state of the fluid is $p = \alpha\mu$, where α is a constant and μ is the total proper energy density. If $\frac{2}{3} < \alpha \leqslant 1$, it is found that the vorticity tends to zero during an isotropic collapse, in agreement with a result obtained previously by Ellis (1973) using a different procedure. Nonisotropic collapses are also considered. The dynamical importance of vorticity in a gravitational collapse is examined by considering the behaviour of ω^2/μ.

Introduction

Cocke (1966) has investigated the problem of a gravitational collapse in relativistic magnetohydrodynamics (MHD) by obtaining expressions in terms of the spatial geometry for the change of the magnetic energy density during the collapse. This useful approach allowed both isotropic and nonisotropic collapses to be analysed. The purpose of the present paper is to apply Cocke's technique to examine, in general relativity, the change in the vorticity of a perfect fluid which undergoes a gravitational collapse. Both isotropic and nonisotropic collapses are considered. We first derive the vorticity propagation equation in a form which is the same as Maxwell's equation for the magnetic field four-vector in relativistic MHD, and this in turn allows us to express the vorticity in terms of the spatial geometry.

We use units in which the velocity of light is unity. Latin indices are understood to run from 0 to 3 and Greek indices from 1 to 3. The time coordinate is denoted by x^0 and we adopt the convention that the metric tensor g_{ab} has signature $(-+++)$. Ordinary partial differentiation is denoted by a comma and covariant differentiation by a semicolon. We use an overhead dot to denote the covariant derivative along the particle world line; so that, for example, $\dot{A}^a \equiv A^a{}_{;b}\,u^b$. Parentheses are used to denote symmetrization and square brackets to denote skew-symmetrization.

Vorticity Propagation Equation

We first derive, for a perfect fluid, a vorticity propagation equation which has the same form as Maxwell's equation for the magnetic field H^a in relativistic MHD, namely (Lichnerowicz 1967),

$$(H^a u^b - H^b u^a)_{;b} = 0. \tag{1}$$

We start with the identity

$$u_{b;cd} - u_{b;dc} = R_{sbcd}\, u^s \tag{2}$$

and operate on it with the totally-skew permutation tensor η^{abcd}. On using the symmetry property $R_{s[bcd]} = 0$, this yields

$$\eta^{abcd}\, u_{b;cd} = 0. \tag{3}$$

Now, we have

$$u_{b;c} = \sigma_{bc} + \tfrac{1}{3}\theta h_{bc} + \omega_{bc} - \dot{u}_b u_c , \tag{4}$$

where σ_{bc} is the shear tensor, $\theta = u^a{}_{;a}$ is the expansion, $h_{bc} = g_{bc} + u_b u_c$ is the projection tensor, ω_{bc} is the vorticity tensor and $\dot{u}_b = u_{b;d}\, u^d$ is the acceleration vector. As σ_{bc} and h_{bc} are symmetric, equation (3) becomes, on using (4),

$$\eta^{abcd}(\omega_{bc} - \dot{u}_b u_c)_{;d} = 0. \tag{5}$$

Now

$$\omega_{bc} = \eta_{bcrs}\, \omega^r u^s , \tag{6}$$

where ω^r is the vorticity vector and, noting that

$$\eta^{bcad}\, \eta_{bcrs} = -4\delta^{[a}_r \delta^{d]}_s , \tag{7}$$

we see that equation (5) can be written as

$$(\omega^a u^b - \omega^b u^a)_{;b} + \tfrac{1}{2}\eta^{abcd}(\dot{u}_b u_c)_{;d} = 0. \tag{8}$$

Equation (8) is quite general. We now show that, for a perfect fluid with an equation of state $p = p(\mu)$, where μ is the total proper energy density, equation (8) can be written as

$$(r\omega^a u^b - r\omega^b u^a)_{;b} = 0, \tag{9}$$

where r is the acceleration potential defined by

$$r = \exp\left(\int_{p_0}^{p} \frac{\mathrm{d}p}{p+\mu}\right), \tag{10}$$

with p_0 some standard pressure. Equation (9) for $r\omega^a$ compares with equation (1) for H^a. As the differential equation of the streamlines of a perfect fluid is

$$(\mu + p)\dot{u}_a + h^b_a p_{,b} = 0, \tag{11}$$

we have in terms of r

$$\dot{u}_a = -h^b_a (\ln r)_{,b}. \tag{12}$$

Thus we have

$$\tfrac{1}{2}\eta^{abcd}(\dot{u}_b u_c)_{;d} = -\tfrac{1}{2} r^{-1}\, \eta^{abcd}\, r_{,b}\, u_{c;d} \tag{13}$$

and, on using equation (4) again, we can rewrite this as

$$\tfrac{1}{2}\eta^{abcd}(\dot{u}_b u_c)_{;d} = \tfrac{1}{2} r^{-1}\, \eta^{abcd}\, r_{,b}\, \dot{u}_c u_d - \tfrac{1}{2} r^{-1}\, \eta^{abcd}\, r_{,b}\, \omega_{cd}. \tag{14}$$

But it follows from equation (12) that the first term on the right-hand side of (14) is zero. Using equations (6) and (7) we obtain finally

$$\tfrac{1}{2}\eta^{abcd}(\dot{u}_b u_c)_{;d} = (r_{,b}/r)(\omega^a u^b - \omega^b u^a). \tag{15}$$

On combining equations (8) and (15), we obtain (9), which is the vorticity propagation equation we require in the following analysis. We see that in general relativity $r\omega^a$ and H^a obey the same equation, rather than ω^a and H^a, as in nonrelativistic theory.

Expressions for ω and μ in Terms of Spatial Geometry

Equation (9) can be written in terms of an ordinary divergence as

$$\left(|g|^{\frac{1}{2}} r(\omega^a u^b - \omega^b u^a)\right)_{,b} = 0, \tag{16}$$

where g is the determinant of the metric tensor g_{ab}. If we now consider a co-moving coordinate system in which the contravariant spatial components of the fluid four-velocity are zero throughout all space–time, then we have

$$u^a = |g_{00}|^{-\frac{1}{2}}\delta^a_0. \tag{17}$$

We will not assume that the system possesses any particular symmetry properties, e.g. we do not assume that $g_{0\alpha} = 0$ ($\alpha = 1, 2, 3$). If we let $a = \alpha$ in equation (16) then it follows (using 17) that, in this co-moving coordinate system, (16) reduces to

$$\left((g/g_{00})^{\frac{1}{2}} r\omega^\alpha\right)_{,0} = 0. \tag{18}$$

Thus it follows that

$$\omega^\alpha = r^{-1}(g_{00}/g)^{\frac{1}{2}} W^\alpha(x^\beta), \tag{19}$$

where $W^\alpha(x^\beta)$ are three functions of the spatial coordinates x^β only. We now introduce the spatial matric tensor $\gamma_{\alpha\beta}$ defined by (Landau and Lifshitz 1971)

$$\gamma_{\alpha\beta} = g_{\alpha\beta} - g_{0\alpha}g_{0\beta}/g_{00}. \tag{20}$$

It can be shown that $g = g_{00}\gamma$, where γ is the determinant of $\gamma_{\alpha\beta}$ (Landau and Lifshitz). Thus equation (19) becomes

$$\omega^\alpha = W^\alpha(x^\beta)/r\sqrt{\gamma}. \tag{21}$$

We now chose coordinate axes so that initially $W^2 = W^3 = 0$ which, as the W^α are time independent, will always be true. It therefore follows from equation (21) that $\omega^2 = \omega^3 = 0$ always. Now it also follows from the identity $\omega_a u^a = 0$ and equation (17) that, in the co-moving coordinate system, $\omega_0 = 0$. Thus $\omega = (\omega_1\omega^1)^{\frac{1}{2}}$, where $\omega = (\omega_a\omega^a)^{\frac{1}{2}}$ is the magnitude of the vorticity. To evaluate ω, we express ω_1 in terms of ω^1:

$$\omega_1 = g_{10}\omega^0 + g_{11}\omega^1. \tag{22}$$

But as $\omega_0 = 0$, it follows from the identity $\omega_0 = g_{0a}\omega^a$ that $\omega^0 = -(g_{01}/g_{00})\omega^1$. Therefore, by equations (20) and (22), we have $\omega_1 = \gamma_{11}\omega^1$ and hence $\omega = \sqrt{\gamma_{11}}\,\omega^1$.

Thus using equation (21) we have

$$\omega = r^{-1}(\gamma_{11}/\gamma)^{\frac{1}{2}}\, W^1(x^\beta). \tag{23}$$

It remains to obtain corresponding expressions for μ and r.

We consider μ first and assume that the equation of state of the fluid is $p = \alpha\mu$, where α is a constant. To ensure that the velocity of sound relative to the fluid does not exceed the velocity of light, we must have $0 \leqslant \alpha \leqslant 1$. With this equation of state, the continuity equation for a perfect fluid, namely,

$$\dot{\mu} + (p+\mu)u^a{}_{;a} = 0 \tag{24}$$

becomes

$$(\mu^{1/(1+\alpha)}\, u^a)_{;a} = 0\,. \tag{25}$$

Using a similar argument to that used above, this equation reduces, in the co-moving coordinate system, to

$$(\sqrt{\gamma}\, \mu^{1/(1+\alpha)})_{,0} = 0\,. \tag{26}$$

Thus it follows that

$$\mu = D(x^\beta)/\gamma^{\frac{1}{2}(1+\alpha)}, \tag{27}$$

where $D(x^\beta)$ is a function of the spatial coordinates x^β only. Also, with the equation of state $p = \alpha\mu$, we have from equation (10)

$$r = (\mu/\mu_0)^{\alpha/(1+\alpha)}, \tag{28}$$

where $\mu_0 = \alpha^{-1} p_0$. Thus, by equation (27) we have

$$r = R(x^\beta)/\gamma^{\frac{1}{2}\alpha}, \tag{29}$$

where $R(x^\beta)$ is a function only of the spatial coordinates. Finally, using equations (23) and (29) we can express ω in terms of the spatial geometry as

$$\omega = \gamma^{\frac{1}{2}(\alpha-1)}\, \gamma_{11}^{\frac{1}{2}}\, W^1(x^\beta)/R(x^\beta). \tag{30}$$

The expressions (27) and (30) allow us to analyse the changes in μ and ω in terms of the spatial geometry as the fluid collapses.

Gravitational Collapse

We use Cocke's (1966) definition of a gravitational collapse: a gravitational collapse is defined as a situation in which at some fixed spatial point (x^β) in the co-moving coordinate system, μ tends to infinity for some sequence of times x^0. From equation (27) we see that geometrically the necessary and sufficient condition for a collapse to occur is that $\gamma \to 0$ for some sequence of times. As $\gamma \to 0$ there are three possibilities for γ_{11}: either γ_{11} remains nonzero or diverges; or $\gamma_{11} \to 0$ but γ_{11}/γ still tends to infinity; or $\gamma_{11} \to 0$ and γ_{11}/γ remains finite or tends to zero. We now consider each situation separately.

Situation 1: γ_{11} *Remains Nonzero or Diverges*

The direction of the collapse is essentially perpendicular to $\boldsymbol{\omega}$, as the length of the fluid element in the direction of $\boldsymbol{\omega}$, which is proportional to $(\gamma_{11})^{\frac{1}{2}}$, does not tend to zero. From equation (30) it follows that, if $0 \leqslant \alpha < 1$, ω will always tend to infinity. For $\alpha = 1$, which is the stiffest possible equation of state, we have $\omega \sim (\gamma_{11})^{\frac{1}{2}}$ and so ω will remain finite or diverge depending on whether γ_{11} remains finite or diverges. A particular case of interest in this situation is that in which γ_{11} remains constant. The collapse is then perpendicular to $\boldsymbol{\omega}$. If γ_{11} is constant then it follows from equation (30) that $\omega \sim \gamma^{\frac{1}{2}(\alpha-1)}$ or, on using equation (27), that $\omega \sim \mu^{(1-\alpha)/(1+\alpha)}$. The variation of ω and ω^2/μ as a power of μ for significant values of α during a collapse in which γ_{11} remains constant is as follows:

α	0	1/3	1
ω	μ	$\mu^{\frac{1}{2}}$	const.
ω^2/μ	μ	const.	μ^{-1}

The situation $\alpha = 0$ corresponds to 'dust' and $\alpha = \frac{1}{3}$ to radiation. The results for dust in general relativity coincide with nonrelativistic results. Zel'dovich (1962) has argued that an equation of state with $\frac{1}{3} < \alpha \leqslant 1$ is possible and could be valid for extremely dense matter such as occurs in the final stages of the collapse. The variation of ω^2/μ, which by Raychaudhuri's equation is a measure of the dynamical importance of vorticity (Ellis 1973), is also given as a power of μ in the above tabulation. We see from the tabulation that ω always diverges to infinity except for $\alpha = 1$ when it remains constant. Thus for no value of α does ω tend to zero, unlike the case of an isotropic collapse as shown below. If $\alpha < \frac{1}{3}$, the vorticity becomes more important dynamically as the collapse proceeds, while if $\frac{1}{3} < \alpha \leqslant 1$ its dynamical importance decreases in the collapse.

Situation 2: γ_{11} *Vanishes and* γ_{11}/γ *Diverges*

As $\gamma_{11} \to 0$, the fluid element is crushed in the direction of $\boldsymbol{\omega}$ as well as in directions perpendicular to $\boldsymbol{\omega}$. A particular case of interest in this situation is the isotropic collapse for which $\gamma_{11} = \gamma^{\frac{1}{3}}$. By equations (27) and (30) we have for an isotropic collapse

$$\omega \sim \gamma^{\frac{1}{2}\alpha-\frac{1}{3}} \qquad \text{and} \qquad \omega \sim \mu^{(2-3\alpha)/3(1+\alpha)}.$$

The variation of ω and ω^2/μ as a power of μ for significant values of α during an isotropic collapse is as follows:

α	0	1/9	1/3	2/3	1
ω	$\mu^{2/3}$	$\mu^{1/2}$	$\mu^{1/4}$	const.	$\mu^{-1/6}$
ω^2/μ	$\mu^{1/3}$	const.	$\mu^{-1/2}$	μ^{-1}	$\mu^{-4/3}$

We see that if $\alpha > \frac{2}{3}$ then ω decreases and will eventually tend to zero as the collapse proceeds, the greatest rate of decrease being proportional to $\mu^{-1/6}$ which corresponds to $\alpha = 1$. For $0 < \alpha < \frac{2}{3}$, ω tends to infinity but at a slower rate than the nonrelativistic prediction ($\alpha = 0$). For $\alpha = \frac{2}{3}$, ω will remain constant during an isotropic collapse. Finally, for $\alpha < \frac{1}{9}$ the dynamical importance of ω increases during the collapse, while for $\frac{1}{9} < \alpha \leqslant 1$ its importance decreases. These results agree with those already obtained by Ellis (1973) using a different procedure.

Situation 3: γ_{11} *Vanishes and* γ_{11}/γ *Remains Finite*

It follows from equation (30) that $\omega \sim \gamma^{\frac{1}{2}\alpha}(\gamma_{11}/\gamma)^{\frac{1}{2}}$ during the collapse. For $\alpha = 0$, $\omega \to 0$ if $\gamma_{11}/\gamma \to 0$ but will otherwise remain finite. For $0 < \alpha \leqslant 1$, ω will always tend to zero. Of particular interest in this situation is the collapse in which γ_{11}/γ remains constant. Physically, the cross-sectional area of an element of a vortex tube remains constant during the collapse. If γ_{11}/γ is constant, we have

$$\omega \sim \gamma^{\frac{1}{2}\alpha} \qquad \text{and} \qquad \omega \sim \mu^{-\alpha/(1+\alpha)}\,.$$

The variation of ω and ω^2/μ as a power of μ for this mode of collapse is as follows:

α	0	1/3	1
ω	const.	$\mu^{-1/4}$	$\mu^{-1/2}$
ω^2/μ	μ^{-1}	$\mu^{-3/2}$	μ^{-2}

We see that for all values of α the dynamical importance of ω decreases as the collapse proceeds. Unlike the case for $\alpha = 0$ (nonrelativistic theory), ω does not remain constant for $\alpha > 0$ but participates in the collapse, eventually tending to zero.

Conclusions

We have seen that the method introduced by Cocke (1966) to analyse the gravitational collapse problem in relativistic MHD can also be applied to the collapse of a rotating perfect fluid in general relativity. Both isotropic and nonisotropic collapses were analysed, and significant departures from the predictions of nonrelativistic theory were found. The results we obtained agree with those derived previously by Ellis (1973). The present theory can also be applied to the expansion of a fluid from an initial singularity. As noted by Cocke, however, this analysis does not give information on which kind of collapse (or expansion) will occur for given initial conditions.

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