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Variation in Magnetic Field Strength During a Specific Anisotropic Gravitational Collapse

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Abstract

The MHD approximation has been made in general relativity to derive expressions in terms of the fluid's total proper energy density and rest-mass density for the variation in the strength of the magnetic field during the anisotropic gravitational collapse in which the condition $\theta_{ab} H^a H^b = 0$ holds throughout the collapse, where θ_{ab} is the expansion tensor. The physical significance of this condition is also examined.

In this paper, we consider the nonisotropic gravitational collapse in relativistic magnetohydrodynamics (MHD) for which

$$\theta_{ab} H^a H^b = 0 \tag{1}$$

holds during the collapse, where θ_{ab} is the expansion (or rate-of-strain) tensor and H^a is the magnetic field four-vector. We derive expressions for the variation in the magnetic field strength during the collapse in terms of the total proper energy density ρ and the rest-mass density r of the fluid. This allows us to determine the relative importance of the magnetic energy density and the fluid energy density during the collapse. We also consider the physical significance of the condition (1) which determines the mode of the collapse.

The results derived here can be regarded as generalizations of those obtained by Carstoiu (1963) in nonrelativistic MHD. Carstoiu showed that if the condition (1) were satisfied, with θ_{ab} the three-dimensional expansion tensor, then as the fluid evolves the magnetic field strength is proportional to the fluid density. The present analysis is similar to that of Yodzis (1971) and Mason (1976), who considered the case of an isotropic gravitational collapse defined by the condition $\sigma_{ab} H^a H^b = 0$, where σ_{ab} is the shear tensor. We use units in which the velocity of light is unity. The signature of space–time is $(-+++)$; covariant and ordinary partial differentiation are denoted by a semicolon and comma respectively; and an overhead dot denotes the covariant derivative along a particle world line so that, for example,

$$(T_{ab})^{\cdot} \equiv T_{ab;c}\, u^c .$$

Maxwell's Equations

In this section we obtain an expression for the magnetic field strength H in terms of the expansion $\theta = u^a{}_{;a}$, on the assumption that the condition (1) is satisfied.

Maxwell's equations in the MHD approximation are (Lichnerowicz 1967)

$$(H^a u^b - H^b u^a)_{;b} = 0. \tag{2}$$

If we contract equation (2) with H_a and use $H_a u^a = 0$, we obtain

$$H\dot{H} + H^2\theta - H^a H^b u_{a;b} = 0, \tag{3}$$

where $H^2 = H_a H^a > 0$. Now we have (Ellis 1971)

$$u_{a;b} = \theta_{ab} + \omega_{ab} - \dot{u}_a u_b, \tag{4}$$

where ω_{ab} is the vorticity tensor and $\dot{u}_a$ is the acceleration. Substituting equation (4) into (3) and noting that $H^b u_b = 0$ and also that, since ω_{ab} is skew-symmetric then $\omega_{ab} H^a H^b = 0$, we obtain

$$H\dot{H} + H^2\theta - H^a H^b \theta_{ab} = 0. \tag{5}$$

Thus, if the condition (1) is satisfied, we have

$$\dot{H}/H = -\theta. \tag{6}$$

Using this equation, we can now derive two expressions for H by obtaining θ first in terms of the fluid's total proper energy density ρ and secondly in terms of its rest-mass density r.

Expression for *H* in Terms of ρ

We assume that the fluid is a perfect fluid with the equation of state $p = (\gamma-1)\rho$, where γ is a constant satisfying $1 \leqslant \gamma \leqslant 2$. The upper limit $\gamma \leqslant 2$ ensures that the speed of sound $(\mathrm{d}p/\mathrm{d}\rho)^{\frac{1}{2}}$ does not exceed the speed of light. The case $\gamma = 1$ corresponds to dust, and $\gamma = 4/3$ corresponds to highly relativistic charged particles or isotropic radiation. Zel'dovich (1962) has argued that an equation of state with $4/3 < \gamma \leqslant 2$ is possible and could be valid for extremely dense matter such as could occur in the final stages of a gravitational collapse. We refer to the limiting case $\gamma = 2$ as 'stiff matter' (Ellis 1973).

The continuity equation in the MHD approximation is (Lichnerowicz 1967)

$$\dot{\rho} + (\rho + p)\theta = 0. \tag{7}$$

With the equation of state $p = (\gamma-1)\rho$, this expression reduces to

$$\theta = -\dot{\rho}/\gamma\rho. \tag{8}$$

Substituting for θ in equation (6), we obtain

$$\dot{H}/H = \dot{\rho}/\gamma\rho. \tag{9}$$

Thus $H\rho^{-1/\gamma}$ is constant on the world line of a fluid element and so, as the fluid evolves, H is proportional to $\rho^{1/\gamma}$. The variation, as a power of ρ, of H and of the magnetic energy density $\frac{1}{2}\mu H^2$, is as follows for the case of dust, isotropic radiation and stiff matter:

γ	1	4/3	2
H	ρ	$\rho^{3/4}$	$\rho^{1/2}$
$\frac{1}{2}\mu H^2$	ρ^2	$\rho^{3/2}$	ρ

In a gravitational collapse in which the condition (1) holds, the magnetic energy density will always grow faster than the total fluid energy density ρ, except for the case of stiff matter when it will grow at the same rate as ρ. Thus, if $1 \leqslant \gamma < 2$, the magnetic energy will eventually become the dominant form of energy independent of how small the magnetic field strength was initially. This compares with the situation in an isotropic gravitational collapse in which $\frac{1}{2}\mu H^2$ will grow more slowly than ρ if $4/3 < \gamma \leqslant 2$ (Yodzis 1971). Finally we note that Carstoiu's (1963) nonrelativistic result coincides with that for dust in general relativity.

Expression for H in Terms of r

Conservation of rest-mass implies that

$$(ru^a)_{;a} = 0\,, \tag{10}$$

so that we have $\theta = -\dot{r}/r$. Using this expression for θ in equation (6), it follows that Hr^{-1} is constant on the world line of a fluid element, and so H is proportional to r as the fluid evolves. This result is independent of the equation of state of the fluid. In a gravitational collapse in which the condition (1) holds, the magnetic energy density will grow at a rate proportional to r^2, and so will always eventually become more important than the rest-mass density. The result of this section coincides with Carstoiu's (1963) nonrelativistic result if we identify the classical fluid density with the rest-mass density in general relativity.

Physical Interpretation

To determine the physical significance of the condition (1) which determines the mode of collapse, we consider the relative position vector $X^a_\perp$ between two neighbouring fluid particles. This vector can be split into a relative distance δl and a direction n^a. The rate of change of relative distance is given by (Ellis 1971)

$$(\delta l)^{\boldsymbol{\cdot}}/\delta l = \theta_{ab}\, n^a n^b\,. \tag{11}$$

If we suppose that δl is the relative distance between two neighbouring particles on a magnetic field line then we obtain $n^a = H^a/H$. Because of the frozen-in property of magnetic field lines (Ellis 1973), these particles will always lie on the field line. Thus it follows from equation (11) that the condition (1) implies that the relative distance between neighbouring particles on magnetic field lines remains constant. If the fluid is collapsing, the direction of the collapse will be perpendicular to the magnetic field lines, there being no compression along the field lines. With this interpretation, it can be checked that the results obtained above agree with those which can be derived by considering the case $\gamma_{11} = $ const. in Cocke's (1966) analysis or by setting $\delta l = $ const. in the work of Ellis (1973).

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