Aust. J. Phys., 1976, **29**, 483–8

Generalized Statistics and the Quark Model

H. S. Green

Department of Mathematical Physics, University of Adelaide, P.O. Box 498, Adelaide, S.A. 5001.

Abstract

It is suggested that all symmetries in particle physics are either dynamical, and connected with Poincaré invariance, or statistical, and connected with invariance under interchange of similar particles. To sustain this view, it is necessary to admit quantum statistics more general than Fermi or Bose statistics. Two different generalizations are considered: parastatistics and a new generalization called modular statistics. A discussion is developed of difficulties which arise with parastatistics of order three or more, but are resolved with the help of the corresponding modular statistics. Applications to the Nagoya model, the standard quark model and 'coloured' quark models are described, together with a dynamical model in which the quarks are identified with different kinds of tachyons.

Dynamical and Statistical Symmetries

Much of high energy nuclear physics is concerned with the invariance of the fundamental laws under various kinds of transformations. Wherever such an invariance is found, it is associated with a particular kind of symmetry, and this in turn implies a particular conservation law. It can be argued that all the information we have concerning the elementary particles can be summarized in conservation laws of this kind. There are two, and perhaps only two, kinds of invariance which can be readily understood:

(1) Poincaré invariance is the invariance under rotations and Lorentz transformations, and translations in space and time. These invariance properties are associated with dynamical symmetries, which in turn imply the conservation of energy and momentum, angular momentum etc. (This group of conserved quantities is denoted by $P_\lambda, J_{\lambda\mu}$ $(\lambda, \mu = 0,1,2,3)$, where P_0, the energy, is the generator of translations in time; $\boldsymbol{P} = (P_1, P_2, P_3)$, the momentum, is the generator of translations in space; $\boldsymbol{J} = (J_{23}, J_{31}, J_{12})$, the angular momentum, is the generator of rotations; and $\boldsymbol{J}_0 = (J_{10}, J_{20}, J_{30})$ is the generator of Lorentz transformations.)

(2) Permutation invariance is the invariance under interchanges of similar particles. Invariance properties of this kind are associated with statistical symmetries, and we are going to examine the hypothesis that these in turn imply various other conservation laws, such as conservation of β^- and μ^- leptons, conservation of baryons and conservation of isospin and hypercharge.

We shall need to modify some theorems of Pauli (1940) according to which similar particles of half-odd-integral spin must satisfy Fermi statistics, and similar particles of integral spin must satisfy Bose statistics. This modification is certainly necessary if we agree that 'similar' shall mean 'belonging to the same representation of the Poincaré group', i.e. having the same mass m and spin s, defined by

$$m^2 = p^\lambda p_\lambda, \qquad s(s+1) = -s^\lambda s_\lambda, \tag{1}$$

with

$$s_\lambda = \tfrac{1}{2}\varepsilon_{\lambda\mu\nu\rho} j^{\mu\nu} n^\rho, \qquad n_\rho = p_\rho/m.$$

For, the two kinds of neutrino ν_β and ν_μ are similar particles in the sense of this definition, and if all neutrinos satisfied *the same* Fermi statistics then there would be no way of distinguishing between them. Another likely difficulty for the Pauli theorem is the fact that quarks do not appear to satisfy Fermi statistics; it is for this reason that different 'colours' for the quarks have been suggested.

Pauli's theorems are based on several assumptions. Firstly, it is assumed that the fields ϕ_α representing the elementary particles correspond to representations of the Poincaré group:

$$[\phi_\alpha, P_\lambda] = \mathrm{i}\,\phi_{\alpha,\lambda} = p_\lambda \phi_\alpha, \qquad [\phi_\alpha, J_{\lambda\mu}] = j_{\lambda\mu}\phi_\alpha. \tag{2}$$

Secondly, various assumptions are made related to the requirements of causality. It is assumed that there is a unique state of lowest energy (the vacuum state), but in addition the permutability of fields at space-like separation is postulated: it is assumed that

$$[\phi_\alpha(x), \phi_\beta(y)] = 0, \quad [(x-y)^2 < 0] \qquad \text{or} \qquad \{\phi_\alpha(x), \phi_\beta(y)\} = 0, \quad [(x-y)^2 < 0] \tag{3}$$

for particles of integral or half-odd-integral spin respectively. This last postulate, which incidentally would rule out the possibility of tachyons, is what we propose to question.

Parastatistics

In parafermi statistics (Green 1953) the requirement (3) is replaced by a weaker postulate:

$$[\phi_\alpha(x), [\phi_\beta(y), \phi_\gamma(z)]] = 0, \qquad [(x-y)^2 < 0, (y-z)^2 < 0, (x-z)^2 < 0]. \tag{4}$$

We shall not discuss the consequences of this in detail, but simply point out here that it does guarantee that physical quantities, which are bilinear in the fields, will commute at space-like separation. There are different types of parafermi statistics, characterized by the order p. For $p = 1$, we have Fermi statistics; for $p = 2$, we have a modification of Fermi statistics which will permit, for instance, the representation of the two different kinds of neutrinos by the same field variable ϕ_α; for $p = 3$, we have a further modification which appears suitable for a quark model. Greenberg (1964) was the first to suggest a quark model using three parafields of order three; since a parafield of order three is in some sense equivalent to three fermion fields, this could be regarded as a model with 'hidden' colours. Since then, Govorkov (1968, 1969) and Bracken and Green (1973) have investigated the possibility of a quark model which uses only one parafield, and thereby dispenses with colours (Han and Nambu 1965; Gell-Mann 1972). It turns out that this involves the following heresy!

If A_{pq} $(p, q = 1, 2, 3)$ are the generators of U(3), we abandon the idea that the isospin is $I_3 = \frac{1}{2}(A_{11} - A_{22})$. Instead, we replace the SU(2) isospin subgroup at the end of the chain

$$\mathrm{U}(3) \supset \mathrm{SU}(3) \supset \mathrm{SO}(3) \supset \mathrm{SU}(2).$$

The SO(3) generators are

$$L_1 = -\mathrm{i}(A_{23} - A_{32}), \qquad L_2 = -\mathrm{i}(A_{31} - A_{13}), \qquad L_3 = -\mathrm{i}(A_{12} - A_{21}) \tag{5}$$

and the isospin can be defined by, e.g.

$$I_+ = (L_1 + \mathrm{i}L_2)^2 f(L_3), \qquad I_- = f(L_3)(L_1 - \mathrm{i}L_2)^2, \qquad I_3 = \tfrac{1}{2}L_3, \tag{6}$$

where $f(L_3)$ is easily determined. The SO(3) multiplets in the SU(3) baryon octet are $(\mathrm{n}, \Lambda, \mathrm{p})$ and $(\Sigma^-, \Xi^-, \Sigma^0, \Xi^0, \Sigma^+)$, corresponding to $l = 1$ and 2 respectively, and it will be seen that each SO(3) multiplet contains two SU(2) submultiplets, corresponding to isospin $I = \frac{1}{2}l$ or $\frac{1}{2}(l-1)$. The SU(3) decuplet contains SO(3) multiplets corresponding to $l = 1$ and 3. The resulting description of the hadrons probably has some relation to the Nagoya model (Sakata 1956).

To understand why we need this method of decomposition of the U(3) multiplets, we have to look at the structure of the parastatistics algebra of order three, which is shown in the following diagram:

$$\begin{array}{ccc}
\mathrm{U}_N & \times & \mathrm{U}_3 \\
\cap & & \cup \\
\mathrm{SO}_N & \times & \mathrm{O}_3 \\
\cap & & \cup \\
\mathrm{O}_{2N} & \times & \mathrm{SO}_3 \\
\cap & & \cup \\
\mathrm{SO}_{2N+1} & \times & \mathrm{C}_3
\end{array}$$

Each line is a characterization of the algebra, and the cups indicate the inclusions. It will be seen that the last line includes an algebra which is not actually SU_2 but the equivalent algebra C_3, whose elements A_1, A_2 and A_3 satisfy the anticommutation relations

$$\{A_2, A_3\} = A_1, \qquad \{A_3, A_1\} = A_2, \qquad \{A_1, A_2\} = A_3. \tag{7}$$

The attempt to apply parastatistics of order three to the quark model is, however, beset by difficulties which appear in full force when the extension to interacting or nonlinear fields is attempted. It does not appear to be possible to satisfy the correspondence principle, because of discrepancies between the unquantized and quantized field equations. In addition, there are difficulties with causality which can only be resolved within a somewhat restrictive formalism.

Modular Statistics

The solution of the above problems appears to need the use of a new kind of quantum statistics, called modular statistics (Green 1975). The difference between modular statistics and parastatistics first appears at order three, and is most easily seen from the relations satisfied by the annihilation operators a_j and the corresponding creation operators a_j^*, for a common fixed time t. These operators satisfy

$$a_i a_j + a_j a_i = 0, \qquad a_i a_j^* + a_j^* a_i = \delta_{ij} \tag{8}$$

for Fermi statistics, and for both parastatistics and modular statistics of order one. They satisfy

$$a_i a_j a_k + a_k a_j a_i = 0, \qquad a_i a_j a_k^* + a_k^* a_j a_i = \delta_{jk} a_i \tag{9}$$

and

$$a_i^* a_j a_k^* + a_k^* a_j a_i^* = \delta_{ij} a_k^* + \delta_{jk} a_i^* \tag{10}$$

for parastatistics and modular statistics of order two. The relation (10) holds for modular statistics of all orders! But, whereas $[a_i, [a_j, a_k]]$ vanishes in parafermi statistics of arbitrary order, this is not true for modular statistics and instead we have

$$a_i a_j a_k a_l + a_l a_j a_k a_i = 0, \qquad a_i a_j a_k a_l^* + a_l^* a_k a_i a_j = \delta_{kl} a_i a_j \tag{11}$$

for order three, which is also a logical extension of equations (8) and (9).

Within any irreducible representation of the modular statistics of order three, it is possible to define a cyclic permutation operator u, such that

$$u\, a_i a_j a_k = a_k a_i a_j\, u, \qquad u^3 = 1. \tag{12}$$

Three different sets of creation and annihilation operators can be defined by

$$a_i^{(r)} = u^{*r} a_i u^r \qquad (r = 0, 1, 2 \bmod 3) \tag{13}$$

and it is easily verified that the relations (10) and (11) are all fulfilled, provided the generalized anticommutation relations

$$\{a_i^{(r)}, a_j^{(s)}\} = a_i^{(r)} a_j^{(s)} + a_j^{(s-1)} a_i^{(r+1)} = 0, \tag{14a}$$

$$\{a_i^{(r)}, a_j^{(s)*}\} = a_i^{(r)} a_j^{(s)*} + a_j^{(s-1)*} a_i^{(r-1)} = \delta_{rs} \delta_{ij} \tag{14b}$$

are satisfied. Products of the creation and annihilation operators which commute or anticommute with one another are called *modules*; again it is easy to verify that

$$a_{ijk}^{(rst)} = a_i^{(r)} a_j^{(s)} a_k^{(t)}, \quad a_{ijk}^{(rst)*} = a_k^{(t)*} a_j^{(s)*} a_i^{(r)*}, \quad b_{ij}^{(rs)} = a_i^{(r)} a_j^{(s)*} \tag{15}$$

have this property. Particles created or annihilated by such modules will satisfy Fermi or Bose statistics, and it is natural to identify these with baryons and mesons respectively. Any module can be factorized uniquely into a product of the submodules given by

$$q_i^{(1)} = u^* a_i, \qquad q_i^{(2)} = u\, a_i u, \qquad q_i^{(3)} = a_i u^*, \tag{16}$$

and their hermitian conjugates. These submodules satisfy anticommutation relations and must therefore represent fermions. However, the $a_i^{(r)}$ cannot be interpreted in this way.

By interpreting the $q_i^{(r)}$ as annihilation operators for p, n or Λ^0, we obtain a theory equivalent to the Nagoya model (Sakata 1956). By interpreting them as quark annihilation operators, we obtain a theory consistent with the original quark model formulated by Gell-Mann (1964). To see this, we note that the modular algebra contains a representation of U(3), with generators

$$A_{rs} = \tfrac{1}{2}\sum_i (a_i^{(r-1)*}a_i^{(s-1)} - a_i^{(s)}a_i^{(r)*}) \tag{17}$$

which satisfy the usual commutation relations

$$[A_{rs}, A_{tu}] = \delta_{ts}A_{ru} - \delta_{ru}A_{ts}. \tag{18}$$

The isospin I_3 and the hypercharge Y can be defined in the usual way by

$$I_3 = \tfrac{1}{2}(A_{11} - A_{22}), \qquad Y = \tfrac{1}{3}(A_{11} + A_{22} - 2A_{33}). \tag{19}$$

Then it is easy to verify that the modular annihilation operators $a_{ijk}^{(rst)}$ can be resolved into components totally antisymmetric in the subscripts, corresponding to a decuplet, and components of mixed symmetry, corresponding to an octet. There are 18 independent modules corresponding to fixed values of i, j and k, exactly the number required for the baryons observed in nature. Similarly, the modules $b_{ij}^{(rs)}$ can be resolved into two components symmetric and antisymmetric in the subscripts, corresponding to meson nonets of opposite parity. However, it is evident from the relations (16) that the quarks of this model satisfy Fermi statistics, so that one of the principal reasons for introducing generalized statistics would appear to have lost its force. There are essentially two ways of circumventing this difficulty. One way would be to introduce 'colours' and 'flavours' for the quarks, as Professor Gell-Mann has suggested in his lectures (present issue, pp. 473–82). This can be done in a very natural way within the present formalism. All that is necessary is to introduce a matrix root of the permutation operator u, corresponding to the number of 'flavours' desired. If $u = v^4$, the operators

$$q_i^{(r,s)} = v^{*s}q_i^{(r)}v^s \qquad (r = 1,2,3;\, s = 1,2,3,4) \tag{20}$$

and their conjugates are sufficient to create and annihilate quarks in three colours and four flavours. Here, however, we shall suggest a different solution to the difficulty, arising from the dynamics of the quarks.

In fact, although the above seems to provide a nice solution for the riddle of quark statistics, which does not require us to introduce nine or more different kinds of fermion creation and annihilation operators, it does not help us with the problem of quark confinement. Since generalized statistics can involve us with problems in relation to causality, it makes it even more important that the quarks should not escape from the custody of their modules. But solutions to this problem of confinement must also be sought within the realm of dynamics, rather than statistics.

Dynamics and Confinement

To conclude, we introduce some ideas about the dynamics of quarks, arising from recent discussions with A. J. Bracken. These ideas are suggested by the hypothesis that the quarks are tachyons, with imaginary masses. Then it would be no surprise that they should give rise to causal anomalies individually; but, if we suppose that they exist only in composite particles or modules with real masses, these anomalies will not trouble us. This hypothesis has perhaps the merit that quarks have not yet been observed, and tachyons have also not been observed; thus, by identifying quarks as tachyons, we reduce the number of different things which have not been observed!

To formulate the hypothesis in more specific terms, we suppose that quarks correspond to unitary representations of the Lorentz group, but nonunitary representations of the Poincaré group which are equivalent to certain unitary representations of O(4, 1), with imaginary mass. The generators $J_{\lambda 5}$ of O(4, 1), additional to the generators $J_{\lambda\mu}$ ($\lambda, \mu = 0, 1, 2, 3$) of the Lorentz group, are defined by

$$J_{\lambda 5} = \tfrac{1}{2}\{J_{\lambda\mu}, P^{\mu}\}/M - aP_{\lambda}. \tag{21}$$

Representations of O(4, 1) can be labelled by $l_1 = -\frac{3}{2} - iaM$ and $l_2 = S$ (the spin), where

$$\tfrac{1}{2}J_{RS}J^{RS} = l_1(l_1+3) + l_2(l_2+1)$$

is the quadratic invariant. It is known that there are three types of unitary representations of O(4, 1), with the following characteristics:

Type A. M real, unquantized; S real, quantized.

Type B. M imaginary, quantized; $S = -\frac{1}{2} - i\beta$, unquantized.

Type C. M imaginary, quantized; S quantized.

In representations of type A, P_λ is hermitian; in representations of types B and C, P_λ is neither hermitian nor antihermitian. It appears that there are at least two stable combinations of particles belonging to the exotic representations. We may have two tachyons of type B belonging to representations with complex conjugate masses and momenta, which could correspond to mesons. We may also have one tachyon of type C and two tachyons of type B, with real total momentum, which could correspond to baryons. The advantage of using these representations of the Poincaré group is precisely that there are three distinguishable types, corresponding to the different varieties of quarks determined by the generalized statistics. A field theory which naturally combines the dynamics and the statistics has still to be developed.

References

Bracken, A. J., and Green, H. S. (1973). *J. Math. Phys.* **14**, 1784.
Gell-Mann, M. (1964). *Phys. Lett.* **8**, 214.
Gell-Mann, M. (1972). *Acta Phys. Austriaca Suppl.* **9**, 733.
Govorkov, A. B. (1968). *Zh. Eksp. Teor. Fiz.* **54**, 1785.
Govorkov, A. B. (1969). *Ann. Phys.* (*N.Y.*) **53**, 349.
Green, H. S. (1953). *Phys. Rev.* **90**, 270.
Green, H. S. (1975). *Aust. J. Phys.* **28**, 115.
Greenberg, O. W. (1964). *Phys. Rev. Lett.* **13**, 598.
Han, M. Y., and Nambu, Y. (1965). *Phys. Rev.* B **139**, 1006.
Pauli, W. (1940). *Phys. Rev.* **58**, 716.
Sakata, S. (1956). *Prog. Theor. Phys.* **16**, 686.

Manuscript received 6 August 1975