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Statistical Concepts in Particle Physics

D. C. Peaslee

Research School of Physical Sciences, Australian National University;
present address: Room 24-508, Massachusetts Institute of Technology,
Laboratory for Nuclear Science, Cambridge, Mass. 02139, U.S.A.

Abstract

A review is given of the level density formula in hadron physics, which has reached a completely developed state on the basis of the bootstrap equation. It is shown how the approximate formula

$$\rho(m) \approx (D/m_0)(m/m_0)^{-3}\exp(0\cdot 8\, m/m_0)$$

can be entirely deduced except for the input parameters: $D \approx 1$ and $m_0 \approx 138$ MeV, the pion mass. Applications are sketched to large-angle pp scattering, to nucleon–antinucleon annihilation and to Ericson fluctuations in πp scattering.

Introduction

The following review is confined to what one might call the 'statics' of particle statistics: namely, discussion and applications of the level density formula. This topic has recently reached a very nice state of completeness and self-consistency, and is therefore very suitable for a short review. We do not consider here the 'dynamical' problems of particle statistics, such as multiparticle production, distribution and analysis, and high energy reactions in nuclear matter. These subjects currently occupy a large and vigorously growing literature.

Although many authors have contributed significantly to our present topic, the treatment here follows mainly that of S. C. Frautschi and C. J. Hamer.

We seek a formula for hadron level densities that is analogous to the well-known statistical treatment of nuclear levels. This has been known since the pioneering work of Hagedorn (1965), the level density $\rho(m)$ for states of mass m being given by

$$\rho(m) = cm^a \exp(bm)\,, \tag{1}$$

where

$$a \approx -3\,, \qquad b \approx 1/m_0 = 1/m_\pi\,, \qquad c \approx (1-3)\,m_0^{-a-1}\,.$$

These parameters are obtained both empirically (Hamer and Frautschi 1971) and theoretically (Frautschi 1971; Nahm 1972). Here $m_\pi \approx 138$ MeV is the pion mass. Two important differences from the nuclear case should be noted: (1) The density rises directly as the total rest mass energy mc^2 of the system and hence becomes large at relatively low excitation on the GeV scale; overlapping resonances being very much the rule in hadron statics. (2) The effective 'temperature' of the hadronic system is to first approximation $T = 1/b$, and hence a constant largely independent of excitation energy.

Self-consistent Bootstrap Equation

This fundamental equation (the self-consistent bootstrap equation) is given here in its nonrelativistic form, but only minor modifications are required to make it covariant (Hamer 1973). The level density $\rho(m)$ at a total mass m is expressed in terms of the integrals over all possible contributing configurations of constituent particles of mass m_i, the density of the constituent states being $\rho(m_i)$. Thus

$$\rho(m) = \rho_{\text{in}} + \sum_{n=2}^{\infty} \frac{1}{n!}\left(\frac{V}{(2\pi)^3}\right)^{n-1} \prod_{i=1}^{n} \int \mathrm{d}^3 p_i\, \delta^3(\boldsymbol{P}) \int \mathrm{d}m_i\, \rho(m_i)\, \delta(E-m), \tag{2}$$

where

$$\boldsymbol{P} = \sum_i \boldsymbol{p}_i, \qquad E = \sum_i E_i, \qquad E_i = m_i + Q_i,$$

with $\rho_{\text{in}} = D\,\delta(m-m_0)$ the density of the input states, Q_i the kinetic energy of the ith particle, m_0 the minimum mass of the particles (that is, m_π) and D the degeneracy of m_0.

The necessity and sufficiency of the exponential term in equation (1) can be proved from equation (2) (Hagedorn 1965), but we only sketch an argument for it here: As $m \to \infty$, we neglect ρ_{in} to obtain

$$\rho(m) = \rho(E) = \rho\left(\sum_i m_i + Q_i\right) \approx \prod_i \rho(m_i), \tag{3}$$

where in the last term we have neglected the integrals over $\mathrm{d}^3 p_i$ as $Q_i \to 0$. Thus the dominant factor in equation (3) must be of exponential form:

$$\rho(m_i) \approx \exp(bm_i).$$

To determine secondary factors in equation (1), like the power of m, requires consideration of $Q_i \neq 0$. In general, the average $\bar{Q}_i \approx m_0 \ll m_i$, so that the situation is nonrelativistic:

$$\int \mathrm{d}^3 p_i\, \rho(m_i) \approx \int \mathrm{d}^3 (2m_i Q_i)^{1/2} \exp(bE_i - bQ_i) = \exp(bE_i)\,(2\pi m_i/b)^{3/2}. \tag{4}$$

This is evaluated at fixed E_i, so that

$$\sum_i E_i = m.$$

This already suggests a power law factor, so we try $\rho(m_i) = cm^a \exp(bm_i)$ and then obtain

$$\int \mathrm{d}m_i \int \mathrm{d}^3 p_i\, \rho(m_i) = \exp(bE_i)\, c(2\pi/b)^{3/2} \frac{m_i^{5/2+a}}{(5/2+a)} = \exp(bE_i)\, h(m_i). \tag{5}$$

An important point is that, as $m \to \infty$, integrals like equation (5) converge only if $a < -5/2$. Closer inspection indicates that $a = -3$ in this model, as is seen by Laplace inversion (Nahm 1972; Hamer 1973): The transform integral

$$\int \exp(-\beta E)\, \rho(E)\, \mathrm{d}E$$

is singular as $\beta \to b^+$; the singularity goes as $\sqrt{(\beta - b)}$, which inverts to give in equation (5) another power of $m^{-\frac{1}{2}}$.

I now present a rough argument (Frautschi 1971) to show for any $a < -5/2$ how the form of $\rho(m)$ is self replicating in the bootstrap. For $a < -5/2$, $h(m_i)$ is maximal for small $m_i > m_0 > 0$, the last condition being necessary for convergence. The most likely pattern for the n-particle term as $m \to \infty$ is all m_i small ($m_i \approx m_0$), except for one $m_i \approx m$ to make up the total $\sum m_i = m$. The major term has

$$\int d^3p_i\, \delta^3(\boldsymbol{P}) \int dm_i\, \delta(E-m)\, \rho(m_i) \approx \rho(m),$$

and any one of the particles can contribute this, and so there is a multiplicity factor n. Thus, neglecting ρ_{in} for $m \gg m_0$,

$$\begin{aligned}\rho(m) &\approx \rho(m) \sum_{n=2} \frac{1}{n!}\left(\frac{V}{(2\pi)^3}\right)^{n-1} n\{h(m_0)\}^{n-1} \\ &= \rho(m) \sum_{n=1} x^n/n! = \rho(m)\{\exp(x)-1\}. \end{aligned} \tag{6a}$$

Therefore, we have

$$Vh(m_0)/(2\pi)^3 = x \approx \ln 2. \tag{6b}$$

This is a second numerical parameter (after $a = -3$) determined from theoretical arguments alone.

Scaling Laws of ρ: Algebraic Dependence on V, D, m_0

To obtain scaling in V and D, we multiply equation (2) throughout by λ to obtain

$$\lambda\rho = \lambda D\, \delta(m-m_0) + \sum_{n=2}^{\infty} \frac{1}{n!}\left(\frac{V}{\lambda(2\pi)^3}\right)^{n-1} \prod_{i=1}^{n} \int d^3p_i\, \delta^3(\boldsymbol{P}) \int dm_i\, \lambda\, \rho(m_i)\, \delta(E-m). \tag{7a}$$

If we now view $\lambda\rho = \rho'$ as a bootstrap function, we have

$$\rho' = \rho(m, V/\lambda, \lambda D) = \lambda\, \rho(m, V, D) \tag{7b}$$

by construction, and therefore $\rho(m, V, D) = D\,\sigma(m, VD) = \tau(m, VD)/V$.

To obtain scaling in m, we put

$$m = m_0\, x, \quad m_i = m_0\, x_i, \quad E_i = m_0\, \varepsilon_i, \quad E = m_0\, \varepsilon, \quad p_i = m_0\, q_i, \quad \boldsymbol{P} = m_0\, \boldsymbol{Q},$$

and define $\bar\rho(x) = m_0\, \rho\,(m_0\, x)$. Then we obtain

$$\bar\rho(x) = D\,\delta(x-1) + \sum_{1}^{n} \frac{1}{n!}\left(\frac{Vm_0^3}{(2\pi)^3}\right)^{n-1} \prod_{i=1}^{n} \int d^3q_i\, \delta^3(\boldsymbol{Q}) \int dx_i\, \bar\rho(x_i)\, \delta(\varepsilon - x). \tag{8}$$

In equation (8), the powers of m_0 all cancel in the integrals except for $\delta(\varepsilon - x)$, which provides the m_0^{-1} that is absorbed in the definition of $\bar\rho$ on the right-hand side. Thus we see that

$$\bar\rho(x, m_0, V, D) = \rho(x, Vm_0^3, D) = D\,\sigma(x, VDm_0^3)$$

as above or, finally, that

$$\rho(m) = \bar\rho/m_0 = (D/m_0)\, f(m/m_0, VDm_0^3). \tag{9}$$

Comparing equation (9) with the general form (1) we see that

$$bm = b(m/m_0)m_0 = b_0(m/m_0)(VDm_0^3)^{1/3} = b_0\, m(VD)^{1/3}, \tag{10}$$

and thus

$$\rho = c_0 \frac{D}{m_0}\left(\frac{m}{m_0}\right)^a \exp\bigl(mb_0(VD)^{1/3}\bigr). \tag{11}$$

Evaluation of Parameters

Equation (11) contains six parameters, but m_0 and D are part of the input. This leaves four to be determined, of which two are known: equation (6b) and $a = -3$. It should be noted that c_0 is trivial in significance relative to b_0, and so we can evaluate it roughly. This we do by requiring equation (11) approximately to reproduce $\rho_{\rm in}$:

$$m_0\, \rho(m_0) \approx D = Dc_0 \exp(b_0\, m_0)$$

or

$$c_0 \approx \mathrm{e}^{-1} \sim 1 \tag{12}$$

to this order of accuracy.

We now rewrite equation (6b) as $\ln 2 = 2c_0\, y^{1/2}(2\pi b_0)^{-3/2}$, whence

$$y = DVm_0^3 \approx 30\, b_0^3. \tag{13}$$

For $D = 1$, we take $V \approx (4/3)\pi m_0^{-3}$, so that $y = (4\pi/3) \approx 4$ and, by equation (13),

$$b_0 \approx 0{\cdot}5 \tag{14a}$$

and thus

$$b = y^{1/3}\, b_0/m_0 \approx 0{\cdot}8/m_0. \tag{14b}$$

This compares exactly with the empirical parameters given in equation (1), except of course for D, which doesn't show empirically.

It may be worth pausing to reflect on the remarkable features of this analysis. Starting simply with the bootstrap conditions in equation (2), we have deduced a completely detailed form for the level density:

$$\rho(m) = \frac{D}{m_0}\left(\frac{m}{m_0}\right)^{-3} \exp\bigl(0{\cdot}8(m/m_0)\bigr), \tag{15}$$

where D and m_0 are input parameters. We expect of course that $m_0 \approx 138$ MeV, the mass of the pion, as it is the lightest and most prolific of the strongly produced particles. Somewhat less strongly, we may also expect that $D \approx 3$ or 1, according as the model is for $\pi^+\pi^-\pi^0$ or for π^0 only (which may approximate the full π charge spectrum with restriction on isotopic spin of the system). Note that $m_0 \approx m_\pi$ for baryon as well as meson level densities, since the pion is the particle of smallest mass emitted in cascade decays of the system.

With regard to cascade decays, the bootstrap model has another remarkable feature (Frautschi 1971), which is contained in equation (6a): in the formula

$$\rho(m) \approx \rho(m) \sum_{(n=2)} \frac{x^{n-1}}{(n-1)!},$$

each term in the summand represents the normalized probability P_n of n-particle emission by the state at mass m. Hence

$$P_2 = 0\cdot 69, \qquad P_3 = 0\cdot 24, \qquad P_4 = 0\cdot 06, \qquad P_5+P_6+P_7+\ldots = 0\cdot 01. \tag{16}$$

This nicely justifies the usual practice of concentration on two- or three-body emission in elementary particle calculations.

Applications

Computer Analysis of Equation (2)

Now let us turn to a few applications of these formulae. A first interesting 'application' is not to a comparison with experimental data but to a computer analysis of equation (2) for the bootstrap made by Hamer and Frautschi (1971).

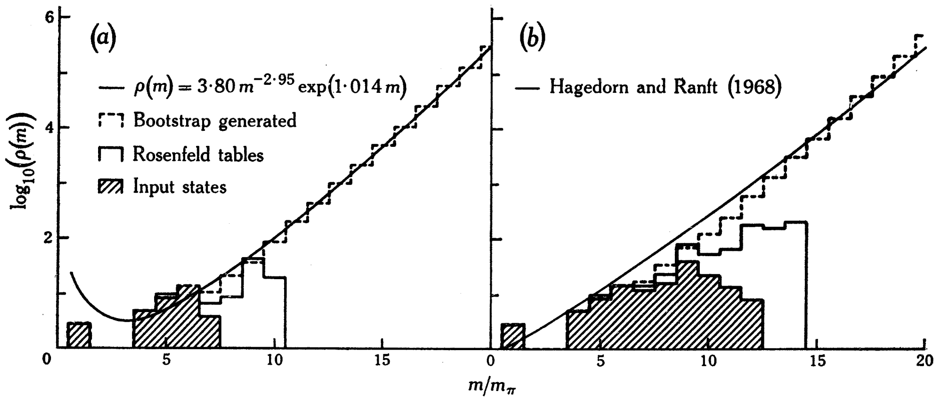

Fig. 1. Bootstrap calculation of (*a*) the meson level density and (*b*) the total hadron (meson plus baryon) state density. The mass (energy) unit is m_π = 138 MeV.

These authors assumed various discrete input states from the data in the Rosenfeld tables (Particle Data Group 1970), and iterated equation (2) many times on a large computer. The results are given in Figs 1*a* and 1*b*. Fig. 1*a* shows the meson state densities computed from the known pseudoscalar and vector meson states as input. The asymptotic fit is in good agreement with the expected parameters discussed in the previous section. Fig. 1*b* shows the same type of calculation for all hadron states, baryons plus mesons. The solid line in Fig. 1*b* refers to a fit (Hagedorn and Ranft 1968) that took $b \approx 0\cdot 86/m_\pi$ in equation (1) but did not have a simple power of m as a preceding factor. The main point of comparison between Figs 1*a* and 1*b* is that the asymptotic level density for mesons alone is the same as for all hadrons, although the empirical inputs are different. This is indirect confirmation of the above remark that $m_0 \approx m_\pi$ for both mesons and baryons.

Wide-angle pp Scattering

The transverse momentum distribution in hadron collisions at high energies should be largely determined by statistical considerations and should in fact be dominated by the exponential factor. The probability of emission of a particle of total energy E by a colliding state of mass m is:

$$P(E) = \rho(m-E)/\rho(m) \approx \exp(-bE) = \exp(-E/T), \tag{17}$$

where T is the (constant) 'temperature'. The transverse momentum distribution is determined by integrating out the longitudinal component,

$$
\begin{aligned}
P(p_\perp)p_\perp \, \mathrm{d}p_\perp &\approx \int_{-\infty}^{\infty} \mathrm{d}p_x \exp(-(p_x^2+\mu^2)^{\frac{1}{2}}/T) \\
&\approx \int_{-\infty}^{\infty} \mathrm{d}p_x \exp(-\mu/T)\exp(-p_x^2/2\mu T) \\
&= (2\pi\mu T)^{\frac{1}{2}} \exp(-\mu/T),
\end{aligned}
$$

where $\mu^2 = p_\perp^2 + m^2$. Most emitted particles will be pions, so that $\mu \to p_\perp$ very rapidly, and so we have essentially:

$$P(p_\perp) \approx \exp(-p_\perp/T). \tag{18}$$

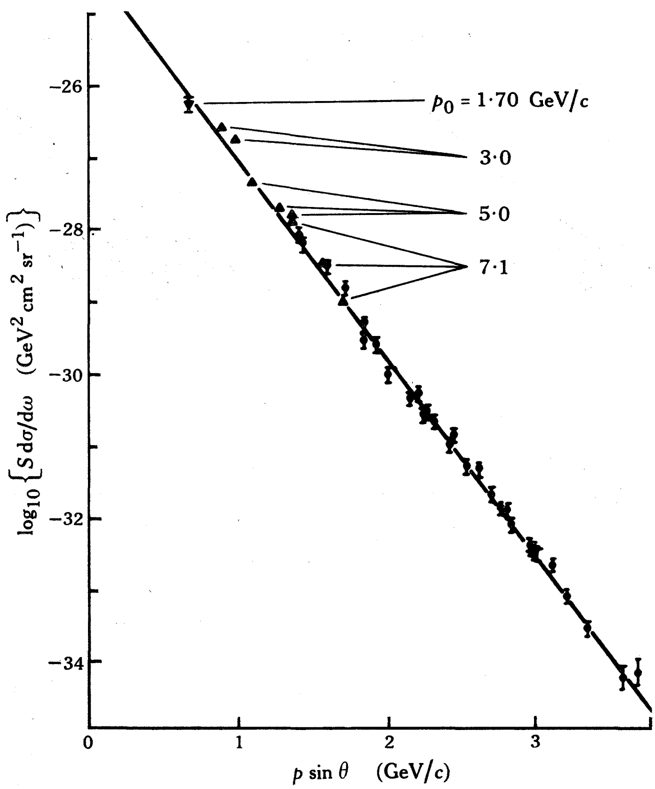

Fig. 2. Proton–proton elastic scattering as a function of transverse momentum $p_\perp = p \sin\theta$.

One way to test the result (18) is with elastic pp scattering at wide angles, which should not reflect grazing incidence or peripheral effects. The scattering is supposed to be due to collision excitation of the protons to a high degree, followed by multipion exchange with the transverse momentum distribution of equation (18), which then becomes the transverse distribution for pp. The data on wide-angle elastic pp scattering are shown in Fig. 2 (Orear 1964). The fit is a remarkably straight line over almost

10 decades, an accuracy far greater than that of equation (18), of course. The value $T = 158$ MeV is obtained from the fitted line. The more obvious case of $p+p \rightarrow p+$(anything) doesn't seem to work as well (Hagedorn and Ranft 1972), but it also gives $T \approx 150$ MeV.

Nucleon–Antinucleon Annihilation ($N\bar{N}$)

On the statistical model (Hamer 1972), the average pion energy is

$$\langle E_\pi \rangle = \tfrac{3}{2}T + m_\pi = m_\pi(1 + 3/2bm_\pi). \tag{19}$$

Then, for a system of mass $m \gg m_\pi$ decaying into pions, the number of pions is

$$n_\pi = m/\langle E_\pi \rangle = m\{m_\pi(1 + 3/2bm_\pi)\}^{-1}. \tag{20}$$

The observation for $\bar{p}p$ at rest is

$$m/m_\pi = 13{\cdot}6, \qquad n_\pi = 4{\cdot}6 \pm 0{\cdot}1, \qquad b \approx 0{\cdot}75/m_\pi, \tag{21}$$

in good agreement with our deduced $0{\cdot}8 = bm_\pi$ above. Note that formula (20) is fairly good even at the mass of a ρ meson, $m_\rho = 0{\cdot}8$ GeV, since it gives $n_\pi(m_\rho) \approx 1{\cdot}9$ as compared with the observed value of 2!

The linear multiplicity formula

$$n_\pi(m) \approx m/3m_\pi \tag{22}$$

should hold as long as $\bar{N}+N$ results in annihilation. At energies high enough to be inelastic in $\bar{N}N$, the growth of $\langle n_\pi \rangle$ with m will diminish. Such a reduction in n_π of course holds from the outset for NN, where only a fraction of the average energy goes into statistical excitation.

It is of interest to note that $\bar{N}N$ annihilation was one of the oldest applications of the original statistical model (Fermi 1950) for elementary particles. Originally the bootstrap was not included, only free pions in a box being considered. This required $V \rightarrow 8V$ to make a fit, which was disastrous. It is a great success of the bootstrap model to eliminate this discrepancy.

Ericson Fluctuations

As a final application, we consider the interesting topic of Ericson fluctuations in particle physics. This idea was developed originally for low energy nuclear physics (Ericson 1960; Ericson and Mayer-Kuckuk 1966) but it has recently been considered in connection with elementary particle reactions.

Although hadron levels become very dense—with a spacing of a few keV at 5 GeV excitation—the average width per level is probably constant, say $\Gamma \approx 200$ MeV. Hence individual levels are not resolvable, and there is tremendous overlap. Thus, in a reaction at energy E, some $N = \Gamma\rho(E)$ levels will be contributing. Statistical fluctuations will thus be of order $N^{-\frac{1}{2}}$ and, if N is not too large, we can observe these 'Ericson fluctuations'. This effect has been well studied in nuclear physics for some years. Earlier efforts to find it in pp scattering failed, but now it appears to be present in πp scattering. For the present discussion our development follows S. Frautschi again (Frautschi 1972).

In the case of elastic scattering at a fixed angle θ, we may write the amplitude as $A_{el}(\theta) = A_C(\theta) + A_F(\theta)$, where A_C and A_F are the coherent and fluctuating amplitudes respectively. If we then represent an average over a macroscopic energy region, say, $(2\to3)\Gamma \sim 500$ MeV by angular brackets then

$$\langle A_C\rangle = A_C, \qquad \langle A_F\rangle = 0 \tag{23}$$

by definition of the coherent and fluctuating amplitudes. We may now write any cross section as $\sigma = G|A|^2$, where G is a geometrical factor for total or differential measurement. Thus, we have

$$\begin{aligned}\langle\sigma\rangle &= G\langle|A|^2\rangle = G\langle|A_F|^2\rangle + G\langle|A_C|^2\rangle \\ &= \sigma_F + \sigma_C,\end{aligned} \tag{24}$$

since the interference terms vanish. The standard deviation C of σ (normalized correlation function) is given by

$$C = \frac{\langle(\sigma-\langle\sigma\rangle)^2}{\langle\sigma\rangle^2} = \frac{\langle|A^4|\rangle - \langle\sigma\rangle^2}{\langle\sigma\rangle^2}. \tag{25a}$$

Now, $\langle|A^4|\rangle$ may be expanded as

$$\begin{aligned}\langle|A^4|\rangle &= \langle|A_C|^4\rangle + 4\langle|A_C^3A_F|\rangle + 6\langle|A_C^2A_F^2|\rangle + 4\langle|A_CA_F^3|\rangle + \langle|A_F|^4\rangle \\ &= \sigma_C^2 \qquad +0 \qquad\qquad +6\sigma_C\sigma_F \qquad +0 \qquad\qquad +3\sigma_F^2,\end{aligned}$$

where the evaluation of the last term depends on a gaussian shape assumption for A_F, and that of the second-last term depends on the antisymmetry of A_F. Thus equation (25a) becomes

$$C = 2(\sigma_F^2 + 2\sigma_F\sigma_C)/(\sigma_F+\sigma_C)^2. \tag{25b}$$

The form (25a) for C can be evaluated experimentally and then compared with equation (25b) to yield σ_F/σ_C. Combining this with $\langle\sigma\rangle = \sigma_F+\sigma_C$, we can then obtain the cross sections individually.

At the particular angle $\theta = 0°$, the amplitudes add coherently for all angular momenta, so that

$$|A_F(0)| \approx |A_C(0)|N^{-\frac{1}{2}}, \qquad |d\sigma_F(0°)/dt| = \{N^{-1}\,d\sigma_C(0°)/dt\}\,y^2. \tag{26}$$

Here y is a reduction factor for the fraction of scattering that goes by direct channel resonances as opposed to pomeron exchange in the t channel. We expect $y\to0$ as $E\to\infty$, and so we try

$$y = \{\sigma_{tot}(E) - \sigma_{tot}^{asymp}\}/\sigma_{tot}(E), \tag{27}$$

where σ_{tot}^{asymp} is the asymptotic total cross section.

Experimental tests of the above ideas have been made with πp elastic scattering at 2→3·5 GeV, and at 5 GeV (Carlson 1973; Schmidt *et al.* 1973). To do this required some modification of detail, since $\rho(E)$ can vary by an order of magnitude across a single level width. The main point was to let $\langle\sigma\rangle = \sigma_F+\sigma_C$ be a smooth monotonic function of energy instead of a constant, as in the nuclear case; with a similar behaviour of course for $y(E)$.

We expect σ_F to be symmetric in θ, while σ_C has a strong forward diffraction peak. However, Carlson (1973) and Schmidt *et al.* (1973) employed data already collected at 0° and 180°. Nevertheless, we can obtain $d\sigma_F/dt$ and $d\sigma_C/dt$, and extract N for both forward and backward scattering. Consistent values of N are obtained, although $d\sigma_F/dt$ and $d\sigma_C/dt$ are very different for 0° and 180°. Figs 3*a* and 3*b* show cross sections for these angles. Note that much greater sensitivity results from using $d\sigma/dt$ rather than σ_{tot}, since $\sigma_{tot} \approx \mathrm{Im}(A(0))$, so that $\sigma_{tot,F}/\sigma_{tot,C} \approx N^{-\frac{1}{2}}$ instead of N^{-1}.

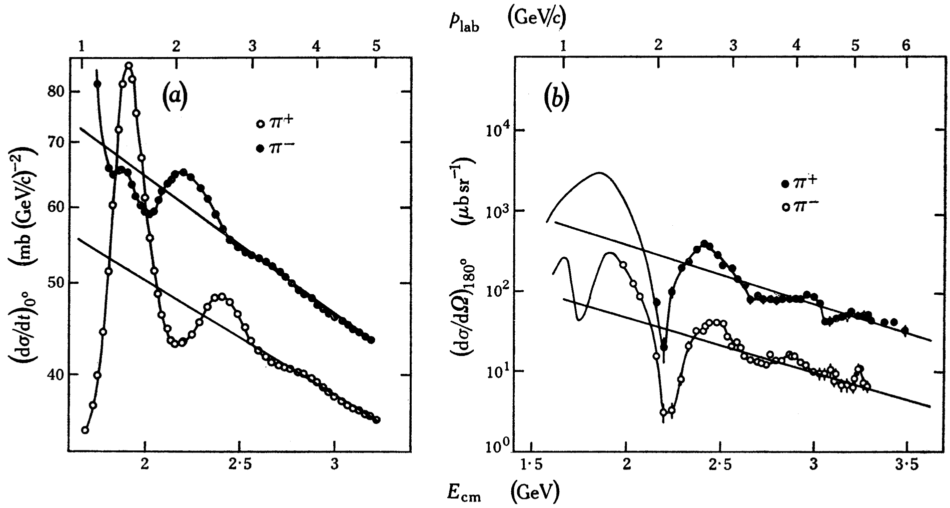

Fig. 3. Elastic $\pi^{\pm}$p differential scattering cross sections in (*a*) the forward (0°) direction and (*b*) the backward (180°) direction.

The average level width can be deduced from the observed spacing Δ of fluctuation peaks by means of $\Gamma \approx \Delta/\sqrt{8}$ (Brink and Stephen 1963). For πp scattering in the 2→3·5 GeV region, we have

$$\Delta \approx 0{\cdot}5 \quad \text{GeV}, \qquad \text{so that} \qquad \Gamma \approx 0{\cdot}2 \quad \text{GeV}. \tag{28}$$

Collecting our previous formulae, we have

$$N = \left(\frac{\Gamma D}{m_0}\right)\left(\frac{m}{m_0}\right)^a \exp(bm). \tag{29}$$

In this analysis (Carlson 1973), $a = -3{\cdot}5$ was assumed to account for the fact that the compound state does not have a random distribution of z component of spin, but rather $S_z = 0$ (see e.g. Hamer 1972). The value $T = 1/b = 160$ MeV was assumed (Hagedorn 1965). Then, with $m_0 = m_\pi$, we have

$$\Gamma D/m_0 = 3{\cdot}6 \qquad \text{for} \quad \pi^-\text{p}, \tag{30a}$$

$$= 0{\cdot}36 \qquad \text{for} \quad \pi^+\text{p}. \tag{30b}$$

With $\Gamma \approx m_0$, this implies

$$(D_+ D_-)^{\frac{1}{2}} = \bar{D} \approx 1\,, \tag{31}$$

which is just as expected from our discussion above. The ratio $D_-/D_+ \approx 10$ is interesting, but its significance is at the level of *I*-spin restrictions on statistics, and has not been analysed.

In summary, however, the very reasonable value of $\bar{D}$ seems to leave little doubt that Ericson fluctuations have been realized in particle reactions. They indirectly confirm the level density formula. They still have not been seen in pp reactions. A search was made in p̄p with limited total counts, which showed suggestive bumps, but not at a statistically significant level (Peaslee *et al.*, to be published).

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