Aust. J. Phys., 1976, **29**, 499–510

Nuclear Sum Rules, Electron Scattering and Giant Resonances

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Abstract

The isoscalar sum rules of Deal and Fallieros (1973) and generalizations of these sum rules are discussed. The isoscalar form factor and transition density for an arbitrary eigenstate of the nucleus are given as sums over the sum rules and, for a particular choice of the operators in the sum rules, are given by a series of which the first terms are the same as the results of the hydrodynamical model. It is shown that caution is needed in making nuclear spin assignments from inelastic electron scattering. The sum rule of Deal and Fallieros is used to clarify the calculation of Bohr and Mottelson (1975) of the energy of the isoscalar giant quadrupole resonance.

Introduction

This paper is concerned with the isoscalar sum rules of Deal and Fallieros (1973), a generalization of these sum rules, and applications of them to giant resonances and inelastic scattering, including electron scattering. The sum rules are also important for inelastic scattering of strongly interacting particles by nuclei. The treatment is confined, for simplicity, to nuclei with zero spin in the ground state, and only excitation of excited states with parity $(-1)^J$, where J is the spin, are considered.

Previously a hydrodynamical model was used in which the nucleus was considered to be composed of an inhomogeneous fluid (Tassie 1956). The fluid flow was assumed to be irrotational and incompressible. This model yielded the transition density:

$$\langle f \,|\, \rho(\boldsymbol{r}) \,|\, 0\rangle \propto r^{J-1}\{\mathrm{d}\rho_{00}(r)/\mathrm{d}r\}\, \mathrm{Y}_{JM}(\Omega) \qquad \text{for} \qquad J = 2, 3, \ldots, \tag{1}$$

where $|0\rangle$ and $|f\rangle$ are the ground state and the excited state of the nucleus, and

$$\rho_{00}(r) = \langle 0 \,|\, \rho(\boldsymbol{r}) \,|\, 0\rangle \tag{2}$$

is the ground state density distribution. The result (1) is usually referred to as the Tassie model. This model, and generalizations of its, have been widely used to analyse inelastic electron scattering (Uberall 1971), while Boridy and Feshbach (1974) have used it successfully to fit high energy proton scattering data.

It is clearly undesirable for such a useful result to be dependent on hydrodynamical assumptions, such as incompressible and irrotational flow. Fortunately, Deal and Fallieros (1973) have derived the result (1) for $\Delta T = 0$ transitions from an energy-weighted sum rule and an assumption that a single doorway state dominates inelastic electric scattering. The treatment given here is confined to isoscalar transitions ($\Delta T = 0$), except for a few remarks about the isovector giant dipole resonance.

The sum rule of Deal and Fallieros can be generalized to an infinite set of sum rules, each being a summation over all the final states. This process is then inverted, and the transition density for excitation of a particular final state can be written as a summation over the sum rules, the first term in this summation being just the Tassie model result (1).

In recent years there has been considerable interest in giant resonances of nuclei and, in particular, in the giant isoscalar electric quadrupole resonance. Bohr and Mottelson (1975) have predicted that the energy of this resonance is given by

$$\hbar\omega \approx 58\,A^{-1/3} \quad \text{MeV}. \tag{3}$$

The sum rule of Deal and Fallieros (1973) is used here to clarify the derivation of this result.

Derivation of Isoscalar Sum Rules

We let

$$A = \sum_i A_i, \qquad B = \sum_i B_i,$$

where A_i and B_i are real functions of the position coordinates of the ith nucleon, and take $E = 0$ as the energy of the ground state, so that

$$H|0\rangle = 0, \tag{4}$$

where H is the nuclear Hamiltonian. We then obtain

$$\langle 0|[A,[H,B]]|0\rangle = \langle 0|(AHB+BHA)|0\rangle.$$

Now AHB is invariant under time reversal, so that $\langle 0|AHB|0\rangle$ is real. Since A, H and B are hermitian, we have

$$\langle 0|AHB|0\rangle = \langle 0|BHA|0\rangle,$$

and

$$\langle 0|AHB|0\rangle = \tfrac{1}{2}\langle 0|[A,[H,B]]|0\rangle. \tag{5}$$

Inserting the complete set of eigenstates of H,

$$H|n\rangle = E_n|n\rangle, \tag{6}$$

yields the sum rule

$$\sum_n E_n\langle 0|A|n\rangle\langle n|B|0\rangle = \tfrac{1}{2}\langle 0|[A,[H,B]]|0\rangle. \tag{7}$$

We define

$$Q_{J\alpha} = \sum_i Q_{J\alpha,i}, \qquad \text{where} \qquad Q_{J\alpha,i} = r_i^{J+2\alpha} Y_{J0}(\Omega_i), \tag{8a, b}$$

and consider $[Q_{J\beta},[H,Q_{J\alpha}]]$.

Now, the nuclear Hamiltonian has the form

$$H = T+V, \qquad \text{with} \qquad T = \sum_i p_i^2/2m. \tag{9a, b}$$

For V velocity independent, we have

$$[V, Q_{J\alpha}] = 0. \tag{10}$$

For a spin–orbit coupling which is linear in the momentum variables,

$$[V, Q_{J\alpha}] \neq 0, \tag{11}$$

but it has no momentum dependence, so that

$$[Q_{J\beta}, [V, Q_{J\alpha}]] = 0. \tag{12}$$

Since $Q_{J\alpha}$ is an isoscalar operator, symmetric in the coordinates of all nucleons, equation (12) holds even in the presence of exchange forces. For an isovector operator, exchange corrections to the sum rules have to be included. This is why the present treatment is only valid for isoscalar transitions.

Neglecting more general velocity-dependent interactions, we have

$$\begin{aligned}
\langle 0 | Q_{J\beta} H Q_{J\alpha} | 0 \rangle &= \tfrac{1}{2} \langle 0 | [Q_{J\beta}, [H, Q_{J\alpha}]] | 0 \rangle \\
&= \tfrac{1}{2} \langle 0 | [Q_{J\beta}, [T, Q_{J\alpha}]] | 0 \rangle \\
&= (2m)^{-1} \sum_{i=1}^{A} \langle 0 | [Q_{J\beta,i}, [p_i^2, Q_{J\alpha,i}]] | 0 \rangle \\
&= (\hbar^2/2m) \sum_i \langle 0 | (\nabla_i r_i^{J+2\beta} Y_{J0}(\Omega_i)) \cdot (\nabla_i r_i^{J+2\alpha} Y_{J0}(\Omega_i)) | 0 \rangle \\
&= (\hbar^2/2m) \int d^3\boldsymbol{r}\, \rho_{00}(r) (\nabla r^{J+2\beta} Y_{J0}(\Omega)) \cdot (\nabla r^{J+2\alpha} Y_{J0}(\Omega)) \\
&= (\hbar^2/2m)(4\pi)^{-1} \{J(2J+2\alpha+2\beta+1) + 4\alpha\beta\} A \langle r^{2J+2\alpha+2\beta-2} \rangle,
\end{aligned} \tag{13}$$

where the density operator

$$\rho(r) = \sum_i \delta(r - r_i) \tag{14}$$

and the ground state density distribution (2) have been used, and

$$\langle r^n \rangle = A^{-1} \int \rho_{00}(r) r^n \, d^3\boldsymbol{r}. \tag{15}$$

The properties of spherical harmonics used in the derivation of (13) are given, for instance, in Appendix A of Uberall (1971). Equation (13) has also been given by Ui and Tsukamoto (1974). For $\alpha = \beta = 0$, equation (13) reduces to the usual energy-weighted sum rule for the isoscalar electric multipole operator.

In the same way, sum rules are obtained for the isoscalar transition density

$$\begin{aligned}
\sum_n E_n \langle 0 | Q_{J\alpha} | n \rangle \langle n | \rho(\boldsymbol{r}) | 0 \rangle &= \langle 0 | Q_{J\alpha} H \rho(\boldsymbol{r}) | 0 \rangle \\
&= \tfrac{1}{2} \langle 0 | [Q_{J\alpha}, [H, \rho(\boldsymbol{r})]] | 0 \rangle,
\end{aligned} \tag{16}$$

but it is easier to work with the form factor

$$F^J(q) = \{4\pi(2J+1)\}^{\frac{1}{2}} \sum_{i=1}^{A} j_J(qr_i)\, Y_{J0}(\Omega_i)$$

$$= \{4\pi(2J+1)\}^{\frac{1}{2}} \int d^3\boldsymbol{r}\, j_J(qr)\, Y_{J0}(\Omega)\, \rho(\boldsymbol{r}). \tag{17}$$

Note that

$$F(q) = \sum_J i^J F^J(q) = \int d^3\boldsymbol{r} \exp(i\boldsymbol{q}.\boldsymbol{r})\, \rho(\boldsymbol{r}), \tag{18}$$

so that the form factor and the transition density are just Fourier transforms of each other. In the first Born approximation, the cross section for longitudinal electric excitation is given by

$$d\sigma/d\Omega = (d\sigma/d\Omega)_P\, |\langle f | F^J(q) | 0\rangle|^2, \tag{19}$$

where $(d\sigma/d\Omega)_P$ is the point nucleus cross section, and the momentum transfer is given by $\hbar q$. Unfortunately, the transition density is sometimes referred to as the radial form factor, or just as the form factor, which can cause some confusion.

Thus, we now have

$$\langle 0 | [Q_{J\alpha}, [H, F^J(q)]] | 0\rangle = (\hbar^2/m)\{(2J+1)/4\pi\}^{\frac{1}{2}} \int d^3\boldsymbol{r}\, \rho_{00}(r)\, r^{J+2\alpha-2}$$

$$\times \{J(J+1) j_J(qr) + (J+2\alpha)\, r\, d j_J(qr)/dr\}.$$

After integration by parts, the sum rule becomes

$$\langle 0 | Q_{J\alpha} H F^J(q) | 0\rangle = \tfrac{1}{2}\langle 0 | [Q_{J\alpha}, [H, F^J(q)]] | 0\rangle$$

$$= -(\hbar^2/2m)\{(2J+1)/4\pi\}^{\frac{1}{2}} \int d^3\boldsymbol{r}\, j_J(qr)\, r^{J+2\alpha-2}$$

$$\times \{2\alpha(2\alpha+2J+1)\rho_{00}(r) + (J+2\alpha) r\, d\rho_{00}/dr\}. \tag{20}$$

Since

$$\langle 0 | Q_{J\alpha} H F^J(q) | 0\rangle = \{4\pi(2J+1)\}^{\frac{1}{2}} \int d^3\boldsymbol{r}\, j_J(qr)\, Y_{J0}(\Omega) \langle 0 | Q_{J\alpha} H \rho(\boldsymbol{r}) | 0\rangle, \tag{21}$$

we have the sum rule for the transition density:

$$\langle 0 | Q_{J\alpha} H \rho(\boldsymbol{r}) | 0\rangle$$

$$= -(\hbar^2/2m) r^{J+2\alpha-2} \{2\alpha(2\alpha+2J+1)\rho_{00} + (J+2\alpha) r\, d\rho_{00}/dr\} Y_{J0}(\Omega). \tag{22}$$

This result has also been given by Ui and Tsukamoto (1974).

For $J \neq 0$ and $\alpha = 0$, equation (22) reduces to

$$\langle 0 | Q_{J0} H \rho(\boldsymbol{r}) | 0\rangle = -(\hbar^2/2m) J r^{J-1} (d\rho_{00}/dr)\, Y_{J0}(\Omega), \tag{23}$$

the sum rule given by Deal and Fallieros (1973), in which the right-hand side has the same radial dependence as equation (1), the hydrodynamical model of Tassie

(1956). For $J = 0$ and $\alpha = 1$, equation (22) reduces to

$$\langle 0 | Q_{01} H \rho(r) | 0 \rangle = -(\hbar^2/2m)\{6\rho_{00} + 2r\, \mathrm{d}\rho_{00}/\mathrm{d}r\} Y_{00}(\Omega), \tag{24}$$

which is the monopole sum rule given by Deal and Fallieros (1973), and for which the right-hand side has the radial dependence given for monopole transitions by Werntz and Uberall (1966).

Giant Isoscalar Electric Quadrupole Resonance

Among the giant resonances discovered in inelastic electron and proton scattering and in other reactions, after the well-known isovector giant dipole resonance, the most investigated resonance is that identified as the E2, $\Delta T = 0$ isoscalar electric quadrupole, for which Bohr and Mottelson (1975) made the prediction (3) above. This prediction is in good agreement with the observed energies (see e.g. Walcher 1973). Apart from the use of the sum rule of Deal and Fallieros (1973), the present treatment follows that of Bohr and Mottelson (1975).

Bohr and Mottelson (1975) considered the operator, which in our notation is Q_{20} and which in the harmonic oscillator shell model induces transitions from the ground state with $\Delta N = 0$ and energy $\hbar\omega^{(0)} = 0$, and with $\Delta N = 2$ and energy $\hbar\omega^{(0)} = 2\hbar\omega_0$, where ω_0 is the harmonic oscillator frequency. The $\Delta N = 2$ levels provide the first approximation to the giant resonance, which is considered to be the first excited state of a collective harmonic oscillator with coordinate α and Hamiltonian

$$E(\alpha, \dot{\alpha}) = \tfrac{1}{2}C\alpha^2 + \tfrac{1}{2}D\dot{\alpha}^2, \tag{25}$$

where

$$C = \hbar\omega/2\alpha_0^2, \qquad D = \hbar/2\omega\alpha_0^2, \tag{26}$$

with

$$\alpha_0 \equiv \langle n=1 | \alpha | n=0 \rangle \tag{27}$$

and ω the frequency of the collective oscillator. In the first approximation, we have $\omega = 2\omega_0$. The normalization of α is chosen so that

$$\alpha_0 = \langle n=1 | Q_{20} | n=0 \rangle . \tag{28}$$

Only the $\Delta N = 2$ level contributes to the sum

$$S = \sum_f E_f |\langle f | Q_{20} | 0 \rangle|^2, \tag{29}$$

so that

$$\alpha_0^2 = S/2\hbar\omega_0, \tag{30}$$

giving as the first approximation

$$C^{(0)} = 2(\hbar\omega_0)^2/S, \qquad D^{(0)} = \hbar^2/2S. \tag{31}$$

In the collective oscillation, the nuclear density distribution oscillates, causing the nuclear potential to oscillate and giving rise to corrections to the equations (31).

The sum rule (23) of Deal and Fallieros (1973) is

$$\sum_f E_f \rho_{f0} \langle f | Q_{20} | 0 \rangle = -(\hbar^2/2m) 2r (\mathrm{d}\rho_{00}/\mathrm{d}r) \mathrm{Y}_{20}(\Omega). \tag{32}$$

If we write

$$\rho = \rho_{00} + \alpha \delta\rho, \tag{33}$$

we obtain

$$\begin{aligned} \sum_f E_f \rho_{f0} \langle f | Q_{20} | 0 \rangle &= 2\hbar\omega_0 \delta\rho \langle n=1 | \alpha | n=0 \rangle \langle n=1 | Q_{20} | n=0 \rangle \\ &= 2\hbar\omega_0 \delta\rho \alpha_0^2 = S \delta\rho. \end{aligned} \tag{34}$$

On comparing the results (34) and (32), we see that

$$\delta\rho = -(\hbar^2/2mS) 2r (\mathrm{d}\rho_{00}/\mathrm{d}r) \mathrm{Y}_{20}(\Omega). \tag{35}$$

Since the transition density in the Deal–Fallieros sum rule is of the same form as that for an incompressible deformation (Tassie 1956), we can write

$$\rho(\boldsymbol{r}) = \rho_{00}(r_0), \tag{36}$$

and the displacement from r_0 to r can be written

$$\boldsymbol{r} = r_0 \{1 + \alpha f(r) \mathrm{Y}_{J0}\}, \tag{37}$$

so that

$$\rho(r) \approx \rho_{00}(r) - r\alpha f \mathrm{Y}_{20} \mathrm{d}\rho_{00}/\mathrm{d}r. \tag{38}$$

Comparing the results (38), (35) and (33), we find

$$f(r) = 2(\hbar^2/2mS). \tag{39}$$

Deforming the potential in the same way, we get

$$\begin{aligned} V(r) &= V_0(r_0) \\ &\approx V_0(r) - \alpha(\hbar^2/2mS) 2r \, \mathrm{Y}_{20} \, \mathrm{d}V_0/\mathrm{d}r \end{aligned} \tag{40}$$

which, for the harmonic oscillator potential, becomes

$$V_0(r) = \tfrac{1}{2} m\omega_0^2 r^2 \tag{41}$$

$$V(r) \approx V_0(r) - \alpha\hbar^2\omega_0^2 S^{-1} r^2 \mathrm{Y}_{20}. \tag{42}$$

As the potential is due to two-body forces, summing over all particles by

$$V = \sum_k V(r_k) \tag{43}$$

counts contributions twice. Thus the correction to the potential energy must be

divided by 2, giving a correction

$$H' = -\tfrac{1}{2}\alpha\hbar^2\omega_0^2 S^{-1}\sum_k r_k^2 Y_{20}(\Omega_k)$$
$$= \tfrac{1}{2}\chi\alpha Q_{20} = \tfrac{1}{2}\chi\alpha^2, \tag{44}$$

where

$$\chi = -\hbar^2\omega_0^2/S = -\tfrac{1}{2}C^{(0)}. \tag{45}$$

Addition of the correction (44) gives

$$C = C^{(0)}+\chi, \qquad D = D^{(0)}, \tag{46}$$

so that

$$\hbar\omega = \hbar(C/D)^{\frac{1}{2}} = \sqrt{2}\,\hbar\omega_0 \approx 58\,A^{-1/3} \text{ MeV}. \tag{47}$$

Excitation of Eigenstate

The states of spin J obtainable from the ground state by inelastic electron scattering are given by $F^J(q)|0\rangle$ for all q. These are also all the states with nonvanishing transition density for transitions to the ground state. From equation (17) and

$$j_J(Z) = \frac{Z^J}{(2J+1)!!}\left(1-\frac{1}{1!(2J+3)}\tfrac{1}{2}Z^2+\ldots\right), \tag{48}$$

the states $Q_{J\alpha}|0\rangle$ give all the states excited by electron scattering. Then any final eigenstate of the nucleus can be written

$$|f\rangle = \sum_\alpha b_\alpha Q_{J\alpha}|0\rangle + |\text{Rem}\rangle, \tag{49}$$

where the remainder $|\text{Rem}\rangle$ does not contribute to inelastic electron scattering, and satisfies

$$\langle\text{Rem}|\rho(r)|0\rangle = 0. \tag{50}$$

Note that the states $Q_{J\alpha}|0\rangle$ are not eigenstates of any Hamiltonian and are not orthogonal. It is more convenient to use

$$M_{J\alpha} = \sum_\beta A_{\alpha\beta} Q_{J\beta} \tag{51}$$

such that

$$\langle 0|M_{J\alpha} H M_{J\beta}|0\rangle = 0 \qquad \text{for} \qquad \alpha \neq \beta. \tag{52}$$

The states $M_{J\alpha}|0\rangle$ are not necessarily orthogonal. In general, we have

$$\langle 0|M_{J\alpha} M_{J\beta}|0\rangle \neq 0 \tag{53}$$

(when the coefficients $A_{\alpha\beta}$ are real, the $M_{J\alpha}$ are real).

We consider the isoscalar excitation of a particular final state $|f\rangle$ with spin J. From equations (51), (48) and (17), we can write

$$F^J(q) = \sum_\beta f_\beta(q) M_{J\beta}. \tag{54}$$

It then follows that

$$\sum_\gamma \langle f|M_{J\gamma}|0\rangle\langle 0|M_{J\gamma}HF^J(q)|0\rangle/\langle 0|M_{J\gamma}HM_{J\gamma}|0\rangle$$
$$= \sum_{\gamma\beta}\langle f|M_{J\gamma}|0\rangle\langle 0|M_{J\gamma}HM_{J\beta}|0\rangle f_\beta(q)/\langle 0|M_{J\gamma}HM_{J\gamma}|0\rangle$$
$$= \sum_\gamma \langle f|M_{J\gamma}|0\rangle f_\gamma(q)$$
$$= \langle f|\sum_\gamma f_\gamma(q)M_{J\gamma}|0\rangle$$
$$= \langle f|F^J(q)|0\rangle, \tag{55}$$

which is the required form factor.

Using equation (51), we have

$$\langle f|F^J(q)|0\rangle = \sum_\gamma B_\gamma\langle 0|Q_{J\gamma}HF^J(q)|0\rangle, \tag{56}$$

and the terms in the form factor are given by equation (20), so that equation (56) could be used to fit electron scattering data. But the coefficients B_γ are not simply related to the properties of the state $|f\rangle$. The expansion in $M_{J\gamma}$ is more useful, especially as we propose to construct $M_{J\gamma}$ so that

$$\langle 0|M_{J\gamma}HF^J(q)|0\rangle \propto q^{J+2\gamma} \quad \text{as} \quad q\to 0, \tag{57}$$

so that the term with $\gamma = 0$ predominates at small q. By comparison, we have the relation

$$\langle 0|Q_{J\gamma}HF^J(q)|0\rangle \propto q^J \quad \text{as} \quad q\to 0 \quad (J\neq 0). \tag{58}$$

Defining

$$C_\gamma = E_f^{\frac{1}{2}}\langle f|M_{J\gamma}|0\rangle/\langle 0|M_{J\gamma}HM_{J\gamma}|0\rangle^{\frac{1}{2}}, \tag{59}$$

we see that

$$|C_\gamma|^2 = E_f|\langle f|M_{J\gamma}|0\rangle|^2/\langle 0|M_{J\gamma}HM_{J\gamma}|0\rangle$$
$$= E_f|\langle f|M_{J\gamma}|0\rangle|^2\Big/\sum_n E_n|\langle n|M_{J\gamma}|0\rangle|^2, \tag{60}$$

which is the fraction of the sum rule satisfied by the state $|f\rangle$. It follows then that

$$\langle f|F^J(q)|0\rangle = E_f^{-\frac{1}{2}}\sum_\gamma C_\gamma\langle 0|M_{J\gamma}HF^J|0\rangle/\langle 0|M_{J\gamma}HM_{J\gamma}|0\rangle^{\frac{1}{2}}. \tag{61}$$

Similarly, the transition density is given by

$$\langle f|\rho(\boldsymbol{r})|0\rangle = E_f^{-\frac{1}{2}}\sum_\gamma C_\gamma\langle 0|M_{J\gamma}H\rho(\boldsymbol{r})|0\rangle/\langle 0|M_{J\gamma}HM_{J\gamma}|0\rangle^{\frac{1}{2}}. \tag{62}$$

A set of M_J is constructed by taking

$$M_J = \sum_{\beta=0}^{\alpha} A_{\alpha\beta}Q_{J\beta}, \tag{63}$$

with $A_{\alpha\alpha} = 1$. The condition (52) is satisfied by

$$\langle 0 | Q_{J\beta} H M_{J\alpha} | 0 \rangle = 0 \qquad \text{for} \qquad \beta < \alpha \tag{64}$$

which, for given α, provides α simultaneous linear equations to determine the α coefficients $A_{\alpha\beta}$ ($\beta < \alpha$). The $A_{\alpha\beta}$ will depend on the ground state density distribution. It can be shown that this set of $M_{J\alpha}$ satisfies the relation (57). Since

$$M_{J0} = Q_{J0}, \tag{65}$$

the first term in the form factor is

$$\langle 0 | Q_{J0} H F^J(q) | 0 \rangle$$

and so the first term in the transition density is

$$\langle 0 | Q_{J0} H \rho(\boldsymbol{r}) | 0 \rangle,$$

which has the same radial dependence as the Tassie model result (1). At small q, the first term predominates, and so at small q we get the same answer as the Tassie model. The remaining terms will give corrections which become more important as q increases.

After solving the equations (64) we then eventually obtain from the equations (13), (22), (62) and (63), writing out only the first two terms,

$$\begin{aligned}\langle f | \rho(\boldsymbol{r}) | 0 \rangle = {} & (2\pi\hbar^2/mE_f A)^{\frac{1}{2}}(2J+1)^{-\frac{1}{2}} Y_{J0}(\Omega) \\ & \times \Big[-C_0 J^{\frac{1}{2}} \langle r^{2J-2} \rangle^{-\frac{1}{2}} r^{J-1} \, \mathrm{d}\rho_{00}/\mathrm{d}r \\ & \quad - C_1 \big\{ (2J+1)\{J(2J+5)+4\} \langle r^{2J+2} \rangle \\ & \qquad - J(2J+3)^2 \langle r^{2J} \rangle^2 \langle r^{2J-2} \rangle^{-1} \big\}^{-\frac{1}{2}} \\ & \quad \times \big\{ (2J+1) r^J \{ 2(2J+3)\rho_{00} + (J+2) r \, \mathrm{d}\rho_{00}/\mathrm{d}r \} \\ & \qquad - (2J+3) \langle r^{2J} \rangle \langle r^{2J-2} \rangle^{-1} J r^{J-1} \, \mathrm{d}\rho_{00}/\mathrm{d}r \big\} + \ldots \Big]. \end{aligned} \tag{66}$$

The first term is the Tassie model (1). It should be noted that, even in the higher terms, the transition density does not involve any derivatives higher than the first of the ground state density distribution.

For the uniform density distribution

$$\langle r^n \rangle = 3(n+3)^{-1} R^n, \tag{67}$$

where R is the nuclear radius, and the form factor is

$$\begin{aligned}\langle f | F^J(q) | 0 \rangle = {} & \{ 3E_f^{-1} (\hbar^2/2m)(2J+1) A \}^{\frac{1}{2}} R^{-1} \\ & \times \left(C_0 J^{\frac{1}{2}} j_J(qR) + \sum_\alpha (-1)^\alpha C_\alpha (2J+4\alpha+1)^{\frac{1}{2}} j_{J+2\alpha}(qR) \right), \end{aligned} \tag{68}$$

which we write as

$$\langle f \,|\, F^J(q) \,|\, 0\rangle = \{3E_f^{-1}(\hbar^2/2m)(2J+1)A\}^{\frac{1}{2}} R^{-1} \sum_\alpha C_\alpha F_\alpha . \tag{69}$$

Fig. 1 shows $F_0, -F_1$, F_2 and $-F_3$ for $J = 2$. If all the C_α are taken to have the same magnitude, it is seen that summation to $\alpha = 2$ is adequate for values of $qR \leqslant 4$. The predominance of the first term for $qR \leqslant 3$ shows why the hydrodynamical model has been so useful in analysing electron scattering data.

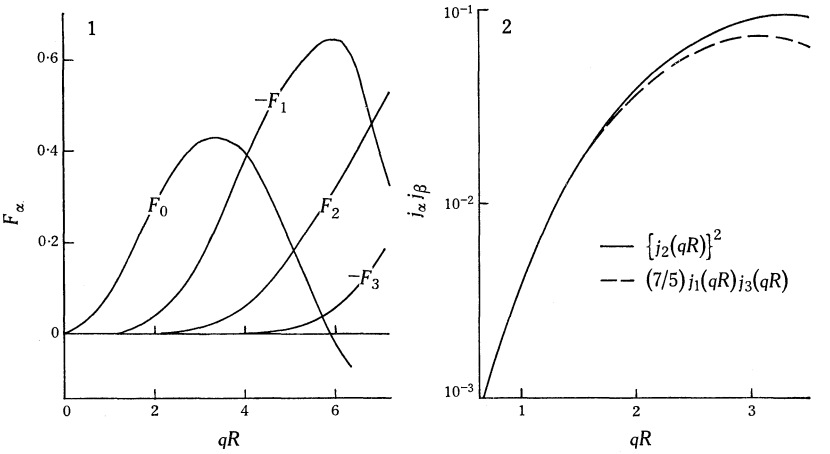

Fig. 1. Plot of the q dependence of the leading F_α terms in the series (69) for the inelastic form factor for electric quadrupole transitions, assuming a uniform density distribution.

Fig. 2. Comparison between the q dependence of $\{j_2(qR)\}^2$ and of $j_1(qR)j_3(qR)$, showing the difficulty of distinguishing between E2 and an additional contribution to E1 electron scattering.

Discussion

Equation (68) shows that higher order terms in the form factor for the excitation of a state of spin J have the same q dependence as the first term for the excitation of a state of higher spin, $J+2$, $J+4$, etc., as was also noted by Ui and Tsukamoto (1974) for $\alpha = 1$ and 2. Although this result does not hold for an arbitrary density distribution, the actual nuclear density distribution for heavy nuclei is roughly like the uniform distribution, and so qualitatively we should expect a similarity between the higher order terms for spin J and the lower order terms for states with higher spin. A well-known special case is the similarity for electric monopole and quadrupole transitions:

$$\langle 0 \,|\, M_{01} HF^J(q) \,|\, 0\rangle = -5^{-\frac{1}{2}} \langle 0 \,|\, M_{20} HF^J(q) \,|\, 0\rangle , \tag{70}$$

which is valid for an arbitrary density distribution.

If the form factors for two transitions are the same, it is only in the first Born approximation that the scattering must be the same. In the DWBA approximation, the effects of distortion depend on the multipolarity of the transition, and this provides some help in determining spins. However, caution is needed in allocating spins to nuclear states according to the shape of the angular distribution of inelastic electron scattering.

Equation (62) can be used to fit inelastic scattering of either electrons or protons by using the procedure of Boridy and Feshbach (1974), with the C_α as adjustable parameters subject to the restriction

$$C_\alpha \leqslant 1 . \tag{71}$$

The values of the C_α determined from experiment can then be compared with the results of nuclear structure calculations, by using some nuclear model to calculate the $\langle f | Q_{J\gamma} | 0 \rangle$.

Deal (1973) has calculated inelastic scattering using what he calls the single doorway and double doorway approximations. The single doorway approximation is just the first term in the sum in equation (66), and the double doorway approximation is equivalent to taking the two terms shown in equation (66). For electron excitation of the 2^+ level in ^{12}C at 4·43 MeV, Deal (1973) obtained a good fit to the experimental data for $q \leqslant 0{\cdot}7$ fm^{-1} with the single doorway approximation, and for $q \leqslant 1{\cdot}8$ fm^{-1} with the double doorway approximation. The data for $q \leqslant 2{\cdot}2$ fm^{-1} deviate from the double doorway approximation, showing the need to include a third term in the sum in equation (66).

If the no-particle–no-hole state is defined as the ground state of the nucleus then inelastic electron scattering excites only one-particle–one-hole states, whereas inelastic scattering by strongly interacting particles can excite higher excitations: two-particle–two-hole states and so on. This arises from the effects of coupling between channels which, for electron scattering, are negligible because of the weakness of the electromagnetic interaction. In principle it is possible to learn more about nuclear wave functions from the inelastic scattering of strongly interacting particles, but in practice this is difficult because of complications due to the strength of the interaction and the sensitivity of the interaction to many nuclear parameters. The analysis of the scattering of strongly interacting particles could be approached more readily if the one-particle–one-hole components of nuclear wave functions were first determined by a systematic study of electron scattering.

There are basically two reasons for the success of the hydrodynamical model based on irrotational and incompressible flow. The first is that inelastic electron scattering excites only the one-particle–one-hole parts of the excited nuclear states. The second is that, no matter how compressible or how viscous a fluid is, it flows incompressibly and irrotationally if it is struck gently enough, which is equivalent to inelastic scattering at low q in the nuclear case.

The treatment given above strictly holds only for isoscalar transitions but, if we ignore exchange corrections, it can be used for isovector transitions. Then, for an isovector E1 transition, the form factor for a uniform charge distribution is

$$F^1(q) = K\{C_0 j_1(qR) - C_1 7^{\frac{1}{2}} j_3(qR) + \ldots\} .$$

In the first Born approximation, the scattering is proportional to

$$|F^1(q)|^2 = K^2\left[C_0^2\{j_1(qR)\}^2 - 2C_0 C_1 7^{\frac{1}{2}} j_1(qR) j_3(qR) + \ldots\right], \tag{72}$$

and the first term is the result of the Tassie model or the Goldhaber–Teller (1948) model (the Tassie model strictly does not apply for $J = 1$ but, as it gives the same result as the Goldhaber–Teller model, it may be used). However, the second term

of equation (72), containing the factor $j_1(qR)j_3(qR)$, has a similar q dependence (for qR not too large) to the first term for an electric quadrupole transition, namely $\{j_2(qR)\}^2$, as is shown in Fig. 2. In analysing electron scattering to the giant $\Delta T = 0$, E2 resonance, the contribution from the giant $\Delta T = 1$, E1 resonance has to be subtracted, and we see that subtracting the E1 contribution according to the hydrodynamical model will leave a contribution with a q dependence similar to an E2 contribution. This result shows that care is needed when making multipole assignments on the basis of the hydrodynamical model.

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Manuscript received 6 August 1975